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The Use of Fuzzy Logic for Description of Spatial Relations between Objects

1. Introduction

In this article we present a review of major methods and algorithms for assessing directional spatial relations between fuzzy and crisp objects in 2D image. The types of relations mentioned are: point – point, point – object, crisp object – crisp object and fuzzy object – fuzzy object. Presented article also takes into account further research on use of fuzzy directional spatial relations for dynamical processes modeling on base of tomograms and thermograms.

Spatial relations between objects are currently one of the major research fields in image processing. The reasons responsible for such a state are growing requirements for robust and precise systems for high-level image analysis, enabling better understanding of its contents. Such a system could be applied in as diverse areas as medicine e.g. to autonomic description of histopathological images [25], image databases to search through pictorial data by matching high-level descriptions, or multi-user interactive games to support human-machine interactions. We can also add the fields of Geographic Information System, which are currently based mainly on spatial relations derived from the mathematical topology [15] and the point set theory [14].

Another area where image analysis involving determination of spatial relations is of special importance is control systems in which computer vision serves as one of the information sources. It is the case, for example, in mobile robot path-planning with obstacles avoidance as well as in various technological processes where physical fields are to be controlled. The authors of this paper are especially interested in the latter field. Obviously, the control action performed in such situation has to be derived on the basis of the observation of the image content i.e. objects' placement and their mutual special relation. Difficulty of the information processing grows when image elements are not sharply defined and so are the relations. Natural human ability to fulfill successfully and efficiently all such tasks is the

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main motivation behind the research on development of formal models of spatial relations. Moreover, since human-being reasoning based on images seen is performed rather in qualitative than quantitative manner, the relevant methods of imprecise data analysis, like fuzzy set theory, are of special interest.

Many different approaches and methods of spatial relation description have been developed so far and reported in literature. The goal of this paper is to present a comprehensive but concise survey of the major qualitative assessment methods for directional spatial relations, applicable to objects on an image interpreted as crisp sets or fuzzy sets. Most of them are based on mathematical formalism only [38], however they all try to mimic human way of reasoning with use of the theory of fuzzy logic and fuzzy sets [39] in most cases.

In this article the emphasis is put on the fuzzy assessment methods of directional spatial relations between two fuzzy objects which, as we believe, can be useful in developing of new 3D material flow type recognition algorithms and temperature control algorithms.

2. Spatial relations between objects

The spatial relations, as it was previously said, are important from the point of view of image processing and recognition and over the years they were used in knowledge-driven image recognition (like e.g. [19, 1, 4, 30, 27, 40]) or applied generally, mostly to segmented images to improve feature recognition, like in [12] or [9]. Their strength depends on the ability to qualitatively assess relations between objects in image, which is done on the basis of fuzzy sets.

One of the first researchers who took effort to discuss the problem and need for spatial relation model was Freeman [17]. He stated, that humans' supreme ability to assess spatial relations is based on their inherently imprecise reasoning. Therefore the "all-or-nothing" approach of the classical mathematical set theory is inadequate to the task of modeling human judgments. After that a proposal was given to use fuzzy sets because of its ability to handle vagueness, inherent to human reasoning.

Freeman claimed, that relations between objects can be divided into two groups. The first, involving comparison between properties of the objects (like smoothness or lightness) and the second, with comparison of their relative positions (to the left, above, far) called *Spatial Relations*. In spatial relations group he pointed 13 distinctive relations: **LEFT OF**, **RIGHT OF**, **BESIDE**, **ABOVE**, **BELOW**, **BEHIND**, **IN FRONT OF**, **NEAR**, **FAR**, **TOUCHING**, **BETWEEN**, **INSIDE**, **OUTSIDE**. In further research also **SURROUND** relation has been added. It is also worth mentioning, that the spatial relations can be divided into three groups on a basis of human perception as presented by Retz [32].

Therefore researchers had started to develop mathematical methods based on different manners and means with however common goal – to provide a model of precise assessment for spatial relations, that will be the most accurate with human imprecise reasoning. The further text will provide review of major assessment methods with emphasis on the division of input data as crisp and fuzzy regions. Similar work from recent years can be found in [3, 23, 7, 11, 8].

Presented methods:

- Aggregation of Angles
- Aggregation of Fuzzy Membership Functions
- Average Angle
- Fuzzy Set Compatibility with use of Histograms and Extension Principle
- Centroid
- Projection and Dominance
- Fuzzy Morphology, based on Dilation and Erosion
- R-Histogram
- Histogram of Forces

3. Spatial relation assessment methods

Aggregation Methods

Krishnapuram *et. al*. [23] stated, that the human perception of position of two objects is closely related to angular information. Thus for two points r, $a \in \Re^2$ they defined spatial relations taking into account the angle θ measured between the line ra and the horizontal line as in Figure 1a. The degree of membership of a relation "*a is to the LEFT of r*" can be obtained as follows

$$\mu_{LEFT}(\theta) = \begin{cases} 1 & |\theta| < \lambda\pi/2 \\ \left(\frac{\pi}{2} - |\theta|\right) / \left(\frac{\pi}{2}(1-\lambda)\right) & \lambda\frac{\pi}{2} \le |\theta| \le \frac{\pi}{2} \\ 0 & |\theta| > \pi/2 \end{cases} \tag{1}$$

In equation (1) large values of λ will give more optimistic results, whereas small values yields pessimistic ones. The simple and fast to calculate membership function μ_{LEFT} (1) is only an example used in [23]. Other functions, which might fit more properly the appropriate task of relation assessment can be used (like e.g. $\cos^2\theta$ or $\sin^2\theta$ – trigonometric based functions in [29], Figure 1b).

A relation between a point $r \in \Re^2$ and a crisp object[1] $A \subset \Re^2$, as depicted in Figure 1a, can also be determined [23, 33]. The point-to-object relation can be obtained by calculation of θ_a for every point a from set A. The resulting set of values $\Theta_a = \{\theta_a\}$ can be averaged to $\theta_{avg} = avg(\Theta_a)$ and assessed by function like (1) (which is then called the Average Method).

Having two crisp objects $R, A \subset \Re^2$, the angle θ_{ra} can be calculated for each pair (r, a), $r \in R$, $a \in A$, resulting in the $\Theta_{RA} = \{\theta_{ra}\}$ set. Similarly as for point-to-object relation, we can use (1) for assessment the object-to-object relation.

[1] Through the whole text we assume, that R/r and A/a are respectively a reference set/point and an argument set/point of examined spatial relation.

If we assume, that R^* and A^* are fuzzy, the object-to-object relation can by determined with same method as for crisp object just using α-cuts[2] of both fuzzy sets for calculations, respectively R^α and A^α. For each pair of points r, a from R^α, A^α we can obtain θ^α, and the set of angles $\Theta^\alpha = \{\theta^\alpha\}$ can be now averaged using equation (1), resulting in μ^α_{LEFT}. Another, but very similar approach proposed in [23] involves aggregation of computed membership values, e.g. μ^α_{LEFT}, for every angle θ^α. So instead of set $\{\theta^\alpha\}$, we use $\{\mu^\alpha_{LEFT}(\theta^\alpha)\}$ and e.g. generalized mean operator with normalized weights (all equal) as shown in [22]. It should be mentioned that $\{\mu^\alpha_{REL}(x)\}$ can be recovered [13] to $\mu_{REL}(x)$ with (2) or (3) in which n is the number of α-cuts.

$$\mu_{REL}(x) = \sup_\alpha \mu^\alpha_{REL}(x) \tag{2}$$

$$\begin{gathered} \mu_{REL}(x) = \int_0^1 \mu^\alpha_{REL}(x)d\alpha \approx \sum(\alpha_i - \alpha_{i+1})\mu^{\alpha_i}_{REL}(x) \\ 1 = \alpha_1 > \alpha_2 > \cdots > \alpha_n > \alpha_{n+1} = 0 \end{gathered} \tag{3}$$

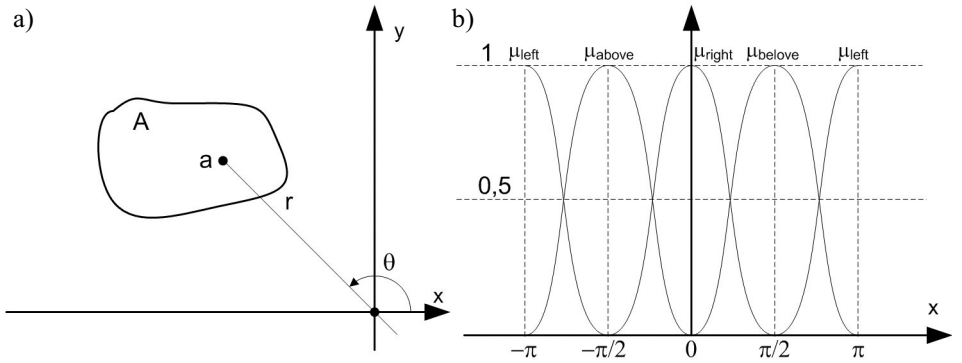

Fig. 1. Spatial relation between arbitrary point *a* from crisp object A (a) fuzzy membership functions for assessing relations in four cardinal directions (b)

Compatibility assessment method

Compatibility method, called also Histogram of Angles was introduced by Miyajima and Ralescu [29, 28]. Its main idea relies on the comparison between fuzzy set describing a relation, which membership function is given by (1) and a histogram (4) as an unlabeled fuzzy set.

Let us assume, that we have two crisp objects, R and A with points, respectively r_i and a_j, where $i = 1..n$ and $j = 1..m$. For each pair (r_i, a_j) an angle has to be calculated, with

[2] An **α-cut** of fuzzy set A is a crisp set of points defined as $A^\alpha = \{x{:}\mu_A(x) \geq \alpha\}$.

formula $\theta_{ij} = \angle(r_i, a_j)$, where the angle is measured between line $r_i a_j$ and horizontal line (the abscissa) passing through the r_i. A multiset[3] $\Theta=\{\theta_{ij}\}$ is obtained as a result. Let $f_\Theta(\theta)$ be the count of angle θ in the multiset Θ, $f_\Theta(\theta) = |\{(r_i, a_j):\angle(r_i, a_j) = \theta\}|$. With such input, we can obtain the Histogram of Angles $H_\Theta(R, A) = \{(\theta, f_\Theta(\theta))\}$, in normalized form given by equation (4).

$$H_\Theta(R, A) = \left\{\left(\theta, \frac{f_\Theta(\theta)}{mn}\right)\right\} \tag{4}$$

Now, we have $H_\Theta(R, A)$, an unlabeled fuzzy set, which can be interpreted as "*the spatial relation between R and A*". Therefore we have to use a compatibility [36] measure of two fuzzy sets. Authors in [29] propose Extension Principle denoted by (5).

$$\mu_{CP(H_\Theta, REL)}(v) = \begin{cases} 0 & \{s : v = \mu_{REL}(s)\} = \varnothing \\ \sup\limits_{\{s:v=\mu_{REL}(s)\}} \mu_{H_\Theta}(s) & \{s : v = \mu_{REL}(s)\} \neq \varnothing \end{cases}; \quad v \in [0; 1] \tag{5}$$

In equation (5) *REL* denotes fuzzy set of assessed relation type, like given by equation (1) or depicted in Figure 1b and $CP(H_\Theta, REL)$ is a fuzzy set for compatibility evaluation between H_Θ and *REL*. The final assessment, a degree to which tested relation holds, can be calculated from function $\mu_{CP}(H_\Theta, REL)(\nu)$ with use of Center of Gravity defuzzification.

Second approach utilizes fuzzy objects. Having two fuzzy objects[4], $R^* = \{(r_i, \mu_{R^*}(r_i))\}$, $A^* = \{(a_j, \mu_{A^*}(a_j))\}$, $i = 1..n$, $j = 1..m$, it is possible to use same algorithm as for crisp objects, until multiset of angles, Θ, is obtained. Now, with each $\theta_{ij} \in \Theta$ a two membership $\mu_{R^*}(r_i)$ and $\mu_{A^*}(a_j)$ are associated so that a membership degree has to be calculated for every θ_{ij}. Authors in [29] suggests fuzzy set $\Theta^* = \{(\theta_{ij}, \mu_\Theta(\theta_{ij})): \mu_\Theta(\theta_{ij}) = \min(\mu_{R^*}(r_i), \mu_{A^*}(a_j)), \theta_{ij} \in \Theta\}$.

To build the histogram, the cardinality of fuzzy set Θ^* and fuzzy frequency $ff(\theta)$ have to be calculated as given by (6). The final histogram is then obtained with use of equation (7).

$$card(\Theta^*) = \sum_{i,j} \mu_{\Theta^*}(\theta_{ij}); \quad ff(\theta) = \sum_{i,j} \left\{\mu_{\Theta^*}(\theta_{ij}) : \theta = \theta_{ij}\right\} \tag{6}$$

$$H_{\Theta^*}(R, A) = \left\{\left(\theta, \frac{ff(\theta)}{card(\Theta^*)}\right)\right\} \quad \theta \in [-\pi; \pi] \tag{7}$$

For obtained histogram $H_{\Theta^*}(R, A)$, the relation assessment method remains the same as described above for crisp objects.

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[3] Multiset can have duplicate elements.

[4] We use asterisk (*) to distinct fuzzy sets from crisp sets.

Centroid Method

The method of centroids presented in [23] and [22] uses points defined in the same space as the image. Each point is called *characteristic point* or *a representation of a crisp object A*. To obtain centroid, a following equation can be used:

$$C_A = \frac{\sum a_i}{|A|} \tag{8}$$

where:

C_A – centroid point,
a_i – point of an object A, $a_i \in A$,
$|A|$ – number of points in object.

To obtain final degree of sentence "*object A is to the left of object R*", an angle between line $C_R C_A$ and horizontal axis has to be calculated and assessed in exactly the same way as in crisp point-to-point relation described in Aggregation Methods, as shown in (9).

$$\mu_{A\_LEFT\_FROM\_R}(x) = \mu_{LEFT}\left(\angle(C_R, C_A)\right) \tag{9}$$

Projection and Dominance Method

Another definition of fuzzy relative position between two crisp objects was given by Kóczy [20]. For a crisp object R a fuzzy projection R^f is given by equations (10) and illustrated in Figure 2a.

$$\begin{aligned} R^f(x) &= \frac{R^{\max}(x) - R^{\min}(x)}{\max(R^{\max}(x) - R^{\min}(x))} \\ R^s &= \{x : R^f(x) > 0\} \end{aligned} \tag{10}$$

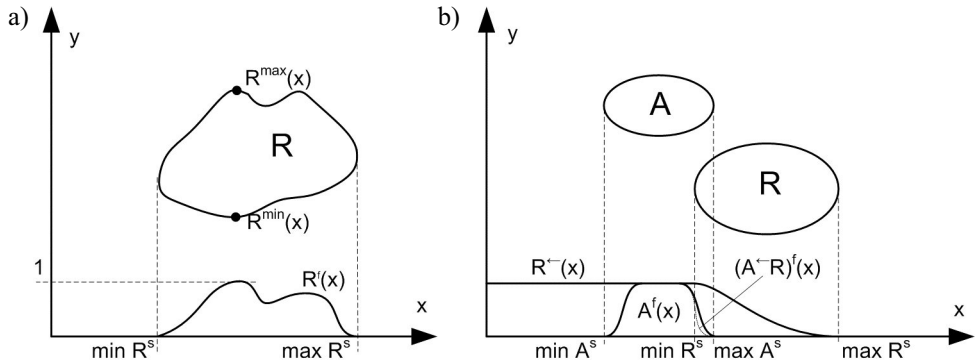

Fig 2. Projection of two crisp objects R and A and two statements $R^{\leftarrow}(x)$ and $(A^{\leftarrow}R)^f(x)$ [20]

In Figure 2b $A^f(x)$ stands for the sentence "*degree an object A is to be found on x*" and $R^{\leftarrow}(x)$ stands for "*in which degree this place is to the left from R*" as it is denoted by (11).

$$R^{\leftarrow}(x) = \frac{\int_x^{+\infty} R^f(\psi)d\psi}{\int_{-\infty}^{+\infty} R^f(\psi)d\psi} \tag{11}$$

In next step Kóczy formulates sentence "*x is in the fuzzy projection of A and to the left of R*" (12) using *prod* operator for conjunction and after that the final degree of relation "***A is to the left from R***" can be defined by (13).

$$(A^{\leftarrow}R)^f(x) = A^f(x) \wedge R^{\leftarrow}(x) = A^f(x) \cdot \frac{\int_x^{+\infty} R^f(\psi)d\psi}{\int_{-\infty}^{+\infty} R^f(\psi)d\psi} \tag{12}$$

$$LEFT(A, R) = \frac{\int_{-\infty}^{+\infty} (A^{\leftarrow}R)^f(x)dx}{\int_{-\infty}^{+\infty} A^f(y)dy} = \frac{\int_{-\infty}^{+\infty} A^f(x)\int_x^{+\infty} R^f(y)dydx}{\int_{-\infty}^{+\infty} R^f(x)dx\int_{-\infty}^{+\infty} A^f(x)dx} \tag{13}$$

Another approach was presented by Keller and Sztandera [21]. In this paper they presented method for modeling relations between crisp or fuzzy objects in 2D image, assuming that fuzzy objects are convex. The method is based on projection of fuzzy objects on U×V axes (principal and orthogonal), conversion of fuzzy objects into crisp objects with use of α-cuts and calculation the ratio of the squared difference between the midpoints of the half-width of intervals, which can be interpreted also as measure of dominance.

Let us have a projections of fuzzy objects A^* and R^* on axis U and V. Assuming, that input sets are convex, so will be the projections [39, 34]:

$$\begin{aligned} \mu_{A^*_U}(x) &= \sup_y\{\mu_{A^*}(x, y)\} \quad \mu_{A^*_V}(y) = \sup_x\{\mu_{A^*}(x, y)\} \\ \mu_{R^*_U}(x) &= \sup_y\{\mu_{R^*}(x, y)\} \quad \mu_{R^*_V}(y) = \sup_x\{\mu_{R^*}(x, y)\} \end{aligned} \tag{14}$$

For each of these four projections an α-cut set has to be obtained, as in (15).

$$A_U^{\alpha} = \{x : \mu_{A^*_U}(x) \geq \alpha\} = [A_U^{\alpha_L}; A_U^{\alpha_R}] \tag{15}$$

As the used sets are convex, each α-cut constitutes an left-right interval as in Figure 3.

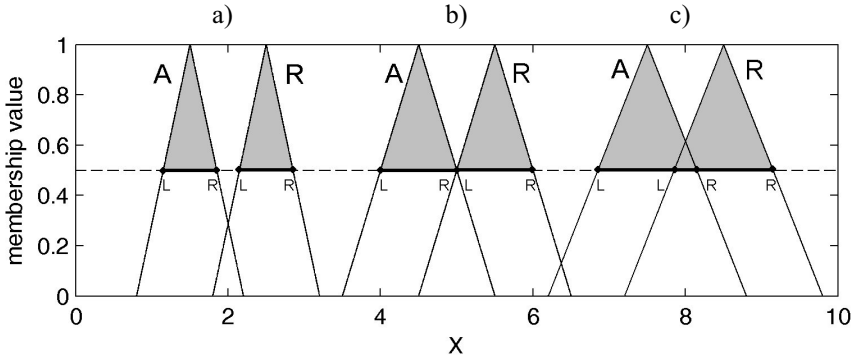

Fig. 3. Three cases of separation. All three cases have α-cut fixed on the same level α = 0.5, however fuzzy sets for each case differ with parameters

Let us now have S_U^α which will be the *ratio of the square of difference between the midpoints of the half-width of intervals*. Similar equations are used to process projections of A^* and R^* on the V axis:

$$S_U^\alpha = \left(\overline{A_U^\alpha} - \overline{R_U^\alpha}\right)^2 / \left(\overline{W_{A_U}^\alpha} - \overline{W_{R_U}^\alpha}\right)^2 \tag{16}$$

$$\begin{aligned} \overline{A_U^\alpha} &= (A_U^{\alpha_L} + A_U^{\alpha_R})/2 \quad & \overline{R_U^\alpha} &= (R_U^{\alpha_L} + R_U^{\alpha_R})/2 \\ \overline{W_{A_U}^\alpha} &= (A_U^{\alpha_R} - A_U^{\alpha_L})/2 \quad & \overline{W_{R_U}^\alpha} &= (R_U^{\alpha_R} - R_U^{\alpha_L})/2 \end{aligned} \tag{17}$$

Basing on S_U^α, one can infer about the type of relation between A and R.

- If $S_U^\alpha > 1$ then both sets are a-*separated*, because $A_U^{\alpha_R} < R_U^{\alpha_L}$ (Fig. 3a).
- If $S_U^\alpha = 1$ then both sets are a-*just separated*, because $A_U^{\alpha_R} = R_U^{\alpha_L}$. In other words, both sets are connected at given value of a in only one point (Fig. 3b).
- If $S_U^\alpha < 1$ then both sets are a-*overlapping*, because $A_U^{\alpha_R} > R_U^{\alpha_L}$ (Fig. 3c).

It has to be noted, that the Projection Methods depends on axes on which those projections are calculated, hence cardinal directions related to projection axes will receive more support than middle directions.

Fuzzy Morphology

Fuzzy Morphology was evolving trough years, starting from topological definitions made by Rosenfeld in [35] and [34]. However the first extensive lecture was published by Bloch and Maitre [10] and further research in assessment of spatial relations with use of mathematical morphology has been done mainly by Bloch.

The two most important operators provided by mathematical morphology to the fuzzy image theory are Erosion and Dilation given by (18) and presented in Figure 5.

$$
\begin{aligned}
D_{(\mu,\nu)}(x) &= \sup_{y \in S} t\left(v(y-x), \mu(y)\right) \\
E_{(\mu,\nu)}(x) &= \inf_{y \in S} T\left(c(v(y-x)), \mu(y)\right)
\end{aligned}
\tag{18}
$$

where:

- μ – fuzzy input image to be dilated or eroded, depicted in Figure 4,
- ν – fuzzy structuring element to be used, depicted in Figure 4,
- D – operation of dilation,
- E – operation of erosion,
- t, T – respectively t-norm and t-conorm,
- $c(\bullet)$ – complementation operator, e.g. $c(x) = 1 - x$, $x \in [0; 1]$.

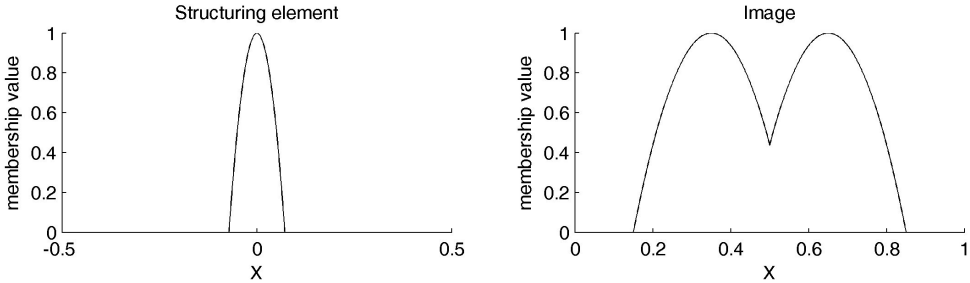

Fig. 4. A structuring element and a sample fuzzy image

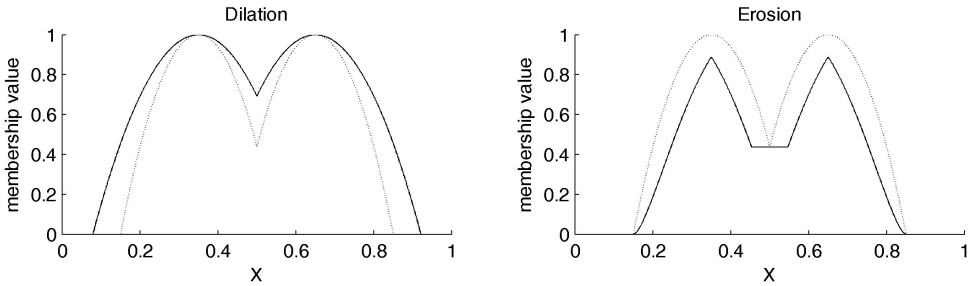

Fig. 5. Results of dilation and erosion with source images form Figure 4

For depicting both morphological operation, we assumed *min* as t-norm, *max* as t-conorm and a standard complementation operator, $c(x) = 1 - x$.

In [5], Bloch proposed two new methods of assessing directional spatial relations between objects based on fuzzy morphology: *Compatibility Fuzzy Pattern Matching* (CFPM) and *Morphological Fuzzy Pattern Matching* (MFPM). CFPM method, designed for crisp objects, combines Histogram of Angles and dilation/erosion operators as follows:

$$\begin{aligned} \Pi(H_\Theta, R) &= D(H_\Theta, R) = \sup_{x\in[-\pi;\pi]} t\left(\mu_{H_\Theta}(x), \mu_R(x)\right) \\ N(H_\Theta, R) &= E(H_\Theta, R) = \inf_{x\in[-\pi;\pi]} T\left(\mu_{H_\Theta}(x), 1-\mu_R(x)\right) \end{aligned} \tag{19}$$

where:

H_Θ – histogram of angles calculated for objects R and A (fuzzy set to be dilated),
R – directional relation (fuzzy structuring element),
Π – degree of possibility which can be interpreted as degree of intersection,
N – degree of necessity which can be interpreted as degree of inclusion.

In MFPM method, two approaches to membership function derivation of desired directional spatial relation were proposed. First[5] one is based on a chosen point of reference (e.g. centroid of reference object) r and calculation of membership value in each point a of the given image as a decreasing function of difference between the angle of relation and the line ar, as depicted in Figure 6a. Second approach consists of creating a band in desired direction. Membership values of points is this direction (inside the band) have to be constant and membership values of points outside the band can be computed in the same way as in previous approach (as decreasing function). A sample band is depicted in Figure 6b.

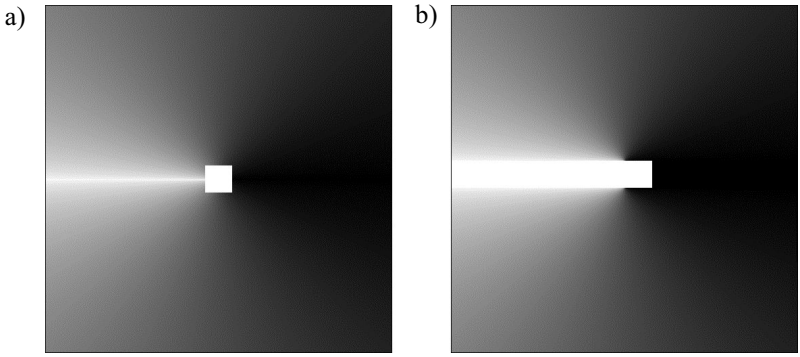

Fig. 6. A two definition of relation "*to the left of*", proposed in [5]. White color corresponds to membership value 1.0 and black color to value 0.0. The squares in the middle of both pictures are the reference objects

[5] This approach is very similar to the point-to-object assessment method.

For both proposed methods (CFPM and MFPM), pair *dilation/possibility/degree of inclusion* and *erosion/necessity/degree of intersection* are the assessment values of tested spatial relation. Now let assume, that $s(O_f)$ is the *support* of fuzzy object O_f and $c(O_f)$ is the *core* of same object. Necessity and possibility can be studied on a basis of following four rules [24]. O_f is a fuzzy object, R is a relation depicted in Figure 6:

- $\Pi(O_f, R) = 0 \Leftrightarrow s(O_f) \cap s(R) = \emptyset$,
- $\Pi(O_f, R) = 1 \Leftrightarrow c(O_f) \cap c(R) \neq \emptyset$
- $N(O_f, R) = 1 \Leftrightarrow s(R) \subseteq c(O_f)$,
- $N(O_f, R) > 0 \Leftrightarrow c(R) \subset s(O_f)$

Other approach for crisp objects was proposed by Gader [18]. He defined dilation operation as the only one which he needed to define spatial relation with use of morphology.

$$R \oplus S = \bigcup_{r \in R} S + r \tag{20}$$

The equation (20) represents dilation as a union of copied shapes by structuring element S at each point r. As a structuring element S, Gader has used a ray denoted as $ray(l, \theta)$ and interpreted as "*a ray of length l cast from point (0,0) in direction* θ". Figure 7 shows examples of the proposed dilation operator. Figure 7a shows the reference element R, Figure 7b shows the object R but after applying dilation operator (20). The structured element used is $ray(l, 45^o)$ denoted as S with l chosen such that its value is greater or equal then the diameter of union $R \cup A$. The angle θ describes expected spatial relation, e.g. for sentence *"A is to the right and above of R",* value of r can be set to 45°.

Figure 7c shows test environment with a reference object R and a test object A. Objects A will be tested for directional relation "*on the right and above of R*". Figure 7d shows result of equation $A\Diamond = (R \oplus S) \cap A$. The final assessment can be done by calculating distance between set $A\Diamond$ and set A.

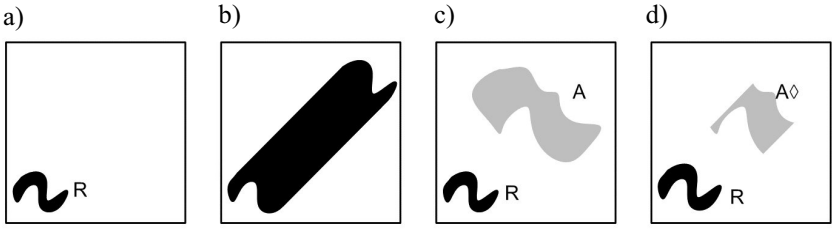

Fig. 7. Images depicting Gader's approach to assessment of spatial relations [18]

Very similar approach to Gader's rays combined with [5] is presented by Bloch in [6], in particular for 3D brain images obtained by magnetic resonance. In this case however, author sets a "landscape" around reference object as a decreasing function of angle between

vector $u(\alpha_1,\alpha_2) = (\cos\alpha_2\cos\alpha_1, \cos\alpha_2\sin\alpha_1, \sin\alpha_2)$ cast from selected point of the reference against a point of tested object. More about landscape functions can be found in [7].

R-Histogram

The idea of R-Histogram was presented by Wang and Makedon in [37] as an extension to the Histogram of Angles for the crisp objects which can handle directional relations only, without topological spatial relations like *inside* or *outside*. R-Histogram introduces labeled distance $LD(p_1, p_2)$ between two points, which in fact is an ordered pair $(d(p_1, p_2), l(p_1, p_2))$ where $d(\bullet,\bullet)$ is Euclidean distance in $\Re^2$ and $l(\bullet,\bullet)$ is the mentioned label of distance d. The interpretation of labels is explained in Table 1.

Table 1
The interpretation of values generated by labeling function $l(\bullet,\bullet)$

Is point r_i inside A?	Is point a_j inside R?	$l(r_i, a_j)$
No	No	**0**
No	Yes	**1**
Yes	No	**2**
Yes	Yes	**3**

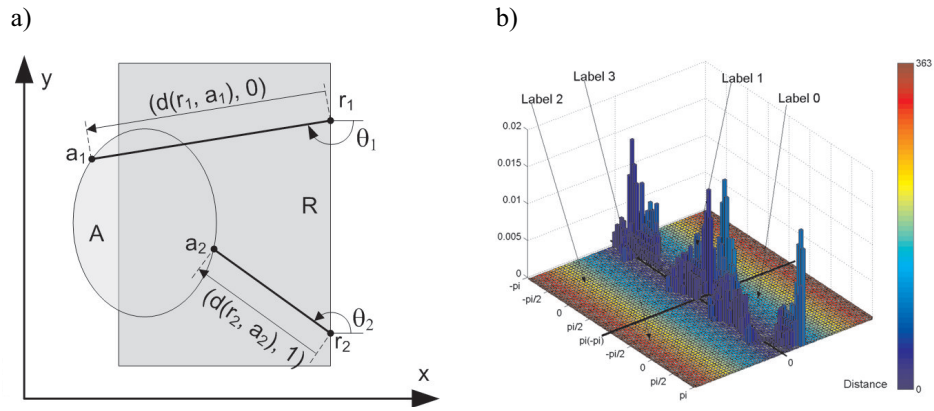

Fig. 8. a) explanation of distances, labels and main idea of gathering data for R-Histogram; b) a sample histogram with all four labels shown [37]

Let us assume, that we have two objects, A and R, as depicted in Figure 8a. There is also given an arbitrary pair of points: r_i on the boundary of R and a_j on the boundary of A. We have to consider three elements: an angle $\theta(r_i, a_j)$ measured between horizontal line passing through point r_i and vector $r_i a_j$ in range $[-\pi; \pi]$, a length of this vector $d(r_i, a_j)$, and a label $l(r_i, a_j)$.

After defining all measures necessary to build the R-Histogram, equation (21) should be applied for every mentioned vector $r_i a_j$.

$$H(I,J,L)=\begin{cases} H(I,J,L)+1 & if \quad \theta(r_i,a_j)\in A_I \\ & \quad \wedge d(r_i,a_j)\in D_J \\ & \quad \wedge l(r_i,a_j)=L \\ H(I,J,L) & otherwise \end{cases} \tag{21}$$

where:

A_I – the range of angles spanned by $H(I, J, L)$; a set of discrete intervals defined by $A_I = [2\pi k/n;\ 2\pi(k+1)/n[$ where $n \in \aleph$, $k \in 0..(n-1)$,

D_J – the range of distance spanned by $H(I, J, L)$. definition very similar to A_I,

L – The set of labels associated with distance $d(\bullet,\bullet)$, $L \in \{0, 1, 2, 3\}$.

The calculated histogram $H(I, J, L)$ has to be normalized for all I, J, L and the final result $h(I, J, L) = Normalize(H(I, J, L))$ is depicted in Figure 8b. Please note the four labeled quadrants corresponding to the set L described in the equation (21).

To assess the spatial relation described by a R-Histogram, authors in [37] proposed a similarity measure (distance measure) between given histogram $h(I, J, L)$ and histograms of images depicting different relations. In Wang *et al*. [38] authors proposed use of the k-NN classifier to improve preciseness of assessment.

Histogram of Forces

The Histogram of Forces, proposed by Matsakis and Wendling [26] differs from previous aggregation/histogram methods. Every one of them was based on a point-to-point relation, hence they handled objects as set of points. The proposed approach handles crisp and fuzzy objects as *longitudinal sections* (Fig. 9a).

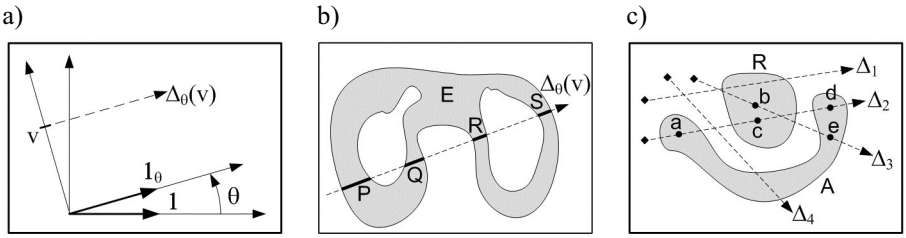

Fig. 9. Definition of longitudinal sections, their application and interpretation of directions

At the beginning, authors define oriented, horizontal line, denoted as $\Delta(v)$, where v is value on the ordinate[6]. This line is defined in the Euclidean coordinate system, which is then rotated by angle θ. Results of those transformations are depicted in Figure 9a. $\Delta_\theta(v)$ is the mentioned oriented horizontal line.

[6] $\Delta(v)$ is identical to linear equation $y = ax+b$, where $b = v$ and $a = 0$.

The longitudinal section $E_\theta(v)$ can be obtained as $E_\theta(v) = E \cap \Delta_\theta(v)$. Result of this calculation can be interpreted as a set of intervals or an union of segments as depicted in Figure 9b. For the case presented, $E_\theta(v) = P \cup Q \cup R \cup S$.

Figure 9c helps to visualize the method of assessment. $\Delta_1..\Delta_4$ are shorten version of $\Delta_{\theta_1}(v_1)..\Delta_{\theta_4}(v_4)$. Diamonds (♦) are observers. Each of the observers searches for argument to support relation "*A is in direction* θ *from R*". Observer 3 can produce such argument: (b, e). Observer 2 can produce two arguments: (a, c) and (c, e) however only (c, e) supports the mentioned relation.

For example, let us assume that we have two objects: R (reference object) and A (argument object; its relations against the R will be assessed). The given values θ and v produces $E_\theta(v) = R_1 \cup A_1$, $R_\theta(v) = \{R_1\}$ and $A_\theta(v) = \{A_1\}$ (similar to Figure 9c and Observer 3). With such information, authors suggest equation (22) for processing the segments:

$$f^{r}_{\theta,v}(d_{R_i}, D_{R_i,A_j}, d_{A_j}) = \int_{D_{R_i,Aj}+d_{A_j}}^{d_{R_i}+D_{R_i,A_j}+d_{A_j}} \left(\int_0^{d_{A_j}} \varphi^r(u-v)dv \right) du \tag{22}$$

where:

- d_{Ri} – length of segment R_i, $|R_i|$,
- d_{Aj} – length of segment A_j, $|Aj|$,
- $D_{Ri,Aj}$ – distance between segments R_i and A_j,
- φ^r – function of the elementary attraction force applied by any point of A to a point of R. In other words, this force wants to pull R in direction θ.

Elementary function φ^r is defined as $\varphi^r(d) = 1/d^r$, where r characterizes the force and d is the distance between points. For $r = 0$ forces are constant and for $r = 2$ forces are gravitational.

Next, every longitudinal section should be processed with equation (23).

$$F^r(\theta, R_\theta(v), A_\theta(v)) = \sum_{\substack{i \in 1..|R_\theta(v)| \\ j \in 1..|A_\theta(v)|}} f^{r}_{\theta,v}(d_{R_i}, D_{R_i,A_j}, d_{A_j}) \tag{23}$$

The value obtained with this equation can be interpreted as a weight $(R_\theta(v), A(v))$ of argument supporting expression "*A is in direction* θ *of R*". Finally, the histogram of forces is represented by (24):

$$F^r_{R,A}(\theta) = \int_{-\infty}^{+\infty} F^r(\theta, R_\theta(v), A_\theta(v))dv \tag{24}$$

where:

- θ – the range of the histogram, $\theta \in [-\pi; \pi]$,
- r – the type of force as in equation (22),
- R, A – reference object and argument object.

Assessment of calculated histogram is done in the same way as in the case of angle histograms – by comparison with fuzzy set, declared on the same universe of discourse as the histogram, that is [–π; π]. In the case of processing fuzzy objects, the standard approach is used, which involves α-cuts. In order to extend presented method, the equation (23) should be modified as follows (however, authors give two examples) [13]:

$$F^r(\theta, R_\theta(v), A_\theta(v)) = \sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j F^r(\theta, R_\theta^{\alpha_i}(v), A_\theta^{\alpha_j}(v)) \qquad (25)$$

$$m_i = \alpha_i - \alpha_{i+1} \quad m_j = \alpha_j - \alpha_{j+1}$$

The histogram of forces approach has been extended to 3D space by Ni *et al.* [31].

4. Fields of interests and further research

Our current work is focused on the development of methods and means for applying spatial relations, as directional relations and distance relations, to two types of images: a 3D volumetric image produced by Electrical Capacitance Tomography ECT [2] (Fig. 10a) which is the major area of research in Computer Engineering Department TUL and to 2D images presenting temperature field distribution at the surface, being for example a result of induction heating (Fig. 11a) [16].

The first case assumes that it is possible to create a model of spatial relations between objects in two-phase gas-liquid flow involving both directions and distances (Fig. 10b). Such a model, a map of spatial relations, not only could allow to infer about the type of flow but also it could become a basis for synthesis of a control system.

a) b)

Fig. 10. An Electrical Capacitance Tomograph: a) the experimental set-up; b) a reconstructed image

In the second application area the spatial relation description can formally depict regions of the charge (steel cylinder) which are over- or under-heated. In this way the efficient temperature control algorithm can be developed. Images for analysis and decision-making process can be obtained with use of a thermovision camera (Fig. 11b).

a) b)

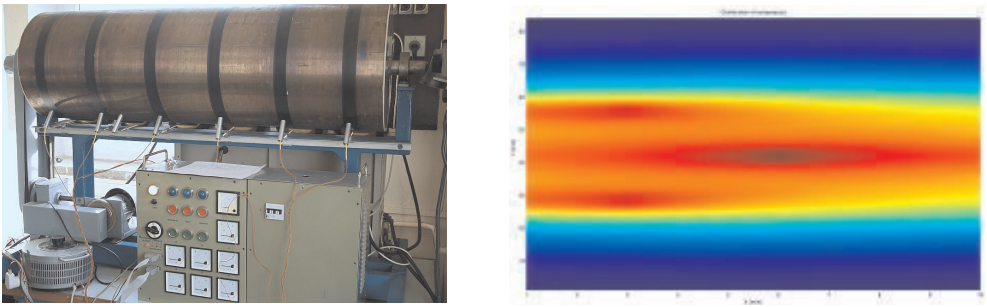

Fig. 11. A steel cylinder heated by the set of inductors with sensors and control panel (a); a theoretical model of temperature distribution with three heating points (b) (two points at $X = 3$ and one at $X = 6$)

It should be noted that both types of images are not crisp but rather ambiguous and imprecise hence can be considered as fuzzy sets. Such assumption leads to the conclusion that the fuzzification step of the fuzzy control structure is not necessary and can be ommited in both applications. This gives no information loss typical for fuzzyfying algorithms.

The challenge of both applications is the dynamic nature of both technological processes, being of special importance if their control is to be concerned. For that we have to review temporal modeling approach in fuzzy logic and combine them with spatial relations.

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