THE MAXIMUM PRINCIPLE
FOR VISCOSITY SOLUTIONS
OF ELLIPTIC
DIFFERENTIAL FUNCTIONAL EQUATIONS

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Abstract. This paper is devoted to the study of the maximum principle for the elliptic equation with a deviated argument. We will consider viscosity solutions of this equation.

Keywords: maximum principle, viscosity solution, elliptic equations.

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1. INTRODUCTION

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We denote by $C(\Omega)$ the space of continuous functions from $\Omega$ into $\mathbb{R}$ with the usual supremum norm. $USC(\Omega)$ is the space of upper semicontinuous functions $u : \Omega \to \mathbb{R}$ and $LSC(\Omega)$ is the space of lower semicontinuous functions $u : \Omega \to \mathbb{R}$. Moreover $C_0(\Omega) = \{ u \in C(\Omega) : u = 0 \text{ on } \partial \Omega \}$. The continuous function $\alpha : \Omega \to \mathbb{R}^n$ is given. We define $I_\Omega : C_0(\Omega) \to C(\mathbb{R}^n)$, $R : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$, $P_\Omega : C(\mathbb{R}^n) \to C(\Omega)$ and $R_\Omega : C_0(\Omega) \to C(\Omega)$ by

$$(I_\Omega u)(x) = \begin{cases} u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega, \end{cases}$$

$$Ru(x) = u(\alpha(x)), \quad P_\Omega u = u_{|\Omega}, \quad R_\Omega = P_\Omega R I_\Omega.$$

We shall discuss the Maximum Principle for viscosity solutions of the following functional differential elliptic problem:

$$\begin{cases} F \left( x, u(x), R_\Omega u(x), Du(x), D^2 u(x) \right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \quad (1.1)$$

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where $F : \Omega \times \mathbb{R} \times C(\Omega) \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ is a given function. Here $S(n)$ is the set of symmetric $n \times n$ matrices. In order to define the viscosity solutions we need some definitions and assumptions.

**Assumption 1.1.** Suppose that the function $F : \Omega \times \mathbb{R} \times C(\Omega) \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ of the variables $(x, r, q, p, X)$ is nondecreasing in $r$ and nonincreasing in $X$.

In order to define the viscosity solutions we need some definitions.

**Definition 1.2.** If $u : \Omega \to \mathbb{R}$, $\hat{x} \in \Omega$ and

$$u(x) \leq u(\hat{x}) + <p, x - \hat{x}> + \frac{1}{2} <X(x - \hat{x}), x - \hat{x}> + o(|x - \hat{x}|)$$

as $\Omega \ni x \to \hat{x}$, then we say that $(p, X) \in J^{2,+}_\Omega u(\hat{x})$.

**Definition 1.3.** If $u : \Omega \to \mathbb{R}$, $\hat{x} \in \Omega$, then we define the sets $J^{2,-}_\Omega u(\hat{x})$, $J^{2,+}_\Omega u(x)$ and $J^{2,-}_\Omega u(x)$ by

$$J^{2,-}_\Omega u(\hat{x}) = -J^{2,+}_\Omega (-u(\hat{x})),$$

$$J^{2,+}_\Omega u(x) = \left\{ (p, X) \in \mathbb{R}^n \times S(n) : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times S(n) \right\},$$

$$J^{2,-}_\Omega u(x) = \left\{ (p, X) \in \mathbb{R}^n \times S(n) : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times S(n) \right\},$$

$$J^{2,+}_\Omega u(x) = \left\{ (p, X) \in \mathbb{R}^n \times S(n) : \exists (x_n, p_n, X_n) \in \Omega \times \mathbb{R}^n \times S(n) \right\}.$$

$J^{2,+}_\Omega u(\hat{x})$ depends on $\Omega$, but it is the same for all sets $\Omega$, for which $\hat{x}$ is an interior point. Let $J^{2,+}_\Omega u(\hat{x})$ denote this common value. Now, we can define the viscosity solutions.

**Definition 1.4.** Let $F$ satisfy Assumption 1.1 and $\Omega \subset \mathbb{R}^n$. A viscosity subsolution of $F = 0$ (equivalently, a viscosity solution of $F \leq 0$) on $\Omega$ is a function $u \in C(\Omega)$ such that

$$F(x, u(x), R_\Omega u(x), p, X) \leq 0 \quad \text{for all } x \in \Omega \text{ and } (p, X) \in J^{2,+}_\Omega u(x).$$

Similarly, a viscosity supersolution of $F = 0$ on $\Omega$ is a function $u \in C(\Omega)$ such that

$$F(x, u(x), R_\Omega u(x), p, X) \geq 0 \quad \text{for all } x \in \Omega \text{ and } (p, X) \in J^{2,-}_\Omega u(x).$$

Finally, $u$ is a viscosity solution of $F = 0$ in $\Omega$ if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in $\Omega$.

Lemma 1.5. Let $\Theta$ be a subset of $\mathbb{R}^n$, $u \in USC(\Theta)$, $v \in LSC(\Theta)$ and
\[
M_\gamma = \sup_{(x,y)\in\Theta\times\Theta} \left( u(x) - v(y) - \frac{\gamma}{2} |x-y|^2 \right)
\] (1.2)
for $\gamma > 0$. Let $M_\gamma < \infty$ for large $\gamma$ and $(x_\gamma,y_\gamma)$ be such that
\[
\lim_{\gamma\to\infty} \left( M_\gamma - \left( u(x_\gamma) - v(y_\gamma) - \frac{\gamma}{2} |x_\gamma - y_\gamma|^2 \right) \right) = 0.
\] (1.3)
Then the following conditions holds:
\[
\lim_{\gamma\to\infty} \gamma |x_\gamma - y_\gamma|^2 = 0 \quad \text{and}
\lim_{\gamma\to\infty} M_\gamma = u(\hat{x}) - v(\hat{x}) = \sup_{x\in\Theta} (u(x) - v(x)),
\] (1.4)
whenever $\hat{x} \in \Theta$ is a limit point of $x_\gamma$ as $\gamma \to \infty$.

Theorem 1.6. Let $\Theta_i$ be a locally compact subset of $\mathbb{R}^{n_i}$ for $i = 1,2,\ldots,k$, $\Theta = \Theta_1 \times \ldots \times \Theta_k$, $u_i \in USC(\Theta_i)$, and $\varphi$ be twice continuously differentiable in a neighborhood of $\Theta$. Set
\[
w(x) = u_1(x_1) + \ldots + u_k(x_k) \quad \text{for} \quad x = (x_1,\ldots,x_k) \in \Theta,
\]
and suppose $\hat{x} = (\hat{x}_1,\ldots,\hat{x}_k) \in \Theta$ is a local maximum of $w - \varphi$ relative to $\Theta$. Then for each $\epsilon > 0$ there exists $X_i \in S(n_i)$ such that
\[
(D_{x_i}\varphi(\hat{x}),X_i) \in J_{\Theta_i}^{\epsilon,+} u_i(\hat{x}_i) \quad \text{for} \quad i = 1,2,\ldots,k,
\]
and the block diagonal matrix with entries $X_i$ satisfies
\[
- \left( \frac{1}{\epsilon} + \|A\| \right) I \leq \begin{bmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{bmatrix} \leq A + \epsilon A^2,
\] (1.6)
where $A = D^2\varphi(\hat{x}) \in S(n)$, $n = n_1 + \ldots + n_k$ and $I$ denotes the unit matrix.

The above lemma and theorem will be used later.

2. THE MAXIMUM PRINCIPLE

Assumption 2.1. Suppose that the function $F: \Omega \times \mathbb{R} \times C(\Omega) \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ of
the variables $(x,r,q,p,X)$ is continuous, nonincreasing in $X$ and such that:
(a) there are constants $L > K > 0$ such that
\[
F(x,r,q,p,X) - F(x,\hat{r},\hat{q},p,X) \geq L(r-\hat{r}) - K(q-\hat{q})
\] (2.1)
for $r \geq \hat{r}$ and $q \geq \hat{q}$,
(b) there is a function \( \omega : [0, \infty] \rightarrow [0, \infty] \) that satisfies \( \omega(0^+) = 0 \) such that

\[
F(y, r, q, \gamma(x - y), Y) - F(x, r, q, \gamma(x - y), X) \leq \omega(\gamma|x - y|^2 + |x - y|),
\]

whenever \( x, y \in \Omega, r \in \mathbb{R}, q \in C(\Omega), X, Y \in S(n) \) and

\[
-3\gamma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq 3\gamma \begin{bmatrix} I & -I \\ -I & I \end{bmatrix},
\]

(c) there is constant \( M > 0 \) such that

\[
|\alpha(x) - \alpha(y)| \leq M|x - y|.
\]

**Remark 2.2.** If the condition (a) holds, then the function \( F \) is nondecreasing in \( r \) and nonincreasing in \( q \).

**Theorem 2.3.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \), the function \( F \) satisfies Assumption 2.1. Let \( u \in C(\Omega) \) (respectively, \( v \in C(\Omega) \)) be a subsolution (respectively, supersolution) of \( F = 0 \) in \( \Omega \) and \( u \leq v \) on \( \partial \Omega \). Then \( u \leq v \) in \( \Omega \).

**Proof.** Let

\[
M_\gamma = \sup_{(x,y) \in \bar{\Omega} \times \Omega} \left( u(x) - v(y) - \frac{\gamma^2}{2}|x - y|^2 \right).
\]

\( M_\gamma \) is finite since \( u - v \) is continuous and \( \bar{\Omega} \) is compact.

Suppose, contrary to our claim, that there is \( z \in \Omega \) such that \( u(z) > v(z) \). From (2.4) we get that

\[
M_\gamma \geq u(z) - v(z) \equiv \delta > 0 \quad \text{for} \quad \gamma > 0.
\]

Choose \( (x_\gamma, y_\gamma) \) such that \( M_\gamma = u(x_\gamma) - v(y_\gamma) - \frac{\gamma^2}{2}|x_\gamma - y_\gamma|^2 \). By Lemma 1.5, we know that \( \lim_{\gamma \to \infty} x_\gamma = \lim_{\gamma \to \infty} y_\gamma \). Let \( g = \lim_{\gamma \to \infty} x_\gamma = \lim_{\gamma \to \infty} y_\gamma \). We show that \( (x_\gamma, y_\gamma) \in \Omega \times \Omega \) for large \( \gamma \). On the contrary, suppose that \( (x_\gamma, y_\gamma) \notin \Omega \times \Omega \) for large \( \gamma \). Then \( g \in \partial \Omega \). From the fact, that \( u \leq v \) on \( \partial \Omega \) and Lemma 1.5 we get \( \lim_{\gamma \to \infty} M_\gamma \leq 0 \). This contradicts (2.5).

Let \( k = 2, \Omega_1 = \Omega_2 = \Omega, u_1 = u, u_2 = -v \) and \( \varphi(x, y) = \frac{\gamma}{2}|x - y|^2 \) in Theorem 1.6. Note that

\[
\tilde{J}^2 - v = -\tilde{J}^{2,+}(-v), \quad D_x \varphi(\hat{x}, \hat{y}) = -D_y \varphi(\hat{x}, \hat{y}) = \gamma(\hat{x} - \hat{y}), \quad A = D^2 \varphi(\hat{x}, \hat{y}) = \gamma \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}, \quad A^2 = 2\gamma A \quad \text{and} \quad \|A\| = 2\gamma.
\]

And now from Theorem 1.6 we get that for every \( \epsilon > 0 \) there exists \( X, Y \in S(n) \) such that

\[
(\gamma(\hat{x} - \hat{y}), X) \in \tilde{J}^{2,+} u(\hat{x}), \quad (\gamma(\hat{x} - \hat{y}), Y) \in \tilde{J}^{2,-} v(\hat{y}) \quad \text{and} \quad -\left( \frac{1}{\epsilon} + 2\gamma \right) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \gamma (1 + 2\epsilon\gamma) \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.
\]
Choosing $\epsilon = \frac{1}{\gamma}$ yields

$$-3\gamma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq 3\gamma \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}. $$

Let $(\hat{x}, \hat{y})$ denote $(x_\gamma, y_\gamma)$. From the definition of the subsolution and supersolution we get

$$F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), X) \leq 0 \leq F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x} - \hat{y}), Y). \tag{2.6}$$

From Lemma 1.5 and (2.5)

$$0 < \delta \leq M_\gamma = u(\hat{x}) - v(\hat{y}) - \frac{\gamma}{2}|\hat{x} - \hat{y}|^2,$$

$$\gamma|\hat{x} - \hat{y}|^2 \rightarrow 0 \text{ as } \gamma \rightarrow \infty.$$  

By the above, we see that $u(\hat{x}) > v(\hat{y})$. And now, we note that

$$L\delta \leq LM_\gamma \leq L[u(\hat{x}) - v(\hat{y})] \leq$$

$$\leq F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), X) - F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x} - \hat{y}), Y) =$$

$$= [F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), X) - F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x} - \hat{y}), Y)] +$$

$$+ [F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x} - \hat{y}), Y) - F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), Y)] +$$

$$+ [F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), Y) - F(\hat{x}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), X)].$$

From (2.6) we get

$$F(\hat{x}, u(\hat{x}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), X) - F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x} - \hat{y}), Y) \leq 0. \tag{2.8}$$

From definitions of $M_\gamma$ and $(\hat{x}, \hat{y})$ we get

$$u(\hat{x}) - v(\hat{y}) - \frac{\gamma}{2}|\hat{x} - \hat{y}|^2 = M_\gamma \geq u(\alpha(\hat{x})) - v(\alpha(\hat{y})) - \frac{\gamma}{2}|\alpha(\hat{x}) - \alpha(\hat{y})|^2.$$  

We thus obtain

$$u(\alpha(\hat{x})) - v(\alpha(\hat{y})) \leq u(\hat{x}) - v(\hat{y}) - \frac{\gamma}{2}|\hat{x} - \hat{y}|^2 + \frac{\gamma}{2}|\alpha(\hat{x}) - \alpha(\hat{y})|^2.$$  

If $v(\alpha(\hat{y})) \leq u(\alpha((\hat{x})))$, then by the above and (2.1), (2.3), we get

$$F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x} - \hat{y}), Y) - F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), Y) \leq$$

$$\leq K[u(\alpha(\hat{x})) - v(\alpha(\hat{y}))] \leq K[u(\hat{x}) - v(\hat{y})] - \frac{K\gamma}{2}|\hat{x} - \hat{y}|^2 + \frac{K\gamma}{2}|\alpha(\hat{x}) - \alpha(\hat{y})|^2 \leq (2.9)$$

$$\leq K[u(\hat{x}) - v(\hat{y})] + \frac{K\gamma M}{2}|\hat{x} - \hat{y}|^2.$$  

$F$ is nonincreasing in $q$, so if $v(\alpha(\hat{y})) \geq u(\alpha(\hat{x}))$, then

$$F(\hat{y}, v(\hat{y}), R_\Omega v(\hat{y}), \gamma(\hat{x} - \hat{y}), Y) - F(\hat{y}, v(\hat{y}), R_\Omega u(\hat{x}), \gamma(\hat{x} - \hat{y}), Y) \leq 0. \tag{2.10}$$
From (2.7)–(2.10) and (2.2), we get
\[ L[u(\hat{x}) - v(\hat{y})] \leq K[u(\hat{x}) - v(\hat{y})] + \frac{K\gamma M}{2} |\hat{x} - \hat{y}|^2 + \omega(\gamma|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|). \]

By the above,
\[ (L - K)[u(\hat{x}) - v(\hat{y})] \leq \frac{K\gamma M}{2} |\hat{x} - \hat{y}|^2 + \omega(\gamma|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|). \]  

(2.11)

We know that \( L > K \) and
\[ \frac{K\gamma M}{2} |\hat{x} - \hat{y}|^2 + \omega(\gamma|\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}|) \to 0 \quad \text{as} \quad \gamma \to \infty. \]

Therefore, from (2.11) we get that \( [u(\hat{x}) - v(\hat{y})] \to 0 \) as \( \gamma \to \infty \). We see from this and (2.7) that \( L\delta \leq 0 \). This contradicts the fact that there is \( z \in \Omega \) such that \( u(z) > v(z) \). This finishes the proof.

Now, we give an example which demonstrates that if \( F \) is increasing in \( q \), then the Theorem 2.3 is false.

**Example 2.4.** We define \( \Omega = [-1, 1] \times [-1, 1] \), \( u(x, y) = e^{1-x^2-y^2} \), \( v(x, y) = 2, 5 \) and
\[ (Lz)(x, y) = -\frac{1}{10} \frac{\partial^2 z}{\partial x^2}(x, y) - \frac{1}{10} \frac{\partial^2 z}{\partial y^2}(x, y) + u \left( \frac{x}{10} + \frac{9}{10} \frac{y}{10} + \frac{9}{10} \right) - 2, 4. \]

We use the program wxMaxima and calculate \((Lv)(x, y), (Lu)(x, y)\) for \((x, y) \in \Omega\). We get \( (Lv)(x, y) = 0, 1 \) for \((x, y) \in \Omega\), and the graph of \( Lu : \Omega \to \mathbb{R} \) is showed on Figure 1. We see that \((Lu)(x, y) < 0\) for \((x, y) \in \Omega\), \( u(x, y) < 1 < v(x, y) \) on \( \partial \Omega \) and \( u(0, 0) = e > 2, 5 = v(0, 0) \). Therefore, the assertion of Theorem 2.3 does not hold.
REFERENCES


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