

RANDOM INTEGRAL EQUATIONS ON TIME SCALES

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Abstract. In this paper, we present the existence and uniqueness of random solution of a random integral equation of Volterra type on time scales. We also study the asymptotic properties of the unique random solution.

Keywords: random integral equations, time scale, existence, uniqueness, stability.

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1. INTRODUCTION

The random integral equations of Volterra type, as a natural extension of deterministic ones, arise in many applications and have been investigated by many mathematicians. For details, the reader may see the monograph [22, 27], the papers [7, 12, 21, 26] and references therein. For the general theory of integral equations see, the monographs [8, 11] and references therein. In recent years, it initiated the study of integral equations on time scales and obtained some significant results see [1, 16, 19, 25]. The stochastic differential equations on time scales was first studied by Sanyal in his Ph.D. Thesis [24]. For other results about stochastic processes see [23].

The aim of this paper is to obtain the general conditions which ensure the existence and uniqueness of a random solution of a random integral equation of Volterra type on time scales and to investigate the asymptotic behavior of such a random solution. The paper is organized as follows: in Section 2 we set up the appropriate framework on random processes on time scales. We also introduce some functional spaces within which the study of random integral equations can be developed. In Section 3 we present the existence and uniqueness of random solutions. Finally, we establish an asymptotic stability result.

2. PRELIMINARIES

A *time scale* \mathbb{T} is an arbitrary nonempty closed subset of the real number \mathbb{R} . Then the time scale \mathbb{T} is a complete metric space with the usual metric on \mathbb{R} . Since a time scale \mathbb{T} may or may not be connected, we need the concept of jump operators. The *forward (backward) jump operator* $\sigma(t)$ at $t \in \mathbb{T}$ for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ for $t > \inf \mathbb{T}$) is given by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (respectively $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$) for all $t \in \mathbb{T}$. If $\sigma(t) > t$, $t \in \mathbb{T}$, we say t is *right scattered*. If $\rho(t) < t$, $t \in \mathbb{T}$, we say t is *left scattered*. If $\sigma(t) = t$, $t \in \mathbb{T}$, we say t is *right-dense*. If $\rho(t) = t$, $t \in \mathbb{T}$, we say t is *left-dense*. Also, define the *graininess function* $\mu : \mathbb{T} \rightarrow [0, \infty)$ as $\mu(t) := \sigma(t) - t$. We recall that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous function* if f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \rightarrow t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. We remark that every *rd-continuous function* is Lebesgue Δ -integrable (see [14]). A *rd-continuous function* $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *positively regressive* if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$. We will denote by \mathcal{R}^+ the set of all positively regressive functions. In the following, assume that \mathbb{T} is unbounded. Without lost the generality, assume that $0 \in \mathbb{T}$ and let $\mathbb{T}_0 = [0, \infty) \cap \mathbb{T}$. Also, assume that there exists a strictly increasing sequence $(t_n)_n$ of elements of \mathbb{T}_0 such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Denote by \mathcal{L} the σ -algebra of Δ -measurable subsets of \mathbb{T}_0 and by λ the Lebesgue Δ -measure of \mathcal{L} . Having the measure space $(\mathbb{T}_0, \mathcal{L}, \lambda)$ one can introduce the Lebesgue-Bochner integral for functions from \mathbb{T}_0 to a Banach space by simply employing the standard procedure from measure theory (see [3, 18]). The Lebesgue-Bochner integral for functions from \mathbb{T}_0 to a Banach space was introduced by Neidhart in [18] and the Henstock-Kurzweil-Pettis integral was introduced by Cichoń in [10]. For details on the construction of the Lebesgue integral for real functions defined on a time scale, see [2, 4, 5, 9, 14, 15]. Further, let (Ω, \mathcal{A}, P) be a complete probability space. A function $x : \Omega \rightarrow \mathbb{R}$ is called a *random variable* if $\{\omega \in \Omega : x(\omega) < a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. Let $1 \leq p < \infty$. A random variable $x : \Omega \rightarrow \mathbb{R}$ is said to be *p-integrable* if $\int_{\Omega} |x(\omega)|^p dP(\omega) < \infty$. Let $\mathcal{L}^p(\Omega)$ be the space of all *p-integrable* random variables. Then $\mathcal{L}^p(\Omega)$ is a vector space and the function $x \mapsto \|x\|_{\mathcal{L}^p(\Omega)}$ defined by

$$\|x\|_{\mathcal{L}^p(\Omega)} = \left(\int_{\Omega} |x(\omega)|^p dP(\omega) \right)^{1/p}$$

is a seminorm on $\mathcal{L}^p(\Omega)$. If $x \in \mathcal{L}^1(\Omega)$, then

$$E[x] := \int_{\Omega} x(\omega) dP(\omega)$$

is called the *expected value* of random variable x . A random variable x is called a *P-essentially bounded* if there exists a $M > 0$ and $A \in \mathcal{A}$ with $P(A) = 0$ such that $|x(\omega)| \leq M$ for all $\omega \in \Omega \setminus A$. Let $\mathcal{L}^\infty(\Omega)$ be the space of all *P-essentially bounded* random variables. Then

$$\|x\|_{\mathcal{L}^\infty(\Omega)} = P\text{-ess sup}_{\omega \in \Omega} |x(\omega)|$$

is a seminorm on $\mathcal{L}^\infty(\Omega)$, where

$$P\text{-ess sup}_{\omega \in \Omega} |x(\omega)| := \inf\{M > 0 : |x(\omega)| \leq M \quad P\text{-a.e. } \omega \in \Omega\}.$$

When a random variable x is p -integrable or P -essentially bounded it is convenient to use notation \widehat{x} to denote the equivalent class of random variables which coincide with x for P -a.e. $\omega \in \Omega$. Let us denote by $L^p(\Omega)$ the space of all equivalence classes of random variables that are p -integrable and by $L^\infty(\Omega)$ the space of all equivalence classes of random variables that are P -essentially bounded. If $x \in \mathcal{L}^p(\Omega)$, $1 \leq p \leq \infty$, we denote by \widehat{x} its equivalence class, that is, $y \in \widehat{x}$ if and only if $y(\omega) = x(\omega)$ for P -a.e. $\omega \in \Omega$. Moreover, we have that $\|y\|_{\mathcal{L}^p(\Omega)} = \|x\|_{\mathcal{L}^p(\Omega)}$. Thus we can define a norm $\|\cdot\|_{L^p(\Omega)}$ on $L^p(\Omega)$ by means of the formula $\|\widehat{x}\|_{L^p(\Omega)} = \|x\|_{\mathcal{L}^p(\Omega)}$, $1 \leq p \leq \infty$. Then $L^p(\Omega)$, $1 \leq p \leq \infty$, is a Banach space with respect to the norm $\|\cdot\|_{L^p(\Omega)}$.

Since, for $1 \leq p \leq \infty$, $L^p(\Omega)$ is a Banach space, then all elementary properties of the calculus (such as continuity, differentiability, and integrability) for abstract functions defined on a subset of \mathbb{T} with values into a Banach space remain also true for the functions defined a subset of \mathbb{T} with values into $L^p(\Omega)$, $1 \leq p \leq \infty$.

Thereby, if $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$ is strongly measurable then the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue measurable on \mathbb{T}_0 . Also, a strongly measurable function $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$ is Bochner Δ -integrable on \mathbb{T}_0 if and only if the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue Δ -integrable on \mathbb{T}_0 (see [3]).

Let $1 \leq p \leq \infty$. A function $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$ is called *rd-continuous function* if X is continuous at every right-dense point $t \in \mathbb{T}_0$, and $\lim_{s \rightarrow t^-} X(s)$ exists in $L^p(\Omega)$ at every left-dense point $t \in \mathbb{T}_0$.

Of particular importance is the fact that every *rd-continuous function* $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$ is Bochner Δ -integrable on \mathbb{T}_0 (see [3, Theorem 6.3]).

If $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$ is a strongly measurable function then for each fixed $t \in \mathbb{T}_0$, $X(t) \in L^p(\Omega)$ is an equivalence class. If for each $t \in \mathbb{T}_0$ we select a particular function $x(t, \cdot) \in X(t)$ then we obtain a function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ such that $\omega \mapsto x(t, \omega)$ is a random variable for each $t \in \mathbb{T}_0$. This resulting function is called a *representation* of X . In fact, such a representation is so called a *random process*. However, is not immediate that this representation function is even a $\mathcal{L} \times \mathcal{A}$ -measurable function. In this sense, we have the following result.

Lemma 2.1. (a) ([13, Theorem III.11.17]). *Let $(\mathbb{T}_0 \times \Omega, \mathcal{L} \times \mathcal{A}, \lambda \times P)$ be the product space of the measure space $(\mathbb{T}_0, \mathcal{L}, \lambda)$ and (Ω, \mathcal{A}, P) . Let $1 \leq p \leq \infty$ and let $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$ be a Bochner Δ -integrable function. Then there exists a $\mathcal{L} \times \mathcal{A}$ -measurable function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ which is uniquely determined except a set of $\lambda \times P$ -measure zero, such that $\widehat{x}(t, \cdot) = X(t)$ for λ -a.e. $t \in \mathbb{T}_0$. Moreover, $x(\cdot, \omega)$ is Lebesgue Δ -integrable on \mathbb{T}_0 for P - a.e. $\omega \in \Omega$ and integral $\int_{\mathbb{T}_0} x(t, \omega) \Delta t$, as a function of ω , is equal to the element $\int_{\mathbb{T}_0} X(t) \Delta t$ of $L^p(\Omega)$, that is,*

$$\int_{\mathbb{T}_0} x(t, \cdot) \Delta t = \left(\int_{\mathbb{T}_0} X(t) \Delta t \right) (\cdot).$$

(b) ([13, Lemma III.11.16]). Let $1 \leq p < \infty$ and let $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ be a $\mathcal{L} \times \mathcal{A}$ -measurable function such that $x(t, \cdot) \in \mathcal{L}^p(\Omega)$ for λ -a.e. $t \in \mathbb{T}_0$. Then the function $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$, defined by $X(t) = \widehat{x}(t, \cdot)$, is strongly measurable on \mathbb{T}_0 .

A $\mathcal{L} \times \mathcal{A}$ -measurable function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ will be called a *measurable random process*.

Remark 2.2. Let $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ be a measurable random process such that, for each fixed $t \in \mathbb{T}_0$, $x(t, \cdot) \in \mathcal{L}^p(\Omega)$. If we denote $\widehat{x}(t, \cdot)$ by $X(t)$, then $X(t) : \Omega \rightarrow \mathbb{R}$ is a random variable such that $X(t) \in L^p(\Omega)$ and $x(t, \omega) = X(t)(\omega)$ for P -a.e. $\omega \in \Omega$. In the following, using a common abuse of notation in measure theory, we will denote $x(t, \cdot)$ by $X(t)$ for each fixed $t \in \mathbb{T}_0$. In this way, a measurable random process $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ such that $x(t, \cdot) \in \mathcal{L}^p(\Omega)$ for all $t \in \mathbb{T}_0$ can be identified with a strongly measurable function $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$.

Let us denote by $C_c = C(\mathbb{T}_0, L^p(\Omega))$ the space of continuous functions $X : \mathbb{T}_0 \rightarrow L^p(\Omega)$ with the compact open topology. We recall that if K is a compact subset of \mathbb{T}_0 and U is an open subset of $L^p(\Omega)$ and we put

$$S(K, U) = \{X : K \rightarrow L^p(\Omega) \mid X(K) \subset U\},$$

then the sets

$$S(K_1, \dots, K_n; U_1, \dots, U_n) = \bigcap_{i=1}^n S(K_i, U_i),$$

where $n \in \mathbb{N}$, form a basis for the compact open topology. In fact, this topology coincides with the topology of uniform convergence on any compact subset of \mathbb{T}_0 . The space C_c is a locally convex space [28] whose topology is defined by means of the following family of seminorms:

$$\|X\|_n = \sup_{t \in K_n} \|X(t)\|_{L^p(\Omega)},$$

where $K_n = [0, t_n] \subset \mathbb{T}_0$, $n \in \mathbb{N}$ and $(t_n)_n$ is a strictly increasing sequence of elements of \mathbb{T}_0 such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$.

A distance function can be defined on C_c by

$$d_c(X, Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|X - Y\|_{L^p(\Omega)}}{1 + \|X - Y\|_{L^p(\Omega)}}.$$

The topology induced by this distance function is the same topology of uniform convergence on any compact subset of \mathbb{T}_0 .

Further, consider a continuous function $g : \mathbb{T}_0 \rightarrow (0, \infty)$. By $C_g = C_g(\mathbb{T}_0, L^p(\Omega))$ we denote the space of all continuous functions from \mathbb{T}_0 into $L^p(\Omega)$ such that

$$\sup_{t \in \mathbb{T}_0} \left\{ \frac{\|X(t)\|_{L^p(\Omega)}}{g(t)} : t \in \mathbb{T}_0 \right\} < \infty.$$

Then

$$\|X\|_{C_g} := \sup_{t \in \mathbb{T}_0} \frac{\|X(t)\|_{L^p(\Omega)}}{g(t)} \tag{2.1}$$

is a norm of C_g .

Lemma 2.3. $(C_g, \|\cdot\|_{C_g})$ is a Banach space.

Proof. Let (X_n) be a Cauchy sequence in C_g . Then for each $\varepsilon > 0$ there exists a $N = N(\varepsilon) > 0$ such that $\|X_n - X_m\|_{C_g} < \varepsilon$ for all $n, m \geq N$. Hence, by (2.1), it follows that

$$\|X_n(t) - X_m(t)\|_{L^p(\Omega)} < \varepsilon g(t), \tag{2.2}$$

for all $t \in \mathbb{T}_0$ and $n, m \geq N$. Since $L^p(\Omega)$ is a complete metric space, it follows that, for any fixed $t \in \mathbb{T}_0$, $(X_n(t))$ is a convergent sequence in $L^p(\Omega)$. Therefore, for any fixed $t \in \mathbb{T}_0$, there exists $X(t) \in L^p(\Omega)$ such that $X(t) = \lim_{n \rightarrow \infty} X_n(t)$ in $L^p(\Omega)$. Moreover, it follows from (2.2) that $X(t) = \lim_{n \rightarrow \infty} X_n(t)$ in $L^p(\Omega)$, uniformly on any compact subset of \mathbb{T}_0 . Hence, X is a continuous function from \mathbb{T}_0 into $L^p(\Omega)$. Further, we show that $X \in C_g$. Let us keep n fixed and take $m \rightarrow \infty$ in (2.2). Then we obtain that $X_n - X \in C_g$ for all $n \geq N$. Since $X = (X - X_n) + X_n$ and $X - X_n, X_n \in C_g$, it follows that $X \in C_g$. \square

Remark 2.4. The topology of C_g is stronger than the topology of C_c . Indeed, if $X_n \rightarrow X$ in C_g as $n \rightarrow \infty$, then for each $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that $\|X_n(t) - X(t)\|_{L^p(\Omega)} < \varepsilon g(t)$, for all $t \in \mathbb{T}_0$ and $n \geq N(\varepsilon)$. Since g is bounded on any compact subset of \mathbb{T}_0 , it allows that $X_n(t) \rightarrow X(t)$ as $n \rightarrow \infty$, uniformly on any compact subset of \mathbb{T}_0 . In other words, convergence in C_g implies convergence in C_c . If $g(t) = 1$ on \mathbb{T}_0 , then C_g becomes the space $C = C(\mathbb{T}_0, L^p(\Omega))$ of all continuous and bounded functions from \mathbb{T}_0 into $L^p(\Omega)$. The norm on C is given by

$$\|X\|_c = \sup_{t \in \mathbb{T}_0} \|X(t)\|_{L^p(\Omega)}.$$

Note that the following inclusions hold $C \subset C_g \subset C_c$.

Let (B, D) be a pair of Banach spaces such that $B, D \subset C_c$ and let \mathcal{T} be a linear operator from C_c to itself. The pair of Banach spaces (B, D) is called *admissible* with respect to the operator $\mathcal{T} : C_c \rightarrow C_c$ if $\mathcal{T}(B) \subset D$ ([13]).

Remark 2.5. If the pair (B, D) is admissible with respect to the linear operator $\mathcal{T} : C_c \rightarrow C_c$ then, by Lemma 2.1.1 from [21], it follows that \mathcal{T} is a continuous operator from B to D . Therefore, there exists a $M > 0$ such that

$$\|\mathcal{T}X\|_D \leq M \|X\|_B, \quad X \in B.$$

3. RANDOM INTEGRAL EQUATION OF VOLTERRA TYPE

In this section we study the existence and uniqueness of a random solution of a random integral equation of Volterra type.

$$x(t, \omega) = h(t, \omega) + \lambda \int_{t_0}^t k(t, s, \omega) f(s, x(s, \omega), \omega) \Delta s, \quad t \in \mathbb{T}_0, \quad (3.1)$$

where P -a.e. $\omega \in \Omega$, $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ is the unknown random process, $h : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ is a measurable random process, $f : \mathbb{T}_0 \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is a random function, $k : \Gamma \times \Omega \rightarrow \mathbb{R}$ is the random kernel, $\lambda \in \mathbb{R}^*$, and $\Gamma := \{(t, s) \in \mathbb{T}_0 \times \mathbb{T}_0 : t_0 \leq s \leq t < \infty\}$.

In what follows, we will use the notations $X(t) = x(t, \cdot)$, $H(t) = h(t, \cdot)$, $K(t, s) = k(t, s, \cdot)$, $F(t, X(t)) = f(t, x(t, \cdot), \cdot)$.

Let us consider the following assumptions:

- (h1) $K(t, s) \in L^\infty(\Omega)$ for all $(t, s) \in \Gamma$, $K(\cdot, \cdot) : \Gamma \rightarrow L^\infty(\Omega)$ continuous in its first variable and rd -continuous in its second variable, there exists $k_0 > 0$ and $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$ such that

$$\|K(t, s)\|_{L^\infty(\Omega)} \leq k_0 e_{-\alpha}(t, \sigma(s))$$

for $(t, s) \in \Gamma$.

- (h2) $f(\cdot, x, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}$ is a $\mathcal{L} \times \mathcal{A}$ -measurable function for each $x \in \mathbb{R}$, and there exist an $a > 0$ and a positive random variable $L : \Omega \rightarrow \mathbb{R}$ such that $P(\{\omega \in \Omega : L(\omega) > a\}) = 0$ and

$$|f(t, x, \omega) - f(t, y, \omega)| \leq L(\omega) |x - y|$$

for all $t \in \mathbb{T}_0$ and $x, y \in \mathbb{R}$.

- (h3) $F(t, 0) \in L^p(\Omega)$ for all $t \in \mathbb{T}_0$ and there exists $\beta \in (0, \alpha)$ with $-\beta \in \mathcal{R}^+$ such that

$$r := \sup_{t \in \mathbb{T}_0} \frac{\|F(t, 0)\|_{L^p(\Omega)}}{e_{-\beta}(t, 0)} < \infty.$$

In what follows, consider $g(t) := e_{-\beta}(t, 0)$, $t \in \mathbb{T}_0$, where $0 < \beta < \alpha$. Also, we will use the notation C_β instead of C_g .

Lemma 3.1. *If (h2) and (h3) hold, then*

$$\sup_{t \in \mathbb{T}_0} \frac{\|F(t, X(t))\|_{L^p(\Omega)}}{e_{-\beta}(t, 0)} \leq a \|X\|_{C_\beta} + r < \infty \quad (3.2)$$

for every $X \in C_\beta$, and

$$\|F(t, X(t)) - F(t, Y(t))\|_{L^p(\Omega)} \leq a \|X(t) - Y(t)\|_{L^p(\Omega)} \quad (3.3)$$

for all $t \in \mathbb{T}_0$ and $X, Y \in C_\beta$.

Proof. If we denote $\{\omega \in \Omega : L(\omega) \leq a\}$ by Ω_a , then from (h2) we have that $P(\Omega_a) = 1$. If $X, Y \in C_\beta$, using the Minkowski's inequality, (h2) and (h3), we have

$$\begin{aligned} \|F(t, X(t))\|_{L^p(\Omega)} &= \|f(t, x(t, \cdot), \cdot)\|_{L^p(\Omega)} \leq \\ &\leq \left(\int_{\Omega} |f(t, x(t, \omega), \omega) - f(t, 0, \omega)|^p dP(\omega) \right)^{1/p} + \left(\int_{\Omega} |f(t, 0, \omega)|^p dP(\omega) \right)^{1/p} \leq \\ &\leq \left(\int_{\Omega_a} |L(\omega)|^p |x(t, \omega)|^p dP(\omega) \right)^{1/p} + \|F(t, 0)\|_{L^p(\Omega)} \leq \\ &\leq a \|X(t)\|_{L^p(\Omega)} + \|F(t, 0)\|_{L^p(\Omega)}. \end{aligned}$$

Dividing both sides of the last inequality by $e_{-\beta}(t, 0) > 0$ and taking the supremum with respect to $t \in \mathbb{T}_0$, we obtain (3.2). Also,

$$\begin{aligned} \|F(t, X(t)) - F(t, Y(t))\|_{L^p(\Omega)} &= \|f(t, x(t, \cdot), \cdot) - f(t, y(t, \cdot), \cdot)\|_{L^p(\Omega)} = \\ &= \left(\int_{\Omega} |f(t, x(t, \omega), \omega) - f(t, y(t, \omega), \omega)|^p dP(\omega) \right)^{1/p} \leq \\ &\leq \left(\int_{\Omega_a} |L(\omega)|^p |x(s, \omega) - y(s, \omega)|^p dP(\omega) \right)^{1/p} \leq a \|X(t) - Y(t)\|_{L^p(\Omega)}. \quad \square \end{aligned}$$

Remark 3.2. It follows from Lemma 3.1 that $F(t, X(t)) \in L^p(\Omega)$ for all $t \in \mathbb{T}_0$ and $X \in C_\beta$. Moreover, (3.2) implies that the function $t \mapsto F(t, X(t))$ belong to C_β for all $X \in C_\beta$.

Lemma 3.3. *Let us consider the integral operator $\mathcal{T} : C_c \rightarrow C_c$ defined by*

$$(\mathcal{T}X)(t) = \int_0^t K(t, s)X(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{3.4}$$

If (h1) holds, then $\mathcal{T}(C_\beta) \subset C_\beta$.

Proof. Let $X \in C_\beta$. We have that

$$\begin{aligned} \|(\mathcal{T}X)(t)\|_{L^p(\Omega)} &\leq \int_0^t \|K(t, s)X(s)\|_{L^p(\Omega)} \Delta s \leq \int_0^t \|K(t, s)\|_{L^\infty(\Omega)} \|X(s)\|_{L^p(\Omega)} \Delta s = \\ &= \int_0^t \|K(t, s)\|_{L^\infty(\Omega)} \frac{\|X(s)\|_{L^p(\Omega)}}{e_{-\beta}(s, 0)} e_{-\beta}(s, 0) \Delta s \leq \\ &\leq \|X\|_{C_\beta} \int_0^t \|K(t, s)\|_{L^\infty(\Omega)} e_{-\beta}(s, 0) \Delta s. \end{aligned}$$

Take into account (h1), we infer that

$$\begin{aligned} \int_0^t \|K(t, s)\|_{L^\infty(\Omega)} e_{-\beta}(s, 0) \Delta s &\leq k_0 \int_0^t e_{-\alpha}(t, \sigma(s)) e_{-\beta}(s, 0) \Delta s = \\ &= \frac{k_0}{\alpha - \beta} [e_{-\beta}(t, 0) - e_{-\alpha}(t, 0)]. \end{aligned}$$

Since $-\alpha, -\beta \in \mathcal{R}^+$ and $-\alpha < -\beta$, then (see [6, Corollary 2.10]) we have that $e_{-\beta}(t, 0) > e_{-\alpha}(t, 0)$, $t \in \mathbb{T}_0$, and it follows that

$$\int_0^t \|K(t, s)\|_{L^\infty(\Omega)} e_{-\beta}(s, 0) \Delta s \leq \frac{k_0}{\alpha - \beta} e_{-\beta}(t, 0), \quad t \in \mathbb{T}_0. \tag{3.5}$$

Consequently,

$$\|(\mathcal{T}X)(t)\|_{L^p(\Omega)} \leq \frac{k_0}{\alpha - \beta} \|X\|_{C_\beta} e_{-\beta}(t, 0), \quad t \in \mathbb{T}_0,$$

and thus $\mathcal{T}X \in C_\beta$ for every $X \in C_\beta$, that is, $\mathcal{T}(C_\beta) \subset C_\beta$. □

Remark 3.4. Since, by Lemma 3.3, the pair (C_β, C_β) is admissible with respect to the linear operator $\mathcal{T} : C_c \rightarrow C_c$ then, by Remark 2.5, it follows that \mathcal{T} is a continuous operator from C_β to C_β . Therefore, there exists a $M > 0$ such that

$$\|\mathcal{T}X\|_{C_\beta} \leq M \|X\|_{C_\beta}, \quad X \in C_\beta.$$

In fact, it easy to see that $M = \frac{k_0}{\alpha - \beta}$ is the norm of \mathcal{T} as a linear operator from C_β into C_β .

A solution $X \in C_\beta$ of the integral equation (3.1) is called *asymptotically exponentially stable* if there exists a $\rho > 0$ and a $\beta > 0$ such that $-\beta \in \mathcal{R}^+$ and

$$\|X(t)\|_{L^p(\Omega)} \leq \rho e_{-\beta}(t, 0), \quad t \in \mathbb{T}_0.$$

Remark 3.5. The admissibility concept is related to stability in various senses (see [17]). Let $\mathcal{T} : C_c \rightarrow C_c$ be a linear operator. Roughly speaking we say that the pair of function spaces $B, D \subset C_c$ is admissible with respect to the equation

$$X = H + \mathcal{T}X, \tag{3.6}$$

if this equation has its solution in the space D , for each $H \in D$. Therefore, if we choose $D = C_\beta$ and if $X \in C_\beta$ is a solution of the equation (3.6), then there exists a $\rho > 0$ such that $\|X\|_{C_\beta} \leq \rho$. Using (2.1) we infer that

$$\|X(t)\|_{L^p(\Omega)} \leq \rho e_{-\beta}(t, 0)$$

for all $t \in \mathbb{T}_0$, that is, the solution of the equation (3.6) is asymptotically exponentially stable. For several results concerning the admissibility theory for Volterra integral equations see [11].

These preliminaries being completed, we shall state the following result.

Theorem 3.6. *If the assumptions (h1)–(h3) hold and $H \in C_\beta$, then the integral equation (3.1) has a unique asymptotically exponentially stable solution, provided that $|\lambda| aM < 1$, where $M > 0$ is the norm of the operator \mathcal{T} .*

Proof. Let us consider the operator $\mathcal{V} : C_\beta \rightarrow C_c$ defined by

$$(\mathcal{V}X)(t) = H(t) + \lambda \int_0^t K(t, s)F(s, X(s))\Delta s, \quad t \in \mathbb{T}_0. \tag{3.7}$$

Then we can rewrite the operator \mathcal{V} as

$$(\mathcal{V}X)(t) = H(t) + \lambda(\mathcal{T}G)(t), \quad t \in \mathbb{T}_0, \tag{3.8}$$

where $G(t) := F(t, X(t))$, $t \in \mathbb{T}_0$ and \mathcal{T} is the operator given by (3.4). Since by Remark 3.2 and Lemma 3.1 we have that $\|G\|_{C_\beta} \leq a \|X\|_{C_\beta} + r$, then

$$\|(\mathcal{T}G)(t)\|_{L^p(\Omega)} \leq bMe_{-\beta}(t, 0), \quad t \in \mathbb{T}_0, \tag{3.9}$$

where $b := a \|X\|_{C_\beta} + r$. From (3.8) and (3.9) we obtain that

$$\|(\mathcal{V}X)(t)\|_{L^p(\Omega)} \leq \|H(t)\|_{L^p(\Omega)} + b|\lambda| Me_{-\beta}(t, 0),$$

for all $t \in \mathbb{T}_0$. Dividing both sides of the last inequality by $e_{-\beta}(t, 0) > 0$ and taking the supremum with respect to $t \in \mathbb{T}_0$, it follows that

$$\|\mathcal{V}X\|_{C_\beta} \leq \|H\|_{C_\beta} + b|\lambda| M, \tag{3.10}$$

and so $\mathcal{V}X \in C_\beta$ for all $X \in C_\beta$. Further, we show that the operator \mathcal{V} is a contraction on C_β . Indeed, using (3.3) and (3.5), we have

$$\begin{aligned} \|(\mathcal{V}X)(t) - (\mathcal{V}Y)(t)\|_{L^p(\Omega)} &\leq |\lambda| \int_0^t \|K(t, s)[F(s, X(s)) - F(s, Y(s))]\|_{L^p(\Omega)} \Delta s \leq \\ &\leq |\lambda| \int_0^t \|K(t, s)\|_{L^\infty(\Omega)} \|F(s, X(s)) - F(s, Y(s))\|_{L^p(\Omega)} \Delta s \leq \\ &\leq a|\lambda| \int_0^t \|K(t, s)\|_{L^\infty(\Omega)} \frac{\|X(s) - Y(s)\|_{L^p(\Omega)}}{e_{-\beta}(s, 0)} e_{-\beta}(s, 0) \Delta s \leq \\ &\leq a|\lambda| \|X - Y\|_{C_\beta} \int_0^t \|K(t, s)\|_{L^\infty(\Omega)} e_{-\beta}(s, 0) \Delta s \leq \\ &\leq \frac{a|\lambda| k_0}{\alpha - \beta} \|X - Y\|_{C_\beta} e_{-\beta}(t, 0) = \\ &= a|\lambda| M \|X - Y\|_{C_\beta} e_{-\beta}(t, 0). \end{aligned}$$

Thus

$$\|(\mathcal{V}X)(t) - (\mathcal{V}Y)(t)\|_{L^p(\Omega)} \leq a |\lambda| M \|X - Y\|_{C_\beta}$$

for all $t \in \mathbb{T}_0$, and so

$$\|\mathcal{V}X - \mathcal{V}Y\|_{C_\beta} \leq a |\lambda| M \|X - Y\|_{C_\beta},$$

with $a |\lambda| M < 1$, that is, \mathcal{V} is a contraction on C_β . From Banach’s Fixed Point Theorem, it follows that there exist a unique solution $X \in C_\beta$ of the integral equation (3.1). From Remark 3.5, we infer that the solution is asymptotically exponentially stable. \square

Corollary 3.7. *If all the hypotheses of Theorem 3.6 hold for $\beta = 0$, then the integral equation (3.1) has a unique solution $X \in C$.*

Corollary 3.8. *If all the hypotheses of Theorem 3.6, then the solution of the integral equation (3.1) is asymptotically stable in mean, that is, $E[|X(t)|] \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. Since $-\beta < 0$, then $e_{-\beta}(t, 0)$ decreases monotonically towards zero as $t \rightarrow \infty$, and therefore $\|X(t)\|_{L^p(\Omega)} \rightarrow 0$ as $t \rightarrow \infty$. Since $E[|X(t)|^p] = \|X(t)\|_{L^p(\Omega)}^p$ then, using the Jensen’s inequality, we infer that $E[|X(t)|] \rightarrow 0$ as $t \rightarrow \infty$. \square

Remark 3.9. Let $\mathbb{T}_0 = [0, \infty)$. Then, for $g(t) = q(t) = e^{-\beta t}$, $t \geq 0$, we obtain Theorem 2.2 from [7]. For $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 3.1 from [26]. Let $\mathbb{T}_0 = \mathbb{N}$. Then, for $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 5.3.1 from [27].

In what follows, using the concept of admissibility, we prove a general result of the existence and uniqueness for the integral equation (3.1). From this result it is possible to derive many existence results, by particularizing the spaces B and D .

Let us consider the integral equation (3.1) under the following conditions:

- (h̃1) $K(t, s) \in L^\infty(\Omega)$ for all $(t, s) \in \Gamma$, $K(\cdot, \cdot) : \Gamma \rightarrow L^\infty(\Omega)$ continuous in its first variable and rd -continuous in its second variable.
- (h̃2) $B, D \subset C_c$ are Banach spaces stronger than C_c such that the pair (B, D) is admissible with respect to the linear operator $\mathcal{T} : C_c \rightarrow C_c$ defined by (3.4).
- (h̃3) For each $X \in D$, the function $t \mapsto F(t, X(t))$ belong to B , and the operator $\mathcal{G} : D \rightarrow B$, defined by $(\mathcal{G}X)(t) = F(t, X(t))$ for all $t \in \mathbb{T}_0$, satisfies the Lipschitz condition

$$\|\mathcal{G}X - \mathcal{G}Y\|_B \leq a \|X - Y\|_D$$

for all $X, Y \in D$ and some $a > 0$.

Theorem 3.10. *If the assumptions (h̃1)–(h̃3) hold and $H \in D$, then the integral equation (3.1) has a unique solution $X \in D$, provided that $|\lambda| aM < 1$, where $M > 0$ is the norm of the operator \mathcal{T} .*

Proof. Let us consider the operator $\mathcal{V} : D \rightarrow C_c$ defined by $\mathcal{V}X = H + \lambda\mathcal{T}\mathcal{G}X$. Since the pair (B, D) is admissible with respect to the linear operator \mathcal{T} , it follows from Remark 2.5 that there exists a $M > 0$ such that $\|\mathcal{T}X\|_D \leq M\|X\|_B$ for all $X \in B$. Using (h3) and the fact that $H \in D$ it follows from Minkowski's inequality that

$$\begin{aligned}\|\mathcal{V}X\|_D &\leq \|H\|_D + |\lambda| M \|\mathcal{G}X\|_B \leq \|H\|_D + |\lambda| M \|\mathcal{G}X - \mathcal{G}0\|_B + |\lambda| M \|\mathcal{G}0\|_B \leq \\ &\leq \|H\|_D + a|\lambda| M \|X\|_D + |\lambda| M \|\mathcal{G}0\|_B < \infty,\end{aligned}$$

that is, $\mathcal{V}X \in D$ for all $X \in D$. Next, all $X, Y \in D$ we have that $\mathcal{V}X - \mathcal{V}Y = \lambda\mathcal{T}(\mathcal{G}X - \mathcal{G}Y)$. Obviously, $\mathcal{G}X - \mathcal{G}Y \in B$ and $\mathcal{V}X - \mathcal{V}Y \in D$. It follows that

$$\|\mathcal{V}X - \mathcal{V}Y\|_D \leq |\lambda| M \|\mathcal{G}X - \mathcal{G}Y\|_B \leq |\lambda| aM \|X - Y\|_D,$$

with $|\lambda|aM < 1$, that is, \mathcal{V} is a contraction on D . From Banach's Fixed Point Theorem, it follows that there exist a unique solution $X \in D$ of the integral equation (3.1). \square

Remark 3.11. If $\mathbb{T}_0 = [0, \infty)$, we obtain Theorem 2.4 from [7]. For $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 2.1.2 from [27]. If $\mathbb{T}_0 = \mathbb{N}$, then, for $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 5.1.2 from [27].

REFERENCES

- [1] M. Adivar, N.Y. Raffoul, *Existence results for periodic solutions of integro-dynamic equations on time scales*, Ann. Mat. Pura Appl. **188** (2009) 4, 543–559.
- [2] R. Agarwal, M. Bohner, D. O'Regan, A. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), 1–26.
- [3] B. Aulbach, L. Neidhart, *Integration on measure chains*, Proc. of the Sixth International Conference on Difference Equations, B. Aulbach, S. Elaydi, G. Ladas, eds., Augsburg, Germany 2001, pp. 239–252.
- [4] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: an Introduction with Applications*, Birkhäuser, Boston, 2001.
- [5] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
- [6] E. Akin-Bohner, M. Bohner, F. Akin, *Pachpatte inequalities on time scales*, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005) 1, 1–23.
- [7] N.U. Ahmed, K.L. Teo, *On the stability of a class of nonlinear stochastic systems*, J. Information and Control **20** (1972), 276–293.
- [8] T.A. Burton, *Volterra integral and differential equations*, vol. 202 of Mathematics in Science and Engineering, Elsevier B.V., Amsterdam, 2nd ed., 2005.
- [9] A. Cabada, D.R. Vivero, *Expression of the Lebesgue Δ -integral on time scales as a usual Lebesgue integral; application to the calculus of Δ -antiderivatives*, Math. Comput. Modelling **43** (2006), 194–207.

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- [10] M. Cichoń, *On integrals of vector-valued functions on time scales*, Commun. Math. Anal. **1** (2011) 11, 94–110.
- [11] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York-London, 1973.
- [12] B.C. Dhage, S.K. Ntouyas, *Existence and attractivity results for nonlinear first order random differential equations*, Opuscula Math. **30** (2010) 4, 411–429.
- [13] N. Dunford, J.T. Schwartz, *Linear Operators I*, Interscience, New York, 1958.
- [14] G.Sh. Guseinov, *Integration on time scales*, J. Math. Anal. Appl. **285** (2003), 107–127.
- [15] S. Hilger, *Analysis on measure chains – a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.
- [16] T. Kulik, C.C. Tisdell, *Volterra integral equations on time scales: basic qualitative and quantitative results with applications to initial value problems on unbounded domains*, Int. J. Difference Equ. **3** (2008) 1, 103–133.
- [17] J.L. Massera, J.J. Schäffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.
- [18] L. Neidhart, *Integration im Rahmen des Maßkettenkalküls*, Diploma Thesis, University of Augsburg, 2001.
- [19] A. Sikorska-Nowak, *Integrodifferential equations on time scales with Henstock-Kurzweil-Pettis delta integrals*, Abstr. Appl. Anal. **1** (2010), 1–17.
- [20] D.B. Pachpatte, *On a nonstandard Volterra type dynamic integral equation on time scales*, Electron. J. Qual. Theory Differ. Equ. **72** (2009), 1–14.
- [21] W.J. Padgett, C.P. Tsokos, *On a stochastic integro-differential equation of Volterra type*, SIAM J. Appl. Math. **23** (1972), 499–512.
- [22] A.T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.
- [23] S. Sanyal, *Mean square stability of Itô Volterra dynamic equation*, Nonlinear Dyn. Syst. Theory **11** (2011) 1, 83–92.
- [24] S. Sanyal, *Stochastic Dynamic Equations*, Ph.D. Thesis, Missouri University of Science and Technology, Rolla, Missouri, 2008.
- [25] C.C. Tisdell, A. Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with an application to economic modelling*, Nonlinear Anal. **68** (2008) 11, 3504–3524.
- [26] C.P. Tsokos, M.A. Hamdan, *Stochastic asymptotic exponential stability of stochastic integral equations*, J. Appl. Prob. **9** (1972), 169–177.
- [27] C.P. Tsokos, W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.
- [28] K. Yosida, *Functional Analysis*, 6th ed., Springer-Verlag, 1980.

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