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THE MAXIMUM PRINCIPLE FOR VISCOSITY SOLUTIONS OF ELLIPTIC DIFFERENTIAL FUNCTIONAL EQUATIONS

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Abstract. This paper is devoted to the study of the maximum principle for the elliptic equation with a deviated argument. We will consider viscosity solutions of this equation.

Keywords: maximum principle, viscosity solution, elliptic equations.

Mathematics Subject Classification: 35J15, 35J60, 35R10.

1. INTRODUCTION

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $C(\Omega)$ the space of continuous functions from Ω into $\mathbb{R}$ with the usual supremum norm. $USC(\Omega)$ is the space of upper semicontinuous functions $u : \Omega \to \mathbb{R}$ and $LSC(\Omega)$ is the space of lower semicontinuous functions $u : \Omega \to \mathbb{R}$. Moreover $C_0(\Omega) = \{u \in C(\Omega) : \quad u = 0 \text{ on } \partial\Omega\}$. The continuous function $\alpha : \Omega \to \mathbb{R}^n$ is given. We define $I_\Omega : C_0(\Omega) \to C(\mathbb{R}^n)$, $R : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$, $P_\Omega : C(\mathbb{R}^n) \to C(\Omega)$ and $R_\Omega : C_0(\Omega) \to C(\Omega)$ by

$$(I_\Omega u)(x) = \begin{cases} u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega, \end{cases}$$

$$Ru(x) = u(\alpha(x)), \quad P_\Omega u = u_{|\Omega}, \quad R_\Omega = P_\Omega R I_\Omega.$$

We shall discuss the Maximum Principle for viscosity solutions of the following functional differential elliptic problem:

$$\begin{cases} F\left(x, u(x), R_\Omega u(x), Du(x), D^2u(x)\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.1}$$

where $F : \Omega \times \mathbb{R} \times C(\Omega) \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ is a given function. Here $\mathcal{S}(n)$ is the set of symmetric $n \times n$ matrices. In order to define the viscosity solutions we need some definitions and assumptions.

Assumption 1.1. Suppose that the function $F : \Omega \times \mathbb{R} \times C(\Omega) \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ of the variables (x, r, q, p, X) is nondecreasing in r and nonincreasing in X.

In order to define the viscosity solutions we need some definitions.

Definition 1.2. If $u : \Omega \to \mathbb{R}$, $\hat{x} \in \Omega$ and

$$u(x) \leq u(\hat{x}) + < p, x - \hat{x} > + \frac{1}{2} < X(x - \hat{x}), x - \hat{x} > + o(|x - \hat{x}|)$$

as $\Omega \ni x \to \hat{x}$, then we say that $(p, X) \in J_\Omega^{2,+} u(\hat{x})$.

Definition 1.3. If $u : \Omega \to \mathbb{R}$, $\hat{x} \in \Omega$, then we define the sets $J_\Omega^{2,-} u(\hat{x})$, $\bar{J}_\Omega^{2,+} u(x)$ and $\bar{J}_\Omega^{2,-} u(x)$ by

$$\begin{aligned}
J_\Omega^{2,-} u(\hat{x}) &= -J_\Omega^{2,+}(-u(\hat{x})), \\
\bar{J}_\Omega^{2,+} u(x) &= \Big\{ (p, X) \in \mathbb{R}^n \times \mathcal{S}(n) : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}(n) \\
&\qquad (p_n, X_n) \in J_\Omega^{2,+} u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X) \Big\}, \\
\bar{J}_\Omega^{2,-} u(x) &= \Big\{ (p, X) \in \mathbb{R}^n \times \mathcal{S}(n) : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times \mathcal{S}(n) \\
&\qquad (p_n, X_n) \in J_\Omega^{2,-} u(x_n) \text{ and } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X) \Big\}.
\end{aligned}$$

$J_\Omega^{2,+} u(\hat{x})$ depends on Ω, but it is the same for all sets Ω, for which $\hat{x}$ is an interior point. Let $J^{2,+} u(\hat{x})$ denote this common value. Now, we can defined the viscosity solutions.

Definition 1.4. Let F satisfy Assumption 1.1 and $\Omega \subset \mathbb{R}^n$. A viscosity subsolution of $F = 0$ (equivalently, a viscosity solution of $F \leq 0$) on Ω is a function $u \in C(\Omega)$ such that

$$F(x, u(x), R_\Omega u(x), p, X) \leq 0 \quad \text{for all } \; x \in \Omega \; \text{ and } \; (p, X) \in J_\Omega^{2,+} u(x).$$

Similarly, a viscosity supersolution of $F = 0$ on Ω is a function $u \in C(\Omega)$ such that

$$F(x, u(x), R_\Omega u(x), p, X) \geq 0 \quad \text{for all } \; x \in \Omega \; \text{ and } \; (p, X) \in J_\Omega^{2,-} u(x).$$

Finally, u is a viscosity solution of $F = 0$ in Ω if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in Ω.

The Maxima Principles for non-functional differential elliptic equations can be found in [2–4]. Existence of solutions for linear differential-functional equations of elliptic type have been studied in [1]. Paper [5] is devoted to viscosity solutions for first order partial differential-functional equations. In [2] we can find the following lemma and theorem.

Lemma 1.5. *Let Θ be a subset of $\mathbb{R}^n$, $u \in USC(\Theta)$, $v \in LSC(\Theta)$ and*

$$M_\gamma = \sup_{(x,y)\in\Theta\times\Theta} \left(u(x) - v(y) - \frac{\gamma}{2}|x-y|^2\right) \tag{1.2}$$

for $\gamma > 0$. Let $M_\gamma < \infty$ for large γ and (x_γ, y_γ) be such that

$$\lim_{\gamma\to\infty} \left(M_\gamma - \left(u(x_\gamma) - v(y_\gamma) - \frac{\gamma}{2}|x_\gamma - y_\gamma|^2\right)\right) = 0. \tag{1.3}$$

Then the following conditions holds:

$$\lim_{\gamma\to\infty} \gamma|x_\gamma - y_\gamma|^2 = 0 \quad \textit{and} \tag{1.4}$$

$$\lim_{\gamma\to\infty} M_\gamma = u(\hat{x}) - v(\hat{x}) = \sup_{x\in\Theta} (u(x) - v(x)), \tag{1.5}$$

whenever $\hat{x} \in \Theta$ is a limit point of x_γ as $\gamma \to \infty$.

Theorem 1.6. *Let Θ_i be a locally compact subset of $\mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, k$, $\Theta = \Theta_1 \times \ldots \times \Theta_k$, $u_i \in USC(\Theta_i)$, and φ be twice continuously differentiable in a neighborhood of Θ. Set*

$$w(x) = u_1(x_1) + \ldots + u_k(x_k) \quad \textit{for} \quad x = (x_1, \ldots, x_k) \in \Theta,$$

and suppose $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_k) \in \Theta$ is a local maximum of $w - \varphi$ relative to Θ. Then for each $\epsilon > 0$ there exists $X_i \in S(n_i)$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \bar{J}^{2,+}_{\Theta_i} u_i(\hat{x}_i) \quad \textit{for} \quad i = 1, 2, \ldots, k,$$

and the block diagonal matrix with entries X_i satisfies

$$-\left(\frac{1}{\epsilon} + \|A\|\right) I \le \begin{bmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{bmatrix} \le A + \epsilon A^2, \tag{1.6}$$

where $A = D^2\varphi(\hat{x}) \in \mathcal{S}(n)$, $n = n_1 + \ldots + n_k$ and I denotes the unit matrix.

The above lemma and theorem will be used later.

2. THE MAXIMUM PRINCIPLE

Assumption 2.1. Suppose that the function $F : \Omega \times \mathbb{R} \times C(\Omega) \times \mathbb{R}^n \times \mathcal{S}(n) \to \mathbb{R}$ of the variables (x, r, q, p, X) is continuous, nonincreasing in X and such that:

(a) there are constants $L > K > 0$ such that

$$F(x, r, q, p, X) - F(x, \tilde{r}, \tilde{q}, p, X) \ge L(r - \tilde{r}) - K(q - \tilde{q}) \tag{2.1}$$

for $r \ge \tilde{r}$ and $q \ge \tilde{q}$,

(b) there is a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0^+) = 0$ such that

$$F(y,r,q,\gamma(x-y),Y) - F(x,r,q,\gamma(x-y),X) \le \omega(\gamma|x-y|^2 + |x-y|), \quad (2.2)$$

whenever $x,y \in \Omega$, $r \in \mathbb{R}$, $q \in C(\Omega)$, $X,Y \in S(n)$ and

$$-3\gamma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \le \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \le 3\gamma \begin{bmatrix} I & -I \\ -I & I \end{bmatrix},$$

(c) there is constant $M > 0$ such that

$$|\alpha(x) - \alpha(y)| \le M|x-y|. \quad (2.3)$$

Remark 2.2. If the condition (a) holds, then the function F is nondecreasing in r and nonincreasing in q.

Theorem 2.3. *Let Ω be a bounded open subset of $\mathbb{R}^n$, the function F satisfies Assumption 2.1. Let $u \in C(\bar{\Omega})$ (respectively, $v \in C(\bar{\Omega})$) be a subsolution (respectively, supersolution) of $F = 0$ in Ω and $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.*

Proof. Let

$$M_\gamma = \sup_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} \left(u(x) - v(y) - \frac{\gamma}{2}|x-y|^2 \right). \quad (2.4)$$

M_γ is finite since $u - v$ is continuous and $\bar{\Omega}$ is compact.
Suppose, contrary to our claim, that there is $z \in \Omega$ such that $u(z) > v(z)$. From (2.4) we get that

$$M_\gamma \ge u(z) - v(z) \equiv \delta > 0 \quad \text{for} \quad \gamma > 0. \quad (2.5)$$

Choose (x_γ, y_γ) such that $M_\gamma = u(x_\gamma) - v(y_\gamma) - \frac{\gamma}{2}|x_\gamma - y_\gamma|^2$. By Lemma 1.5, we know that $\lim_{\gamma\to\infty} x_\gamma = \lim_{\gamma\to\infty} y_\gamma$. Let $g = \lim_{\gamma\to\infty} x_\gamma = \lim_{\gamma\to\infty} y_\gamma$. We show that $(x_\gamma, y_\gamma) \in \Omega \times \Omega$ for large γ. On the contrary, suppose that $(x_\gamma, y_\gamma) \notin \Omega \times \Omega$ for large γ. Then $g \in \partial\Omega$. From the fact, that $u \le v$ on $\partial\Omega$ and Lemma 1.5 we get $\lim_{\gamma\to\infty} M_\gamma \le 0$. This contradicts (2.5).

Let $k = 2$, $\Omega_1 = \Omega_2 = \Omega$, $u_1 = u$, $u_2 = -v$ and $\varphi(x,y) = \frac{\gamma}{2}|x-y|^2$ in Theorem 1.6. Note that

$$\bar{J}^{2,-}v = -\bar{J}^{2,+}(-v), \quad D_x\varphi(\hat{x},\hat{y}) = -D_y\varphi(\hat{x},\hat{y}) = \gamma(\hat{x}-\hat{y}),$$
$$A = D^2\varphi(\hat{x},\hat{y}) = \gamma \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}, \quad A^2 = 2\gamma A \quad \text{and} \quad \|A\| = 2\gamma.$$

And now from Theorem 1.6 we get that for every $\epsilon > 0$ there exists $X, Y \in \mathcal{S}(n)$ such that

$$(\gamma(\hat{x}-\hat{y}), X) \in \bar{J}^{2,+}u(\hat{x}), \quad (\gamma(\hat{x}-\hat{y}), Y) \in \bar{J}^{2,-}v(\hat{y}) \quad \text{and}$$
$$-\left(\frac{1}{\epsilon} + 2\gamma\right) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \le \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \le \gamma(1 + 2\epsilon\gamma) \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

Choosing $\epsilon = \frac{1}{\gamma}$ yields

$$-3\gamma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \le \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \le 3\gamma \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

Let $(\hat{x}, \hat{y})$ denote (x_γ, y_γ). From the definition of the subsolution and supersolution we get

$$F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), X) \le 0 \le F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x}-\hat{y}), Y). \tag{2.6}$$

From Lemma 1.5 and (2.5)

$$\begin{aligned} &0 < \delta \le M_\gamma = u(\hat{x}) - v(\hat{y}) - \frac{\gamma}{2}|\hat{x}-\hat{y}|^2, \\ &\gamma|\hat{x}-\hat{y}|^2 \to 0 \quad \text{as} \quad \gamma \to \infty. \end{aligned}$$

By the above, we see that $u(\hat{x}) > v(\hat{y})$. And now, we note that

$$\begin{aligned} &L\delta \le LM_\gamma \le L[u(\hat{x}) - v(\hat{y})] \le \\ &\le F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), X) - F(\hat{x}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), X) = \\ &= [F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), X) - F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x}-\hat{y}), Y)] + \\ &+ [F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x}-\hat{y}), Y) - F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), Y)] + \\ &+ [F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), Y) - F(\hat{x}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), X)]. \end{aligned} \tag{2.7}$$

From (2.6) we get

$$F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), X) - F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x}-\hat{y}), Y) \le 0. \tag{2.8}$$

From definitions of M_γ and $(\hat{x}, \hat{y})$ we get

$$u(\hat{x}) - v(\hat{y}) - \frac{\gamma}{2}|\hat{x}-\hat{y}|^2 = M_\gamma \ge u(\alpha(\hat{x})) - v(\alpha(\hat{y})) - \frac{\gamma}{2}|\alpha(\hat{x}) - \alpha(\hat{y})|^2.$$

We thus obtain

$$u(\alpha(\hat{x})) - v(\alpha(\hat{y})) \le u(\hat{x}) - v(\hat{y}) - \frac{\gamma}{2}|\hat{x}-\hat{y}|^2 + \frac{\gamma}{2}|\alpha(\hat{x}) - \alpha(\hat{y})|^2.$$

If $v(\alpha(\hat{y})) \le u(\alpha((\hat{x})))$, then by the above and (2.1), (2.3), we get

$$\begin{aligned} &F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x}-\hat{y}), Y) - F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), Y) \le \\ &\le K[u(\alpha(\hat{x})) - v(\alpha(\hat{y}))] \le K[u(\hat{x}) - v(\hat{y})] - \frac{K\gamma}{2}|\hat{x}-\hat{y}|^2 + \frac{K\gamma}{2}|\alpha(\hat{x}) - \alpha(\hat{y})|^2 \le \\ &\le K[u(\hat{x}) - v(\hat{y})] + \frac{K\gamma M}{2}|\hat{x}-\hat{y}|^2. \end{aligned} \tag{2.9}$$

F is nonincreasing in q, so if $v(\alpha(\hat{y})) \ge u(\alpha(\hat{x}))$, then

$$F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x}-\hat{y}), Y) - F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x}-\hat{y}), Y) \le 0. \tag{2.10}$$

From (2.7)–(2.10) and (2.2), we get

$$L[u(\hat{x}) - v(\hat{y})] \le K[u(\hat{x}) - v(\hat{y})] + \frac{K\gamma M}{2}|\hat{x} - \hat{y}|^2 + \omega(\gamma|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|).$$

By the above,

$$(L - K)[u(\hat{x}) - v(\hat{y})] \le \frac{K\gamma M}{2}|\hat{x} - \hat{y}|^2 + \omega(\gamma|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|). \tag{2.11}$$

We know that $L > K$ and

$$\frac{K\gamma M}{2}|\hat{x} - \hat{y}|^2 + \omega(\gamma|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|) \to 0 \quad \text{as} \quad \gamma \to \infty.$$

Therefore, from (2.11) we get that $[u(\hat{x}) - v(\hat{y})] \to 0$ as $\gamma \to \infty$. We see from this and (2.7) that $L\delta \le 0$. This contradicts the fact that there is $z \in \Omega$ such that $u(z) > v(z)$. This finishes the proof. □

Now, we give an example which demonstrates that if F is increasing in q, then the Theorem 2.3 is false.

Example 2.4. We define $\Omega = [-1, 1] \times [-1, 1]$, $u(x, y) = e^{1-x^2-y^2}$, $v(x, y) = 2,5$ and

$$(Lz)(x, y) = -\frac{1}{10}\frac{\partial^2 z}{\partial x^2}(x, y) - \frac{1}{10}\frac{\partial^2 z}{\partial y^2}(x, y) + u\left(\frac{x}{10} + \frac{9}{10}, \frac{y}{10} + \frac{9}{10}\right) - 2,4.$$

We use the program wxMaxima and calculate $(Lv)(x, y)$, $(Lu)(x, y)$ for $(x, y) \in \Omega$. We get $(Lv)(x, y) = 0,1$ for $(x, y) \in \Omega$, and the graph of $Lu : \Omega \to \mathbb{R}$ is showed on Figure 1. We see that $(Lu)(x, y) < 0$ for $(x, y) \in \Omega$, $u(x, y) < 1 < v(x, y)$ on $\partial\Omega$ and $u(0, 0) = e > 2,5 = v(0, 0)$. Therefore, the assertion of Theorem 2.3 does not hold.

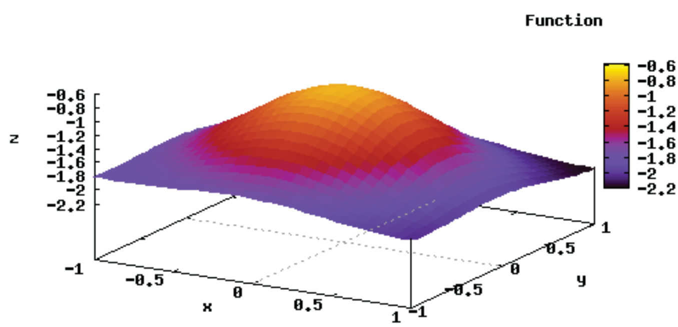

Fig. 1. Graph Lu on Ω

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