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VIBRATION CONTROL OF FUNCTIONALLY GRADED PIEZOELECTRIC PLATES

SUMMARY

The study develops the modelling of active laminated plates with homogeneous piezopolimer sensor layers and newly proposed piezoelectric functionally graded actuator layers. In the considered model the transverse distribution of electromechanical properties is defined by a variation of the PZT ceramic constituent, which is described by the exponential function. The effective electromechanical properties of the composite are obtained based on the rule of mixtures. The governing equations for laminated plates are formulated. The control strategy is based on the constant gain velocity feedback. The effects of the applied distribution of piezoceramic fraction within the actuator layers on the structural response of the plate including changes in the natural frequencies and reduction of the resonant amplitudes are numerically examined.

Keywords: *laminated plate, piezoelectric control, functionally graded actuator*

STEROWANIE DRGANIAMI GRADIENTOWYCH, PIEZOELEKTRYCZNYCH PŁYT

Praca jest rozwinięciem modelowania aktywnych, laminowanych płyt zawierających jednorodne, piezopolimerowe warstwy pomiarowe oraz nowego typu warstwy wykonawcze (aktuatory) z piezokompozytu o strukturze gradientowej. Rozkład właściwości elektromechanicznych w kierunku poprzecznym aktuatora wyznaczony jest przez zmienny udział piezoceramiki PZT, który w rozważanym modelu opisany jest funkcją wykładniczą. Efektywne elektromechaniczne właściwości kompozytu określono na podstawie prawa mieszanin. Sformułowano równanie ruchu płyty, zakładając sterowanie z prędkościowym sprzężeniem zwrotnym o stałym współczynniku wzmocnienia. Zbadano symulacyjnie wpływ parametrów rozkładu składnika piezoelektrycznego w warstwach aktuatorów na częstości drgań własnych i charakterystyki amplitudowo-częstotliwościowe aktywnej płyty, ze szczególnym uwzględnieniem redukcji amplitud rezonansowych w określonym przedziale częstotliwości.

Słowa kluczowe: *płyty laminowane, sterowanie piezoelektykiem, gradientowe aktuatory*

1. INTRODUCTION

Active laminated composites with integrated piezoelectric sensor and actuator layers have found a relevant role in vibration control of light-weight flexible structures. In order to achieve the satisfied control effectiveness a relatively large deformation of piezoelectric actuator layers is generated during the control process. Thus, the interaction between the piezoelectric layer and the main structure creates severe interfacial shear stresses, which initiate micro-cracks, and finally may lead to the edge delamination and failure of the system. The effect of the shear stress concentration is enhanced by the sharp change in elastic properties of bonded layers i.e. piezoceramic layers, which are widely used for actuation, and classic layers of the laminate (e.g. graphite-epoxy, glass-epoxy). The damage hazard may be reduced significantly by using piezocomposite layers with varied electromechanical properties across the thickness direction. The concept of functionally graded materials (FGMs) has been introduced (applied) as a thermal barrier to reduce the high thermal stress field at the interface between ceramic and metal, and prevent reduction of strength and stiffness of the structure. Thus, the most familiar FGMs are a mixture of ceramic and metal with a continuously varying volume fraction and they are widely used in air- and spacecrafts, reactor vessels and other applications in a high-temperature environment. The mechanical behaviour of FGMs has been the topic of interest of a large number of researches during the last few years. For example, the response of functionally graded ceramic-metal plates is investigated in [6], the vibration analysis of functionally graded cylindrical shells is presented in [4], the analytical solution for large deflections of FGM plates and shells is derived in [11], FGM plates subjected to transverse loading are studied in [2] and [12]. Active control of the FGM plate with integrated monolithic piezoelectric sensor/actuator layers is discussed in [3] and a finite element model of the FGM plate with piezoelectric fiber composite (PFC) for static analysis can be find in [7]. In order to increase the reliability of piezoelectric devices a new kind of materials, named piezoelectric functionally graded (PFG) materials, has been developed. The static transverse displacements and stress field in PFG laminates stacked of layers with electromechanical properties varied from layer to layer is analysed in [1]. In the paper [5] the vibration control of la-

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minated plates using multi-layered functionally graded PFC actuators is studied. The dynamic stability analysis of the PFG plate under the time-dependent thermal load is presented in [9]. The finite element modelling for the static and dynamic analysis of the PFG bimorph is proposed in [10].

In the quest of developing smart structures the modelling of laminated plates with newly proposed actuator layers made of piezoelectric functionally graded material is presented in this study. The electroelastic properties of each PFG layer vary continuously in the thickness direction due to the PZT (lead-zirconate-titanate) fraction exponential distribution. The dynamic analysis is based on the classical laminated plate theory and concerns a steady-state out-of plane vibration. The vibration reduction is achieved applying the velocity feedback control. The effects of the transversal distribution of the volume fraction of piezoelectric and matrix constituents on the PFG actuator properties and the structural response of the plate are examined and discussed.

2. ACTIVE LAMINATED PLATE MODELING

A simply supported rectangular symmetrically laminated plate of dimensions a and b is considered. The plate is composed of conventional orthotropic fiber reinforced layers and piezoelectric sensor/actuator layers with the transverse poling direction. The sensor layers are assumed as the PVDF (polyvinylidene fluoride) film, while the actuators are of the piezoelectric functionally graded material. The electroelastic properties of each PFG lamina vary continuously in the thickness direction due to the exponential distribution of the PZT fraction. To produce the control bending action the midplane symmetric actuators are both the PFG with either opposite polarization or opposite applied electric field. The actuators are driven by a voltage generated by the sensor layers and transformed according to the velocity feedback control. The dynamic analysis is based on Kirchhoff's simplifications of the plate deformation cf [8].

2.1. Formulation of the piezoelectric functionally graded actuator

Assuming a plane stress state in the actuator layer the constitutive relation of a piezoelectric material in the principal material axes 1, 2, 3 can be reduced to the following matrix form

$$\sigma = \mathbf{c}\varepsilon - \varepsilon\mathbf{E} \tag{1}$$

where:

- σ – stress, $\sigma = [\sigma_1, \sigma_2, \tau_{12}]^T$,
- ε – strain, $\varepsilon = [\varepsilon_1, \varepsilon_2, \gamma_{12}]^T$,
- $\mathbf{E}$ – external electric field, $\mathbf{E} = [E_1, E_2, E_3]^T$ (T denotes matrix transposition),
- $\mathbf{c}$ – stiffness matrix,
- $\mathbf{e}$ – matrix of piezoelectric coefficients, which for the PZT material may be reduced to the form

$$\mathbf{e} = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

In the considered geometry of piezoelectric layers the poling direction is along the 3-axis and coincides with the layer thickness and the z-axis of the laminate. Thus, the electric field components E_1 and E_2 are of zero value. Since, the piezoelectric layers are relatively thin, it is assumed that each layer has a linear electric potential function through the thickness and the constant electric field E_3.

The piezoelectric functionally graded (PFG) material of the actuator is a two-phase isotropic composite and can be defined by a variation of the volume fraction of piezoelectric and matrix constituents. It is assumed that the distribution of the PZT (lead-zirconate-titanate) ceramic volume fraction is the exponential function through the layer thickness h_a given by

$$v(z_l) = v_0 \exp\left(\lambda\left(z_l + \frac{h_a}{2}\right)\right) \quad \text{with } \lambda = \frac{1}{h_a} \ln \frac{v_1}{v_0} \tag{3}$$

where:

- z_l – local co-ordinate measured from the middle surface of the layer,
- v_0 – PZT volume fraction at the bottom surface, which is near to the midplane of the laminate ($z_l = -h_a/2$),
- v_1 – PZT volume fraction at the top surface ($z_l = h_a/2$).

The parameter λ describes the inhomogeneity of the PFG material across the thickness.

The effective properties of the two-phase PFG material, considered in the study, are determined due to the rule of mixtures and by using the following general formula

$$P_{eff}(z_l) = (P_p - P_m)v(z_l) + P_m \tag{4}$$

where:

- P_p – properties of pure piezoelectric material (PZT),
- P_m – properties of matrix material (epoxy resin).

The effect of Poisson's ratio gradation in the actuator layer on the laminate mechanics is much less than that of Young's modulus. Thus, Poisson's ratio is assumed to be constant in this study.

Based on the formula (4) combining with the distribution low of the PZT volume fraction, Equation (3), and taking into account electroelastic isotropy of the PFG material, Young's modulus and piezoelectric coefficient varying in the thickness direction can be given by the relations, respectively:

$$Y(z_l) = (Y_p - Y_m)v_0 \exp\left(\lambda\left(z_l + \frac{h_a}{2}\right)\right) + Y_m \tag{5}$$

$$e_{31}(z_l)=\frac{1}{1-v}\left[(Y_p-Y_m)d_{31}v_0^2\exp\left(2\lambda\left(z_l+\frac{h_a}{2}\right)\right)+\right.$$
$$\left.+d_{31}Y_m v_0\exp\left(\lambda\left(z_l+\frac{h_a}{2}\right)\right)\right] \tag{6}$$

where:

Y_p, Y_m – Young's moduli of piezoelectric and matrix material, respectively,

d_{31} – piezoelectric strain constant for the PZT material.

The piezoelectric coefficient function Eq. (6) is formulated for a non-piezoelectric matrix and the elements of the stiffness matrix related to the mechanically isotropic material:

$$c_{11}=c_{22}=\frac{Y(z_l)}{1-v^2},\qquad c_{12}=c\frac{vY(z_l)}{1-v^2}$$
$$c_{66}=\frac{Y(z_l)}{2(1+v)} \tag{7}$$

The control moment resultant $\mathbf{M}^E$ for m PFG actuators symmetrically located about the laminate midplane can be defined by

$$\mathbf{M}^E=\begin{bmatrix}M_x^E\\ M_y^E\\ M_{xy}^E\end{bmatrix}=\sum_{k=1}^{m}\int_{z_{k-1}}^{z_k}\mathbf{e}^k\,\mathbf{E}^k\,z\,dz \tag{8}$$

The moment resultant $\mathbf{M}^E$ is formulated with respect to the x, y, z reference axes of the plate with the z co-ordinate measured from the midplane in the thickness direction.

After substituting Eq. (2) and assuming for each layer the external electric field in the thickness direction the actuator interaction may be reduced to the following moment components

$$M_x^E=M_y^E=\sum_{k=1}^{m}E_3^k\int_{z_{k-1}}^{z_k}e_{31}^k(z)\,z\,dz \tag{9}$$

In the Eq. (9) the distribution of piezoelectric coefficient $e_{31}^k(z)$ for each actuator layer is given by Eq. (6) transformed to the global z-axis.

Since, the PFG material isotropy is considered, the twisting moment component vanishes and the two-dimensional bending actuation occurs.

2.2. Sensor relations

The sensor equation is formulated due to the constitutive relation describing the direct piezoelectric effect, which for the transversally polarized layer with the material axes 1, 2, 3 parallel to the plate reference axes x, y, z, respectively, after eliminating the external electric field becomes

$$D_3=\mathbf{e}^T\,\varepsilon \tag{10}$$

where D_3 – electric displacement in the 3-axis direction.

The voltage generated by deformation of the kth sensor layer is obtained after integrating the charge stored on the electrodes, which cover the sensor faces. Using the standard equation for capacitance, and the geometric relation between strain and transverse displacement leads to the following formula

$$V_s^k=-\frac{h_s z_0^k}{\in_{33}\,A_s}\mathbf{e}^T\int_0^a\int_0^b\left[\frac{\partial^2 w}{\partial x^2},\frac{\partial^2 w}{\partial y^2},\frac{\partial^2 w}{\partial x\partial y}\right]^T dxdy \tag{11}$$

where:

A_s – sensor effective electrode area,

h_s – sensor layer thickness,

$\in_{33}$ – permittivity constant,

z_0^k – distance of the kth sensor layer from the laminate midplane.

The voltage produced by each PVDF sensor is transformed due to the control function and then drives the actuator.

2.3. Governing equation of the motion

Based on the classical laminated plate theory (CLPT) the transverse vibration $w(x, y, t)$ of the active symmetrically laminated orthotropic plate subjected to the external distributed load $q(x, y, t)$ can be described as follows

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+$$
$$+D_{22}\frac{\partial^4 w}{\partial y^4}+\tilde{\rho}\frac{\partial^2 w}{\partial t^2}=q(x,y,t)-p(x,y,t) \tag{12}$$

where:

D_{ij} – elements of the bending stiffness matrix, $i, j = 1, 2, 6$,

$\tilde{\rho}$ – equivalent mass density parameter,

$p(x, y, t)$ – loading produced by the piezoelectric control system.

The bending stiffness D_{ij} for the n-layered laminate is defined as the following sum of integrals

$$D_{ij}=\sum_{k=1}^{n}\int_{z_{k-1}}^{z_k}\bar{c}_{ij}^k z^2\,dz \tag{13}$$

where $\bar{c}_{ij}^k$ stiffness coefficients related to the kth layer of thickness $t_k = z_k - z_{k-1}$ and determined with respect to plate reference axes x, y.

Viscoelastic behaviour of the laminate is approximated according to the Voigt-Kelvin model and is based on the elastic-viscoelastic correspondence principle to predict complex moduli for the composite material. In general case the complex modulus can be written in the form $Y^* = Y(1 + j\mu\omega)$, were Y and μ indicate elastic modulus and retardation time, respectively. In consequence the stiffness matrix elements D_{ij} become complex.

In the present analysis the control loading $p(x, y, t)$ produced by the PFG actuator layers located symmetrically about the middle of the plate is given by

$$p(x, y, t) = \frac{\partial M_x^E}{\partial x^2} + \frac{\partial M_y^E}{\partial y^2} \tag{14}$$

The solution to the governing equation Eq. (12) with the simply supported boundary conditions is obtained for the steady-state case and assuming the velocity feedback control strategy. The plate behaviour is presented in terms of frequency response functions.

3. RESULTS AND DISCUSSION

Numerical study is performed for a rectangular laminated plate of the dimensions $a = 600$ mm, $b = 400$ mm and the total thickness h = 2.8 mm. The plate is simply supported and composed of classic graphite-epoxy layers of thickness h_1 = 0.2 mm and PVDF (polyvinyliolene fluoride) sensor layers and PFG actuator layers of thickness h_s = 0.1 mm and h_a = 1 mm, respectively. The symmetric stacking order is applied. The symbols "S" and "A" indicate the sensor and actuator, respectively. The stiffness parameters of the graphite-epoxy composite are following: Y_{11} = 150 GPa, Y_{22} = 9 GPa, G_{12} = 7.1 GPa, and the equivalent mass density is equal ρ = 1600 kg/m^3. The electromechanical properties of the piezoelectric materials and matrix are listed in Table 1.

Table 1
Material properties

Parameter	ρ kgm^{-3}	Y GPa	ν	d_{31} m/V	d_{32} m/V	$\in_{33}/\in_0$
PVDF	1780	2	0.3	$23\cdot10^{-12}$	$3\cdot10^{-12}$	12
PZT	7650	63	0.28	$190\cdot10^{-12}$	$190\cdot10^{-12}$	1200
Matrix	1300	6	0.3	–	–	–

Internal damping is involved for a limitation of the resonant amplitudes. For simplification the equivalent damping of the plate composite material refers to the orthotropic graphite-epoxy layers and is described by the Voigt-Kelvin model with the following retardation time values: $\mu_1 = 10^{-6}$ s, $\mu_2 = \mu_{12} = 4\cdot10^{-6}$ s.

The harmonic excitation of the amplitude intensity q_0 = = 1 Nm^{-2} is uniformly distributed over the plate surface. The velocity feedback of the constant gain k_d = 0.02 s is applied.

The variation of the PZT material in the PFG actuator is described by the exponential function Eq. (3) with the maximal volume fraction v_1 = 0.8 at the top surface and varying the inhomogenity parameter defined as the ratio of the limiting volume fractions $R = v_1/v_0$.

Calculations are performed to show the influence of the applied parameters of the PZT material gradation on the elastic and piezoelectric properties and their distribution across the PFG layer thickness, and also the change of the plate dynamic responses.

Figures 1 and 2 show the distribution of Young's modulus Y and the piezoelectric coefficient d_{31} in the thickness direction, respectively. In these figures the non-dimensional co-ordinate are used. The non-dimensional Young modulus and piezoelectric coefficient relate to the maximal values Y_{max} and d_{31max}, respectively, which are obtained due to the rule of mixture for the largest PZT volume fraction v_1 = 0.8. According to the applied distribution of the PZT inclusions the inhomogenity of the considered electroelastic properties increases significantly when the volume fraction ratio R becomes greater.

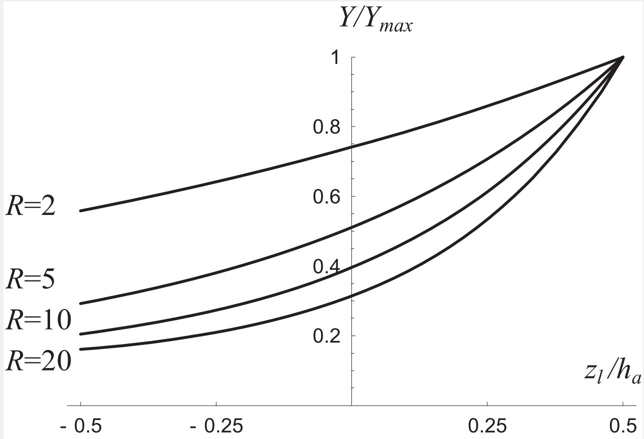

Fig. 1. Variation of Young's modulus in the PFG layer depending on the volume fraction ratio R

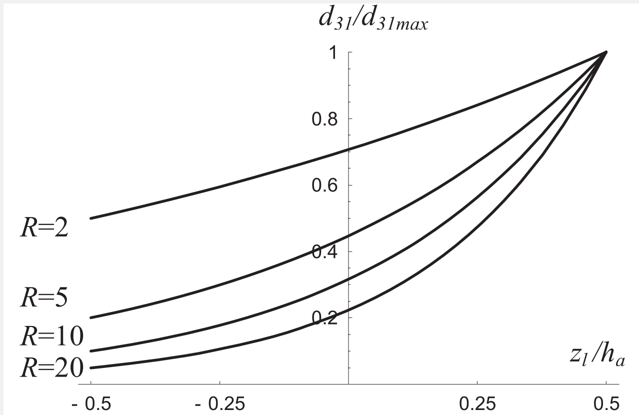

Fig. 2. Variation of the piezoelectric coefficient in the PFG layer depending on the volume fraction ratio R

The stiffness and mass density, which vary through the PFG layer thickness (according to the applied pattern), change the natural frequencies of the laminate depending on the parameters of the PZT fraction distribution. For example, in Figure 3 the plots of the first natural frequency ω_{11} versus the PZT volume fraction v_1 are presented. They are obtained for a few values of the inhomogenity parameter R. Comparing the plots it can be seen that with an increase in both the maximal volume fraction v_1 and the volume fraction ratio R the frequency ω_{11} increases. The influence of the parameter R is more significant for greater values of v_1 but the differences between the curves compared diminish as the PZT volume fraction ratio v_1/v_0 increases.

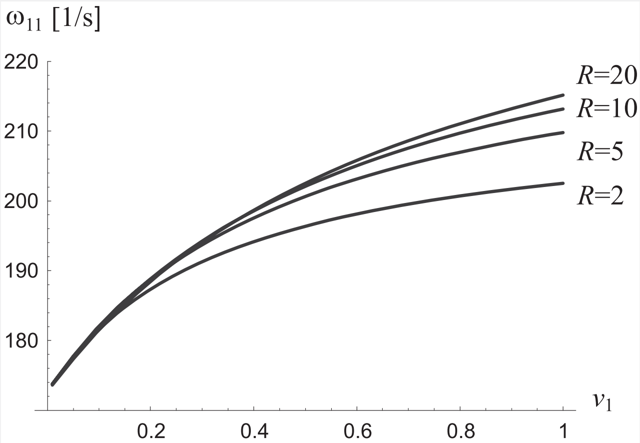

Fig. 3. Natural plate frequency versus the PZT volume fraction v_1. Influence of the volume fraction ratio R

The modification of frequency response functions created by the inhomogenity parameter R for the first (1–1) mode of the actively damped plate is shown in Figure 4. The dynamic responses are calculated at point $x = y = 100$ mm assuming the maximal PZT fraction $v_1 = 0.8$. The plots confirm the natural frequency increase, which is observed for greater values of the ratio R. Comparing the resonance curves it can be seen that with increasing the inhomogenity of the considered PFG actuators the resonant amplitudes rise and the control effectiveness becomes lower. The similar effect may be observed for another resonance regions of the plate.

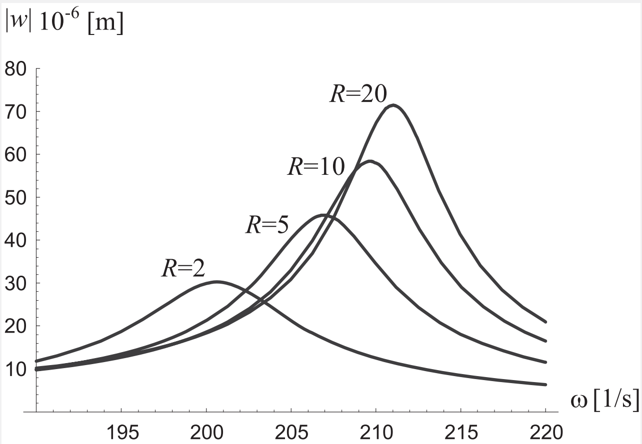

Fig. 4. Effects of variations in the volume fraction ratio R near the first resonance ($v_1 = 0.8$)

The frequency response functions of the plate with the PFG actuators of parameters $v_1 = 0.8$ and $R = 10$ calculated within a wide frequency range is presented in Figure 5.

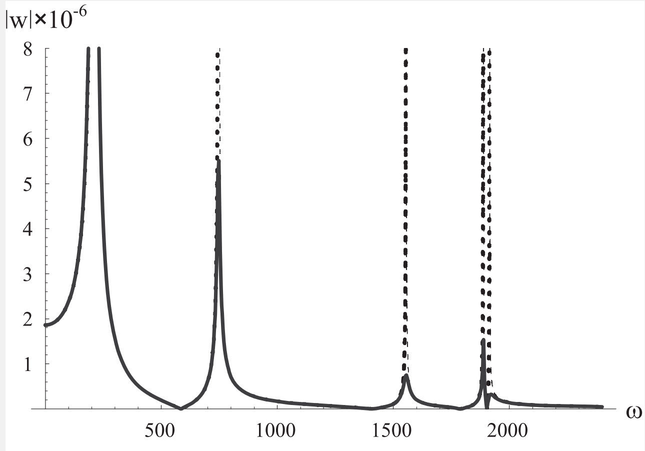

Fig. 5. Active vibration reduction for the laminated plate with PFG actuators ($v_1 = 0.8$, $R = 10$). Dotted line – uncontrolled plate.

Due to the plate geometry and the external load the resonance picks occur at frequencies corresponding to the natural modes: $\omega_{11} = 210$, $\omega_{31} = 746$, $\omega_{13} = 1554$, $\omega_{33} = 1887$, $\omega_{51} = 1914\ \mathrm{s}^{-1}$. The uncontrolled response of the plate is indicated by a dotted line. It is evident that the active reduction of the plate transverse vibration is realized by the applied control system (solid line). The amplitudes of the higher modes are strongly reduced due to both the active damping and the passive form of energy dissipation relating to the applied material damping model.

4. FINAL REMARKS

The model of the actuator layer with functionally graded mechanical and piezoelectric properties has been formulated and applied for transverse vibration control of the laminated plates. The concept of the PFG actuator is based on the two-phase material with the exponential gradation of the PZT fillers through the thickness, which matches the desired distribution of electromechanical properties. The results of simulation show the influence of the parameters of the PZT volume fraction distribution on the plate dynamics. It is also shown that the PFG actuator layers offer a satisfying operational effectiveness of the control system. Therefore, they can be used to minimize the risk of damage by the edge delamination caused by the interface stress concentration comparing with the traditional monolithic actuators.

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