Opuscula Math. 33, no. 2 (2013), 337–344
http://dx.doi.org/10.7494/OpMath.2013.33.2.337 Opuscula Mathematica

A NEW METHOD FOR SOLVING ILL-CONDITIONED LINEAR SYSTEMS

Fazlollah Soleymani

Communicated by Zdzisław Jackiewicz

Abstract. An accurate numerical method is established for matrix inversion. It is shown theoretically that the scheme possesses the high order of convergence of seven. Subsequently, the method is taken into account for solving linear systems of equations. The accuracy of the contributed iterative method is clarified on solving numerical examples when the coefficient matrices are ill-conditioned. All of the computations are performed on a PC using several programs written in Mathematica 7.

Keywords: matrix inversion, linear systems, Hilbert matrix, ill-conditioned, approximate inverse.

Mathematics Subject Classification: 15A09, 65F10.

1. INTRODUCTION

It is widely known that the solutions of linear systems of equations are sensitive to the round-off error. For some linear systems a small change in one of the values of the coefficient matrix or the right-hand side vector causes a large change in the solution vector. When the solution is highly sensitive to the values of the coefficient matrix A or the right-hand side constant vector b, the equations are said to be ill-conditioned. Therefore, we cannot easily rely on the solutions coming out of an ill-conditioned system.

Ill-conditioned systems pose particular problems when the coefficients are estimated from experimental results [6]. For a system with condition number $\kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$ on its coefficient matrix, one can expect a loss of roughly $\lg_{10} \kappa(A)$ decimal places in the accuracy of the solution.

In other words, solving very ill-conditioned linear systems by classical methods is not usual. The Krylov subspace methods (without preconditioners) [8] and also the iterative methods of the AOR family, both require a remarkable time to find a reliable solution. Note that the Accelerated Over Relaxation family or the AOR method includes two free parameters, which is the extension of the Successive Over Relaxation method.

On the other hand, one way is to construct iterative methods of high order of convergence to find approximate inverses of the ill-conditioned matrices by applying high floating point arithmetic. Several methods of various orders were proposed for approximating (rectangular or square) matrix inverses, such as those according to the minimum residual iterations [1] and the Hotelling-Bodewig algorithm [9].

The Hotelling-Bodewig algorithm is defined as $V_{n+1} = V_n(2I - AV_n)$, $n = 0, 1, \ldots,$ where I is the identity matrix.

In 2011, Li *et al.* in [4] presented the following third-order method

$$V_{n+1} = V_n(3I - AV_n(3I - AV_n)), \quad n = 0, 1, \ldots, \tag{1.1}$$

and also proposed another cubical method for finding A^{-1}:

$$V_{n+1} = \left[I + \frac{1}{4}(I - V_nA)(3I - V_nA)^2\right] V_n, \quad n = 0, 1, \ldots. \tag{1.2}$$

Notice that the iterative method (1.1) can also be found in [3]. As an another example, Krishnamurthy and Sen suggested the following sixth-order iteration method [3, 10] for the above purpose

$$V_{n+1} = V_n(2I - AV_n)(3I - AV_n(3I - AV_n))(I - AV_n(I - AV_n)), \tag{1.3}$$

where $n = 0, 1, \ldots$. In what follows, we first present a proof for the method (1.3).

Theorem 1.1. *Let $A = [a_{ij}]_{N\times N}$ be a nonsingular matrix. If the initial approximation V_0 satisfies $\|I - AV_0\| < 1$, then the iterative method* (1.3) *converges with sixth-order convergence to A^{-1}.*

Proof. Let $\|I - AV_0\| < 1$, and $E_0 = I - AV_0$. Subsequently $E_n = I - AV_n$. Then for (1.3), we have

$$\begin{aligned} E_{n+1} &= I - AV_{n+1} = \\ &= I - A(V_n(2I - AV_n)(3I - AV_n(3I - AV_n))(I - AV_n(I - AV_n))) = \\ &= I - (-AV_n(-2I + AV_n)(3I - 3AV_n + (AV_n)^2)(I - AV_n + (AV_n)^2) = \\ &= (I - AV_n)^6 = (E_n)^6. \end{aligned} \tag{1.4}$$

Moreover, since $\|E_0\| < 1$, by the relation (1.4) we obtain that

$$\|E_{n+1}\| \le \|E_n\|^6 \le \|E_{n-1}\|^{6^2} \le \ldots \le \|E_0\|^{6^{n+1}}, \tag{1.5}$$

where (1.5) tends to zero when $n \to \infty$, that is,

$$I - AV_n \to 0,$$

when $n \to \infty$, and thus for (1.3), we attain

$$V_n \to A^{-1} \quad \text{as} \quad n \to \infty. \tag{1.6}$$

Now we show the sixth order of convergence. To do this, we denote $e_n = V_n - A^{-1}$, as the error matrix in the iterative procedure (1.3). We have

$$\begin{aligned} A^{-1} + e_{n+1} = V_{n+1} &= V_n(2I - AV_n)(3I - AV_n(3I - AV_n))(I - AV_n(I - AV_n)) = \\ &= (e_n + A^{-1})(2I - A(e_n + A^{-1}))(3I - A(e_n + A^{-1})(3I - A(e_n + \\ &\quad + A^{-1})))(I - A(e_n + A^{-1})(I - A(e_n + A^{-1}))) = \\ &= -(e_n + A^{-1})(Ae_n - I)(I + (Ae_n)^2 + (Ae_n)^4) = \\ &= (e_n + A^{-1})(I - Ae_n + (Ae_n)^2 - (Ae_n)^3 + (Ae_n)^4 - (Ae_n)^5) = \\ &= A^{-1} - A^5(e_n)^6, \end{aligned}$$

which yields

$$e_{n+1} = -A^5(e_n)^6,$$

and consequently

$$\|e_{n+1}\| \le \|A\|^5 \|e_n\|^6. \tag{1.7}$$

Thus the iteration (1.3) locally converges with sixth order to A^{-1}. This concludes the proof. □

For further reading, we refer the readers to [5, 7]. In this article, in order to work with very ill-conditioned matrices, we will propose a method for finding A^{-1} iteratively. The theoretical convergence of the method will also be studied.

The rest of the paper is organized as follows. The main contribution of this article is given in Section 2. Subsequently, the method is examined in Section 3 numerically. Finally, concluding remarks are presented in Section 4.

2. A HIGH-ORDER METHOD

This section contains a new high order algorithm for finding A^{-1} numerically. In order to deal with very ill-conditioned linear systems, or to find robust approximate inverses of the coefficient matrices, we suggest the following iteration method

$$\begin{aligned} V_{n+1} = \frac{1}{4} V_n(32I + AV_n \times \\ \times (-113I + AV_n(231I + AV_n(-301I + AV_n(259I + AV_n(W_n)))))), \end{aligned} \tag{2.1}$$

where $W_n = -147I + AV_n\,(53I + AV_n\,(-11I + AV_n))$, for any $n = 0, 1, \ldots$, wherein I is the identity matrix, and the sequence of iterates $\{V_n\}_{n=0}^{\infty}$ converges to A^{-1} for a good initial guess. Such a guess, V_0, will be given in Section 3.

Theorem 2.1. *Let $A = [a_{ij}]_{N\times N}$ be a nonsingular matrix. If the initial approximation V_0 satisfies*

$$\|I - AV_0\| < 1, \tag{2.2}$$

then the iterative method (2.1) converges with seventh order to A^{-1}.

Proof. Let $\|I - AV_0\| < 1$, $E_0 = I - AV_0$, and $E_n = I - AV_n$. For (2.1), we have

$$\begin{aligned}
E_{n+1} &= I - AV_{n+1} = \\
&= I - A\Big[\frac{1}{4}V_n(32I + AV_n(-113I + AV_n(231I + AV_n(-301I+ \\
&\quad + AV_n(259I + AV_n(-147I + AV_n(53I + AV_n(-11I + AV_n))))))))\Big] = \\
&= \frac{1}{4}AV_n(32I - 113AV_n + 231(AV_n)^2 - 301(AV_n)^3 + 259(AV_n)^4 - \\
&\quad - 147(AV_n)^5 + 53(AV_n)^6 - 11(AV_n)^7 + (AV_n)^8) = \\
&= -\frac{1}{4}(-2I + AV_n)^2(-I + AV_n)^7 = \frac{1}{4}(I + (I - AV_n))^2(I - AV_n)^7 = \\
&= \frac{1}{4}(I + 2E_n + (E_n)^2)(E_n)^7 = \frac{1}{4}(E_n^7 + 2E_n^8 + E_n^9).
\end{aligned}$$

Thus, we obtain

$$\|E_{n+1}\| = \frac{1}{4}(\|E_n^7 + 2E_n^8 + E_n^9\|) \leq \frac{1}{4}(\|E_n^7\| + 2\|E_n^8\| + \|E_n^9\|). \tag{2.3}$$

Moreover, since $\|E_0\| < 1$, and $\|E_1\| \leq \|E_0\|^7 < 1$, hence we get that (we consider $\|E_n\| < 1$)

$$\|E_{n+1}\| \leq \|E_n\|^7 \leq \|E_{n-1}\|^{7^2} \leq \ldots \leq \|E_0\|^{7^{n+1}} < 1, \tag{2.4}$$

where (2.4) tends to zero when $n \to \infty$, that is,

$$I - AV_n \to 0,$$

when $n \to \infty$, and thus for (2.1), we attain

$$V_n \to A^{-1} \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Now we show the seventh order of convergence. To do this, we denote $e_n = V_n - A^{-1}$, as the error matrix in the iterative procedure (2.1). We have

$$I - AV_{n+1} = \frac{1}{4}[(I - AV_n)^7 + 2(I - AV_n)^8 + (I - AV_n)^9]. \tag{2.6}$$

We can now easily obtain

$$A(A^{-1} - V_{n+1}) = \frac{1}{4}[A^7(A^{-1} - V_n)^7 + 2A^8(A^{-1} - V_n)^8 + A^9(A^{-1} - V_n)^9], \tag{2.7}$$

and

$$-A(V_{n+1} - A^{-1}) = \frac{1}{4}[-A^7(V_n - A^{-1})^7 + 2A^8(V_n - A^{-1})^8 - A^9(V_n - A^{-1})^9]. \tag{2.8}$$

Simplifying (2.8) results in

$$e_{n+1} = -\frac{1}{4}[-A^6(e_n)^7 + 2A^7(e_n)^8 - A^8(e_n)^9], \tag{2.9}$$

which, by taking the norm of both sides, yields

$$\|e_{n+1}\| \leq \frac{1}{4}[\|A^6 e_n^7\| + 2\|A^7 e_n^8\| + \|A^8 e_n^9\|], \tag{2.10}$$

and consequently

$$\|e_{n+1}\| \leq \left(\frac{1}{4}[\|A\|^6 + 2\|A\|^7\|e_n\| + \|A\|^8\|e_n\|^2]\right)\|e_n\|^7. \tag{2.11}$$

Thus, the iteration (2.1) locally converges to A^{-1} with at least seventh order of convergence based on the error inequality (2.11). This concludes the proof. □

3. COMPUTATIONAL TESTS

In this section, experiments are presented to demonstrate the capability of the suggested method. For solving a square real linear system of equations of the general form $Ax = b$, wherein $A \in \mathbb{R}^{N\times N}$, we can now propose the following efficient algorithm

$$x_{n+1} = V_{n+1}b \quad \text{while} \quad n = 0, 1, \ldots.$$

The programming package Mathematica 7 [11] has been used in this section. For numerical comparisons, we have used the Hotelling-Bodewig algorithm denoted by (HBA), the method (1.1) which is also known as Chebyshev's method, and the sixth-order method of Krishnamurthy and Sen (1.3) and the new accurate algorithm (2.1). We use 256 digits floating point arithmetic in our calculations using the command

```
SetAccuracy [expr,256]
```

to keep the effect of round-off error at minimum.

We here take into account the very efficient way of producing V_0 as given by Codevico *et al.* in [2] as follows:

$$V_0 = \frac{A^T}{\|A\|_1\|A\|_\infty}. \tag{3.1}$$

This choice will easily provide guesses enough close to the A^{-1} in order to preserve the convergence order. However, another way to do this is

$$V_0 = \frac{A^T}{Trace(AA^T)}. \tag{3.2}$$

Test Problem. Consider the linear system $Hx = b$, wherein H is the Hilbert matrix defined by $H_{N\times N} = [h_{i,j} = \frac{1}{i+j-1}]$. The right hand side vector will be considered as $b = (10, 10, \ldots, 10)^T$ in this paper. We use Hilbert matrices of various orders as test problems in this section.

Table 1. The condition number of different Hilbert matrices

Order of Hilbert matrix	10	15	20
Condition number	3.5×10^{13}	1.5×10^{21}	6.2×10^{28}

Table 1 simply shows that H is getting ill-conditioned by increasing the dimension of the matrix, and subsequently the system cannot be solved easily. Note that the condition numbers in Table 1 have been obtained using the command

```
N[LinearAlgebra‘MatrixConditionNumber[HilbertMatrix[n]], 5]
```

in Mathematica. We expect to find robust approximations of the Hilbert inverses in less iterations by high-order iterative methods.

Table 2 shows the number of iterations for different methods in order to reveal the efficiency of the proposed iteration. In Table 2, IN and RN stand for iteration number and residual norm for the solution of the linear system, respectively. Note that in this test problem, we have used an initial value constructed by (3.1).

There is a clear reduction in computational steps for the proposed method (2.1) in contrast to the other existing well-known methods of various orders in the literature based on Table 2.

Table 2. Results of comparisons for the Test Problem

Methods	HBA	(1.1)	(1.3)	(2.1)
Order of convergence	2	3	6	7
Hilbert mat. dimension is 10				
IN	96	61	38	33
RN	1.4×10^{-53}	1.4×10^{-82}	2.8×10^{-230}	1.8×10^{-250}
Hilbert mat. dimension is 15				
IN	146	93	57	50
RN	2.2×10^{-41}	5.5×10^{-94}	1.3×10^{-90}	5.7×10^{-250}
Hilbert mat. dimension is 20				
IN	197	124	76	66
RN	7.9×10^{-54}	1.1×10^{-42}	4.8×10^{-41}	3.6×10^{-79}

4. SUMMARY

Iterative methods are often effective especially for large scale systems with sparsity and Hotelling-Bodewig-type methods for well-conditioned linear systems in double precision arithmetic and ill-conditioned ones by using high precision floating points. The Hotelling-Bodewig algorithm is simple to describe and to analyze, and is numerically stable for nonsingular input matrices. This was the idea of developing iterative methods of this type for the solution of linear systems.

In this article, we have developed an iterative method in inverse-finding of matrices. Note that such high order iterative methods are efficient for very ill-conditioned linear systems. We have shown that the suggested method (2.1) reaches the seventh order of convergence. Moreover, the efficiency of the new scheme was illustrated numerically in Section 3.

Acknowledgments

Special thanks go to the anonymous referee for some valuable suggestions, which have resulted in the improvement of this work.

REFERENCES

[1] E. Chow, Y. Saad, *Approximate inverse preconditioners via sparse-sparse iterations*, SIAM J. Sci. Comput. **19** (1998), 995–1023.

[2] G. Codevico, V.Y. Pan, M.V. Barel, *Newton-Like iteration based on a cubic polynomial for structured matrices*, Numer. Algorithms **36** (2004), 365–380.

[3] E.V. Krishnamurthy, S.K. Sen, *Numerical Algorithms – Computations in Science and Engineering*, Affiliated East-West Press, New Delhi, 1986.

[4] H.-B. Li, T.-Z. Huang, Y. Zhang, V.-P. Liu, T.-V. Gu, *Chebyshev-type methods and preconditioning techniques*, Appl. Math. Comput. **218** (2011), 260–270.

[5] W. Li, Z. Li, *A family of iterative methods for computing the approximate inverse of a square matrix and inner inverse of a non-square matrix*, Appl. Math. Comput. **215** (2010), 3433–3442.

[6] C.-S. Liu, W. Yeih, S.N. Atluri, *On solving the ill-conditioned system $Ax = b$: General-purpose conditioners obtained from the boundary-collocation solution of the Laplace equation, using Trefftz expansions with multiple length scales*, CMES Comput. Model. Eng. Sci. **44** (2009), 281–311.

[7] K. Moriya, T. Noderab, *A new scheme of computing the approximate inverse preconditioner for the reduced linear systems*, J. Comput. Appl. Math. **199** (2007), 345–352.

[8] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 2nd ed., SIAM, 2003.

[9] G. Schulz, *Iterative Berechnung der Reziproken Matrix*, Z. Angew. Math. Mech. **13** (1933), 57–59.

[10] S.K. Sen, S.S. Prabhu, *Optimal iterative schemes for computing Moore-Penrose matrix inverse*, Int. J. Systems Sci. **8** (1976), 748-753.

[11] S. Wolfram, *The Mathematica Book*, 5th ed., Wolfram Media, 2003.

Fazlollah Soleymani
fazlollah.soleymani@gmail.com

Islamic Azad University
Department of Mathematics
Zahedan Branch, Zahedan, Iran

Received: March 13, 2012.
Revised: June 1, 2012.
Accepted: June 14, 2012.