Zaure B. RAKISHEVA*

STABILITY AND CONTROL OF THE HEAVY SOLID BODY MOTION

SUMMARY

The solid body motion with one fixed point under gravity action has been studied. The linear change of the variables, which reduced the initial system to the system of the non-dimensional equations with explicitly expressed linear part, had been suggested. The first integral of the norm type was found. The transformed system of the equations was investigated on stability on the first approximation. Analysis of the characteristic roots of this system was carried out. For the case of the instable by Lyapunov solution the control problem was solved. The small forces were found, by addition of which the body motion became asymptotically stable. Change of the first integrals of the controlled system was researched. Nature of the controlling forces was determined and their influence upon the body motion was revealed.

Keywords: *heavy solid body, Euler-Poisson equations, motion control, asymptotical stability by Lyapunov*

STABILIZACJA I KONTROLA RUCHU STAŁYCH, CIĘŻKICH CIAŁ

W artykule przedstawiono zagadnienie ruchu stałego ciężkiego ciała z jednym punktem mocowania pod wpływem sił grawitacji. Zastosowano liniowe przekształcanie początkowych zmiennych w system bezwymiarowych równań liniowych, jednoznacznie objaśniających część liniową. Określono całkę i normę. Przetestowano stabilność przetransformowanego systemu. Rozwiązano problemy niestabilnego sterowania w sensie Lapunowa. Znaleziono wielkość małych sił, po których dodaniu asymptotycznie stabilizowany jest ruch ciała. Badano zmiany funkcji całkowych sterowanego systemu. Właściwości sterujących sił i ich wpływ na ruch ciał zostały określone.

Słowa kluczowe: *ciężkie ciało stałe, równania Eulera-Poissona, sterowanie ruchem, stabilność asymptotyczna w sensie Lapunowa*

1. INTRODUCTION

Any body's displacement from one position to another may be performed as sum of two motions: translational motion of the body and the body's rotation round its one fixed point. As regards translational motion, it is the simplest type of the solid body motion and usually there are no problems with its research. But it is opposite situation with the body's rotation round one fixed point. In addition, solid body with one fixed point is the mechanical model of many real technical apparatuses, devices, real motions of natural and artificial celestial bodies. For example, motion of gyroscope, very important part of many technical instruments, is the solid body rotation round one fixed point. Motions of Earth, Moon, planets and artificial satellites may be considered in many cases as motion of the solid body with one fixed point. That is why this motion and this mechanical model are of great importance in theoretical mechanics.

2. THEORY

Let's consider motion of the heavy solid body with one fixed point O. "Heavy" means that the body is under gravity action $M\bar{g}$, where M is the body mass, g is an acceleration of the gravitation force. Let's choose two Cartesian coordinate systems with the general beginning in the point O: the coordinate system $OXYZ$ fixed in space and the spinning with the body coordinate system $Oxyz$ with the axes, directed along the body's main axes of inertia (Fig. 1).

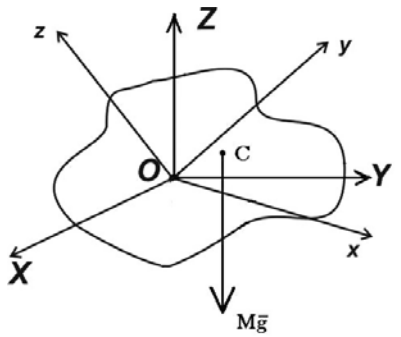

Fig. 1. Training data for NN predictive control

Gravitation force is applied in the body's centre of mass with coordinates a, b, c concerning the mobile coordinate system. Let α, β, γ – direct cosines of the mobile axes with axis OZ.

* Al-Faraby Kazak National University, Almaty, Kazakstan; zaure_ra@mail.ru

Motion of the body is described by the well-known system of the ordinary differential equations of Euler and Poisson (Arkhangelsky 1977):

$$\begin{aligned} &A\dot{p}+(C-B)qr=Mg(c\beta-b\gamma)\\ &B\dot{q}+(A-C)rp=Mg(a\gamma-c\alpha)\\ &C\dot{r}+(B-A)pq=Mg(b\alpha-\alpha\beta)\\ &\dot{\alpha}=r\beta-q\gamma\\ &\dot{\beta}=p\gamma-r\alpha\\ &\dot{\gamma}=q\alpha-p\beta \end{aligned} \tag{1}$$

where:

p, q, r – projections of an angular speed of the body rotation on the mobile axes,

A, B, C – body's main moments of inertia.

This problem has three first integrals (see, for example, in the work (Arkhangelsky 1977)). First two of them are given by general theorems of theoretical mechanics. Integral of energy

$$\frac{1}{2}(Ap^2+Bq^2+cr^2)+Mg(a\alpha+b\beta+c\gamma)=h,$$

where h is the constant of integral.

Integral of squares

$$Ap\alpha+Bq\beta+Cr\gamma=\text{const}.$$

And geometrical integral or integral of cosines

$$\alpha^2+\beta^2+\gamma^2=1.$$

It was ascertained that the system (1) may be solved analytically if we find only one additional first integral. This additional first integral was found only in three cases, which became classical ones: case of Euler, case of Lagrange and case of Kovalevskaya. In all these cases there are restrictions on distribution of mass in the body, i.e. on the form of an ellipsoid of inertia, and on the position of the body's centre of mass. Besides them there are about twenty cases with specific initial conditions, they are described well in the work (Gorr *et al*. 1978). But in 1975 Kozlov V.V. proved that analytical solution couldn't exist in the general case of the problem, when $A \neq B \neq C$ and position of the body's centre of mass was arbitrary (Kozlov 1975). Thus, the question of qualitative research of the problem appears. In this work we carry out the qualitative analysis of the motion, investigate the motion on Lyapunov's stability on the first approximation and find the motion control with a help of additional force moments.

3. NEW RESULTS DISCUSSION

Let's apply the linear change of variables $(p, q, r, \alpha, \beta, \gamma) \to (u_1, u_2, u_3, u_4, u_5, u_6)$ of the next sort:

$$p=\frac{1}{\sqrt{lA}}u_1,\qquad q=\frac{1}{\sqrt{lB}}u_2,\qquad r=\frac{1}{\sqrt{lC}}u_3,$$

$$\alpha=u_4-Mgal,\quad \beta=u_5-Mgbl,\quad \gamma=u_6-Mgcl,$$

where l is the unit, having the following dimension

$$[l]=\frac{\mathrm{s}^2}{\mathrm{kg}\cdot\mathrm{m}^2}.$$

Then the equations (1) are reduced to the system of the non-dimensional equations with explicitly expressed linear part:

$$\begin{aligned} \dot{u}_1 &= \sqrt{\frac{l}{A}}Mgcu_5-\sqrt{\frac{l}{A}}Mgbu_6-\frac{C-B}{\sqrt{lABC}}u_2u_3\\ \dot{u}_2 &= \sqrt{\frac{l}{B}}Mgau_6-\sqrt{\frac{l}{B}}Mgcu_4-\frac{A-C}{\sqrt{lABC}}u_3u_1\\ \dot{u}_3 &= \sqrt{\frac{l}{C}}Mgbu_4-\sqrt{\frac{l}{C}}Mgau_5-\frac{B-A}{\sqrt{lABC}}u_1u_2\\ \dot{u}_4 &= -\sqrt{\frac{l}{C}}Mgbu_3+\sqrt{\frac{l}{B}}Mgcu_2+\frac{1}{\sqrt{lC}}u_3u_5-\frac{1}{\sqrt{lB}}u_2u_6\\ \dot{u}_5 &= -\sqrt{\frac{l}{A}}Mgcu_1+\sqrt{\frac{l}{C}}Mgau_3+\frac{1}{\sqrt{lA}}u_1u_6-\frac{1}{\sqrt{lC}}u_3u_4\\ \dot{u}_6 &= -\sqrt{\frac{l}{B}}Mgau_2+\sqrt{\frac{l}{A}}Mgbu_1+\frac{1}{\sqrt{lB}}u_2u_4-\frac{1}{\sqrt{lA}}u_1u_5 \end{aligned} \tag{2}$$

Let's investigate this system on stability on the first approximation. The characteristic equation of the system of the first approximation will be:

$$\lambda^6+N\lambda^4+P^2\lambda^2=0 \tag{3}$$

Here under the constants N and P^2 the following expressions were designated:

$$N=M^2g^2l\left[\frac{a^2+b^2}{C}+\frac{b^2+c^2}{A}+\frac{c^2+a^2}{B}\right],$$

$$P^2=M^4g^4l^2(a^2+b^2+c^2)\cdot\left[\frac{a^2}{BC}+\frac{b^2}{CA}+\frac{c^2}{AB}\right].$$

The equation (3) gives two zero characteristic roots and the biquadrate equation:

$$\begin{aligned} &\lambda_1=\lambda_2=0\\ &\lambda^4+N\lambda^2+P^2=0 \end{aligned} \tag{4}$$

The roots of equation (4) depend on its discriminant sign. If discriminant is more or equal to zero, all the characteristic roots have zero real parts. According to the Lyapunov's theorem (Malkin 1966) about stability on the first approximation we can't say something about motion stability in these cases. If discriminant less then zero, the equation (4) breaks up into two quadratic equations:

$$\begin{aligned} &\lambda^4+N\lambda^2+P^2=\\ &=(\lambda^2+\sqrt{2P-N}\lambda+P)(\lambda^2-\sqrt{2P-N}\lambda+P)=0 \end{aligned}$$

And its roots will be the following:

$$\lambda_{3,4}=\frac{-\sqrt{2P-N}\pm\sqrt{2P+Ni}}{2} \tag{5}$$

$$\lambda_{5,6}=\frac{\sqrt{2P-N}\pm\sqrt{2P+Ni}}{2} \tag{6}$$

It may be estimated that expression $2P - N$ is more than zero. It means that the characteristic roots (5) have negative real parts and characteristic roots (6) have positive real parts. As Re $\lambda_{5,6} > 0$, according to the Lyapunov's theorem the non-perturbed motion is unstable.

Now we'll try to make unstable motion in the last case the stable one with a help of some additive moments of forces. Let's add to the right parts of the equations of the system (2) the terms of the form $\mu K u_i$, $i = 1, ..., 6$, where $\mu < 1$, $K = -\sqrt{2P - N}$.

We'll have the following system of the differential equations:

$$\begin{aligned}
\dot{u}_1 &= \sqrt{\frac{l}{A}} Mgcu_5 - \sqrt{\frac{l}{A}} Mgbu_6 - \frac{C-B}{\sqrt{lABC}} u_2 u_3 + \mu K u_1 \\
\dot{u}_2 &= -\sqrt{\frac{l}{C}} Mgbu_3 + \sqrt{\frac{l}{B}} Mgcu_2 + \frac{1}{\sqrt{lC}} u_3 u_5 - \\
&\quad - \frac{1}{\sqrt{lB}} u_2 u_6 + \mu K u_4
\end{aligned} \tag{7}$$

Here only the first and forth equations are written, the others can be obtained by the circle permutation of the quantities (A, B, C), (a, b, c), (u_4, u_5, u_6).

The characteristic equation for the system (7) will be:

$$(\mu K - \lambda)^2 [(\mu K - \lambda)^4 + N(\mu K - \lambda)^2 + P^2] = 0 \tag{8}$$

The roots λ_i of equation (8) are:

$$\lambda_1 = \lambda_2 = \mu K,$$

$$\lambda_{3,4} = \frac{-\sqrt{2P-N}\,(2\mu - 1) \pm \sqrt{2P + Ni}}{2},$$

$$\lambda_{5,6} = -\frac{\sqrt{2P-N}\,(2\mu + 1) \pm \sqrt{2P + Ni}}{2}.$$

Taking into account that $K < 0$ and constraining μ within the limits $0.5 < \mu < 1$, we shall obtain the negative real parts for all the roots of the characteristic equation. Hence, the Lyapunov's theorem about asymptotical stability of the non-perturbed motion of the controlled system (7) is true.

In order to define the nature of controlling forces let's realize reverse change of the variables:

$$(u_1, u_2, u_3, u_4, u_5, u_6) \to (p, q, r, \alpha, \beta, \gamma).$$

In the right parts of first three equations of the system (7) in comparison with initial system (1) the small additional moment with the vector, opposite to the vector of kinetic moment $\overline{K}_O = Ap\bar{i} + Aq\bar{j} + Crk$, is observed. As is known, the kinetic moment of the mechanical system concerning the point O is defined by the formula:

$$\overline{K}_O = \sum_{\nu=1}^{N} \bar{r}_\nu \times m_\nu \bar{v}_\nu .$$

Then, considering properties of the vector product, the additional moment can be written down in the form:

$$\overline{M}_O(\overline{F}) = \mu K \sum_{\nu=1}^{N} \bar{r}_\nu \times m_\nu \bar{v}_\nu = \sum_{\nu=1}^{N} \bar{r}_\nu \times \mu K m_\nu \bar{v}_\nu .$$

Hence, the force, creating the controlling moment, has the following expression:

$$F_\nu = \mu K m_\nu \bar{v}_\nu .$$

As in our problem the potential does not depend explicitly on time, capacity of the additional forces can be defined by the formula:

$$N^* = \sum_{\nu=1}^{N} \overline{F}_\nu \cdot \bar{v}_\nu = \mu K \sum_{\nu=1}^{N} m_\nu \bar{v}_\nu \cdot \bar{v}_\nu = \mu K \sum_{\nu=1}^{N} m_\nu \bar{v}_\nu^2 .$$

As $\mu > 0$, $K < 0$, capacity N^* is negative, and, hence, the additional forces are the dissipative ones.

Small additional members in the right parts of last three equations of the system (7) characterize a small deviation of the direction of the gravity force from the axis OZ.

The first integrals of the system (1) change in new variables. Integral of energy has the next form:

$$\begin{aligned}
&\frac{1}{l}(u_1^2 + u_2^2 + u_3^2) + 2Mg(au_4 + bu_3 + cu_6) = \\
&= 2h + 2M^2 g^2 l (a^2 + b^2 + c^2)
\end{aligned} \tag{9}$$

Geometrical integral:

$$\begin{aligned}
&u_4^2 + u_5^2 + u_6^2 - 2Mgl(au_4 + bu_3 + cu_6) = \\
&= 1 - M^2 g^2 l^2 (a^2 + b^2 + c^2)
\end{aligned} \tag{10}$$

Now if we multiply the integral (9) by l and add the integral (10), we'll have as a result the integral of the norm type:

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 = C_1 \tag{11}$$

where

$$C_1 = 1 + 2hl + M^2 g^2 l^2 (a^2 + b^2 + c^2).$$

The same integral can be obtained from the motion equations. Multiplying the first equation by u_1, the second – by u_2 and so on and summing the obtained expressions, we'll have:

$$\dot{u}_1 u_1 + \dot{u}_2 u_2 + \dot{u}_3 u_3 + \dot{u}_4 u_4 + \dot{u}_5 u_5 + \dot{u}_6 u_6 = 0$$

or

$$\frac{1}{2}\frac{d}{dt}(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2) = 0,$$

which gives us again the expression (11).

If make the same procedure for the controlled system (7), we will have as a result:

$$\begin{aligned}
&\dot{u}_1 u_1 + \dot{u}_2 u_2 + \dot{u}_3 u_3 + \dot{u}_4 u_4 + \dot{u}_5 u_5 + \dot{u}_6 u_6 = \\
&= \mu K (u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2)
\end{aligned} \tag{12}$$

Let's designate $I = u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2$, then the expression (12) gives:

$$\frac{dI}{dt} = 2\mu KI.$$

Solving this ordinary differential equation, we'll have:

$$I = C_2 e^{2\mu Kt},$$

where C_2 is some constant. As a factor $K < 0$, when t tends to infinity, I vanishes, i.e. the body motion is slowed down and, finally, it stops.

4. CONCLUSIONS

1. The linear change of the variables, which reduced the initial system to the system of the non-dimensional equations with explicitly expressed linear part, had been suggested. The transformed system of the equations was investigated on stability on the first approximation.
2. For the case of the instable by Lyapunov solution the control problem was defined, i.e. the small forces were searched, by addition of which the body motion was stabilized. It was shown that such forces existed and could be determined.
3. The first integral of the norm type was found. Change of the first integrals of the controlled system was researched. Nature of the controlling forces was determined and their influence upon the body motion was revealed.
4. Thereby, the problem of the motion control was solved for the problem of dynamics of the heavy solid body with one fixed point.

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