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THE FORWARDING INDICES OF GRAPHS – A SURVEY

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Abstract. A routing R of a connected graph G of order n is a collection of $n(n-1)$ simple paths connecting every ordered pair of vertices of G. The vertex-forwarding index $\xi(G,R)$ of G with respect to a routing R is defined as the maximum number of paths in R passing through any vertex of G. The vertex-forwarding index $\xi(G)$ of G is defined as the minimum $\xi(G,R)$ over all routings R of G. Similarly, the edge-forwarding index $\pi(G,R)$ of G with respect to a routing R is the maximum number of paths in R passing through any edge of G. The edge-forwarding index $\pi(G)$ of G is the minimum $\pi(G,R)$ over all routings R of G. The vertex-forwarding index or the edge-forwarding index corresponds to the maximum load of the graph. Therefore, it is important to find routings minimizing these indices and thus has received much research attention for over twenty years. This paper surveys some known results on these forwarding indices, further research problems and several conjectures, also states some difficulty and relations to other topics in graph theory.

Keywords: graph theory, vertex-forwarding index, edge-forwarding index, routing, networks.

Mathematics Subject Classification: 05C38, 68R10.

1. INTRODUCTION

In a communication network, the message delivery system must find a route along which to send each message from its source to its destination. The time required to send a message along the fixed route is approximately dominated by the message processing time at either end-vertex, intermediate vertices on the fixed route relay messages without doing any extensive processing. Metaphorically speaking, the intermediate vertices pass on the message without having to open its envelope. Thus, to a first approximation, the time required to send a message along a fixed route is independent of the length of the route. Such a simple forwarding function can be built into fast special-purpose hardware, yielding the desired high overall network performance.

For a fully connected network, this issue is trivial since every pair of processors has direct communication in such a network. However, in general, it is not the situation. The network designer must specify a set of routes for each pair (x, y) of vertices in advance, indicating a fixed route which carries the data transmitted from the message source x to the destination y. Such a choice of routes is called a *routing*.

We follow [61] for graph-theoretical terminology and notation not defined here. A graph $G = (V, E)$ always means a simple and connected graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of G, $|V|$ is the *order* of G and $|E|$ is the *size* of G. It is well known that the underlying topology of a communication network can be modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

Let G be a connected graph of order n. A *routing* R in G is a set of $n(n-1)$ fixed paths for all ordered pairs (x, y) of vertices of G. The path $R(x, y)$ specified by R carries the data transmitted from the source x to the destination y. A routing R in G is said to be *minimal*, denoted by R_m, if each of the paths specified by R is shortest; *symmetric* or *bidirectional* if for all vertices x and y, the path $R(y, x)$ is the reverse of the path $R(x, y)$ specified by R; *consistent* if for any two vertices x and y, and for each vertex z belonging to the path $R(x, y)$ specified by R, the path $R(x, y)$ is the concatenation of the paths $R(x, z)$ and $R(z, y)$.

It is possible that the fixed paths specified by a given routing R going through some vertex are too many, which means that the routing R loads the vertex too much. The load of any vertex is limited by the capacity of the vertex, for otherwise it would affect the efficiency of transmission and even result in the malfunction of the network.

It seems quite natural that a "good" routing should not load any vertex too much, in the sense that not too many paths specified by the routing should go through it. In order to measure the load of a vertex, in 1987, Chung *et al.* [16] proposed the concept of the forwarding index.

Let $\mathscr{R}(G)$ and $\mathscr{R}_m(G)$ be the sets of routings and minimum routings in a graph G, respectively. For a given $R \in \mathscr{R}(G)$ and $x \in V(G)$, the *load of x with respect to R*, denoted by $\xi_x(G, R)$, is defined as the number of paths specified by R going through x. The parameter

$$\xi(G, R) = \max\{\xi_x(G, R) : \; x \in V(G)\}$$

is called *the forwarding index of G with respect to R*, and the parameter

$$\xi(G) = \min\{\xi(G, R) : \; R \in \mathscr{R}(G)\}$$

is called *the forwarding index of G*.

Similar problems were studied for edges by Heydemann *et al.* [32] in 1989. The *load of an edge e with respect to R*, denoted by $\pi_e(G, R)$, is defined as the number of the paths specified by R which go through it. The *edge-forwarding index of G with respect to R*, denoted by $\pi(G, R)$, is the maximum number of paths specified by R going through any edge of G, i.e.,

$$\pi(G, R) = \max\{\pi_e(G, R) : \; e \in E(G)\},$$

and *the edge-forwarding index of* G is defined as

$$\pi(G) = \min\{\pi(G, R) : R \in \mathscr{R}_m(G)\}.$$

For the minimal routing R_m, let

$$\xi_m(G) = \min\{\xi(G, R_m) : R_m \in \mathscr{R}_m(G)\}$$

and

$$\pi_m(G) = \min\{\pi(G, R_m) : R_m \in \mathscr{R}_m(G)\}.$$

Clearly,

$$\xi(G) \leq \xi_m(G) \quad \text{and} \quad \pi(G) \leq \pi_m(G).$$

The equalities however do not always hold.

The original research of the forwarding indices is motivated by the problem of maximizing network capacity [16]. Maximizing network capacity clearly reduces to minimizing vertex-forwarding index or edge-forwarding index of a routing. Thus, whether or not the network capacity could be fully used will depend on the choice of a routing. Beyond a doubt, a "good" routing should have small vertex-forwarding index and edge-forwarding index. Thus it becomes very significant, theoretically and practically, to compute the vertex-forwarding index and the edge-forwarding index of a graph and has received much attention for over twenty years.

Generally, computing the forwarding index of a graph is very difficult. In this paper, we survey some known results on these forwarding indices so far, further research problems, several conjectures, difficulty, and relations to other topics in graph theory.

Since forwarding indices are first defined for a graph, that is, an undirected graph [16], most of the results in the literature are given for graphs instead of digraphs, but they can be easily extended to digraphs. Nevertheless, we give here most of the results for graphs as they appear in the literature.

2. BASIC PROBLEMS AND RESULTS

2.1. NP-COMPLETENESS

In 1987, Chung *et al.* [16] asked whether the problem of computing the forwarding index of a graph is an NP-complete problem. Following [22], we state this problem as the following decision problem.

Problem 2.1. *Forwarding Index Problem:*
Instance*: A graph* G *and an integer* k*.*
Question*:* $\xi(G) \leq k$*?*

In 1993, Saad [47] proved that Problem 2.1 is NP-complete for general graphs even if the diameter of the graph is 2. In 1994, Heydemann *et al.* [35] showed that Problem 2.1 is NP-complete for graphs with diameter at least 4 when the routings considered are restricted to be minimal, consistent and symmetric; a P-problem for graphs with diameter 2 when the routings considered are restricted to be minimal.

However, Problem 2.1 has not yet been solved for graphs with diameter 3 when the routings considered are restricted to be minimal and/or, consistent and/or symmetric.

The same problem for edges was also suggested by Heydemann *et al.* [32] in 1989.

Problem 2.2. *Edge-Forwarding Index Problem:*
Instance: *A graph G and an integer k.*
Question: $\pi(G) \le k$?

In 1994, Heydemann *et al.* [35] showed that Problem 2.2 is NP-complete for graphs with diameter at least 3 when the routings considered are restricted to be minimal, consistent and symmetric; a P-problem for graphs with diameter 2 when the routings considered are restricted to be minimal. In 2009, Kosowski [39] also showed that the problem of deciding whether $\pi(G) \le 3$ is NP-complete.

Some authors presented algorithms for computing lower or upper bounds of the vertex-forwarding index or the edge-forwarding index for general graphs or special graphs, such as Arráiz *et al.* [2], Ružička and Štefankovič [46], Yuan and Zhou [70].

2.2. BASIC BOUNDS AND RELATIONS

For a given connected graph G of order n and size ε, set

$$A(G) = \frac{1}{n} \sum_{x \in V} \left(\sum_{y \in V \setminus \{x\}} (d_G(x, y) - 1) \right),$$

and

$$B(G) = \frac{1}{\varepsilon} \sum_{(x,\, y) \in V \times V} d_G(x, y),$$

where $d_G(x, y)$ denotes the distance from the vertex x to the vertex y in G.

The following bounds of $\xi(G)$ and $\pi(G)$ were first established by Chung *et al.* [16] and Heydemann *et al.* [32], respectively.

Theorem 2.3 (Chung *et al.* [16], 1987)**.** *Let G be a connected graph of order n. Then*

$$A(G) \le \xi(G) \le (n-1)(n-2). \tag{2.1}$$

The equalities $\xi_G = \xi_m(G) = A(G)$ are true if and only if there exists a minimal routing in G which induces the same load on every vertex. The upper bound is reached for a star $K_{1,n-1}$.

Theorem 2.4 (Heydemann *et al.* [32], 1989)**.** *Let G be a connected graph of order n. Then*

$$B(G) \le \pi(G) \le \left\lfloor \frac{1}{2} n^2 \right\rfloor. \tag{2.2}$$

The equalities $\pi(G) = \pi_m(G) = B(G)$ are true if and only if there exists a minimal routing in G which induces the same load on every edge. The upper bound is reached for a graph obtained by vertex-disjoint union of two connected graphs of order $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$, respectively, with an edge between them.

Xu *et al.* [63] showed the star $K_{1,n-1}$ is a unique graph that attains the upper bound in (2.1). Noting that the upper bound given in (2.2) can be attained, we suggest the following problem.

Problem 2.5. *Give a characterization of graphs whose edge-forwarding index attains the upper bound in* (2.2)*.*

Although the two concepts of vertex-forwarding index and edge-forwarding index are similar, no interesting relationships are known between them except the following trivial inequalities.

Theorem 2.6 (Heydemann *et al.* [32], 1989)**.** *For any connected graph G of order n, maximum degree Δ, minimum degree δ,*

(a) $2\,\xi(G) + 2(n-1) \leq \Delta\pi(G)$;
(b) $\pi(G) \leq \xi(G) + 2(n-1)$;
(c) $\pi_m(G) \leq \xi_m(G) + 2(n-\delta)$.

The inequality in (a) *is also valid for minimal routings.*

Up to date, no graphs have been found for which the forwarding indices satisfy one of the above equalities. Thus, it is necessary to further investigate the relationships between $\pi(G)$ and $\xi(G)$ or between $\pi_m(G)$ and $\xi_m(G)$.

2.3. OPTIMAL GRAPHS

A graph G is said to be *vertex-optimal* if $\pi(G) = A(G)$, and *edge-optimal* if $\xi(G) = B(G)$. Note that for a minimal routing R_m of G if $\xi(G, R_m) = A(G)$, then

$$\xi(G) = \sum_{y \in V} d_G(x,y) - (n-1), \quad \forall\, x \in V. \tag{2.3}$$

It is proved that the equality (2.3) is valid for any Cayley graph.

Theorem 2.7 (Heydemann *et al.* [32], 1989)**.** *Let G be a connected Cayley graph with order n. Then*

$$\xi(G) = \xi_m(G) = \sum_{y \in V} d_G(x,y) - (n-1), \quad \forall\, x \in V. \tag{2.4}$$

Heydemann *et al.* [35] constructed a class of graphs for which the vertex-forwarding index is not given by a minimal consistent routing. Thus, they suggested the following problem.

Problem 2.8 (Heydemann *et al.* [35], 1994)**.** *For which graph or digraph G does there exist a minimal consistent routing R such that $\xi_m(G) = \xi(G,R)$ or a consistent routing R such that $\xi(G) = \xi(G,R)$?*

From Theorem 2.7, Cayley graphs are vertex-optimal. Some results and problems on the forwarding indices of vertex-transitive or Cayley graphs, an excellent survey on this subject was given by Heydemann [30] in 1997.

In 1994, Heydemann *et al.* [35] conjectured that in any vertex-transitive graph G, there exists a minimal routing R_m in which the equality (2.4) holds.

The conjecture has attracted many researchers without complete success until 2002. Shim *et al.* [49] disproved this conjecture. A simple counterexample is the generalized Petersen graph $P(10,2)$, which is vertex-transitive but not Cayley. An infinite family of counterexamples is $K_q \boxtimes P(10,2)$ for any $q \not\equiv 0\,(\mathrm{mod}\ 3)$, which is vertex-transitive but not Cayley, where K_q is a complete graph of order q and the symbol $\boxtimes$ denotes the strong product. Note that $K_q \boxtimes P(10,2)$ for $q \equiv 0\,(\mathrm{mod}\ 3)$ is a Cayley graph [49].

Solé [51] constructed an infinite family of graphs, the so-called *orbital regular graphs*, which are edge-optimal. Gauyacq [25–27] defined a class of *quasi-Cayley graphs*, a new class of vertex-transitive graphs (based on quasi-groups), which contains Cayley graphs, and are vertex-optimal. We state the results of Gauyacq and Solé as the following theorem.

Theorem 2.9 (Solé [51], 1994; Gauyacq [25], 1997)**.** *Any orbital regular graph is edge-optimal, and the quasi-Cayley graph is vertex-optimal.*

However, we do not yet know whether a quasi-Cayley graph is edge-optimal and do not know whether an orbital regular graph is vertex-optimal. Thus, we suggest investigating the following problem.

Problem 2.10. *Investigate whether a quasi-Cayley graph is edge-optimal and an orbital regular graph is vertex-optimal.*

In 1998, Fang *et al.* [18] defined a class of graphs called *Frobenius graphs* based on Frobenius groups Γ, which are connected orbital graphs and Cayley graphs. They showed that a graph is orbital-regular if and only if it is a cycle C_n, a star $K_{1,n-1}$, or a Frobenius graph, and determined the edge-forwarding indices for all Frobenius graphs with Γ of rank at most 5. In 2006, Wang *et al.* [59] determined the edge-forwarding indices of all Frobenius graphs with Γ of rank at most 50. Further discussions on the edge-forwarding indices of Frobenius graphs are referred to Wang [57, 58] and Zhou [72].

Considering $\pi(K_2 \square K_p)$ for $p \geq 3$, where $K_2 \square K_p$ is the Cartesian product of K_2 and K_p, Heydemann *et al.* [32] found that the equality (2.4) is not valid for $\pi(G)$, and proposed the following conjectures.

Conjecture 2.11 (Heydemann *et al.* [32], 1989)**.** *For any distance-transitive graph G with order n and size ε, there exists a minimal routing R_m for which*

$$\pi(G) = \pi(G, R_m) = \left\lceil \frac{n}{\varepsilon} \sum_{y \in V} d_G(x, y) \right\rceil, \quad \forall\ x \in V.$$

Conjecture 2.12 (Heydemann *et al.* [32], 1989)**.** *For any distance-transitive graph G with order n and size ε, there exists a minimal routing in which we have both*

(a) *the load of all vertices is the same, and then,*

$$\xi(G) = \xi_m(G) = \sum_{y \in V} d_G(x, y) - (n-1), \quad \forall\, x \in V,$$

(b) *the load of all the edges is almost the same (difference of at most one) and then,*

$$\pi(G) = \pi_m(G) = \left\lceil \frac{n}{\varepsilon} \sum_{y \in V} d_G(x, y) \right\rceil, \quad \forall\, x \in V.$$

So far, there are no results on the two conjectures.

Note that for a connected graph G with order n and size ε, $A(G)$ is at least $n - 1 - 2\,\varepsilon/n$. Combining this fact with Theorem 2.3, we note that, in order to reach a vertex-forwarding index as small as $(1-c)n$ for some constant $c > 0$, the graph has to contain about $\frac{c}{2}\, n^2$ edges, i.e., only dense graphs satisfy this. The following result shows that the forwarding index can be linear even if the graph has much fewer than n^2 edges.

Theorem 2.13 (Manoussakis and Tuza [41], 1996)**.** *For every positive real c, there are infinitely many graphs with n vertices and no more than $O(n^{1+c})$ edges, whose vertex- and edge-forwarding indices are at most $O(n)$ and $O(n^{1-c})$, respectively, as $n \to \infty$.*

At the same time, Manoussakis and Tuza also investigated classes of sparse graphs with an asymptotically optimal vertex-forwarding index. For every prime power q, they constructed a graph of diameter 2 with $n = q^2 + q + 1$ vertices, maximum degree $q + 1$, vertex-forwarding index $q(q+1) = n - 1$. Furthermore, they gave the following result.

Theorem 2.14 (Manoussakis and Tuza [41], 1996)**.** *For every n, there is a graph of order n with at most $\frac{1}{2}\, n^{3/2} + o(n^{3/2})$ edges and with vertex-forwarding index at most $n + o(n)$, as $n \to \infty$. Moreover, $(\frac{1}{2} - c)n^{3/2}$ edges are not sufficient for a vertex-forwarding index less than or equal to $n + o(n)$ for any constant $c > 0$.*

This result gives a partial solution of the problem raised by Chung *et al.* [16]: designing and analyzing networks with given maximum degree which have minimum or near-minimum forwarding indices (see Section 4 in this paper for details).

2.4. CARTESIAN PRODUCT GRAPHS

As an operation of graphs, the Cartesian product, denoted by "$\Box$" in this paper, can preserve many desirable properties of the factor graphs. A number of important graph-theoretic parameters, such as degree, diameter and connectivity, can be easily calculated from the factor graphs. In particular, the Cartesian product of vertex-transitive (resp. Cayley) graphs is still vertex-transitive (resp. Cayley) (see Section 2.3 in [60]), the Cartesian product of quasi-Cayley graphs is still a quasi-Cayley graph. Thus, studying the relations between forwarding indices of a Cartesian product and forwarding indices of factor graphs is of interest. Heydemann *et al.* [32] obtained the following results.

Theorem 2.15 (Heydemann *et al.* [32], 1989)**.** *For each $i = 1, 2$, let G_i be a connected graph of order n_i, $\xi_i = \xi(G_i)$ and $\pi_i = \pi(G_i)$. Then:*

(a) $\xi(G_1 \Box G_2) \leq n_1\ \xi_2 + n_2\ \xi_1 + (n_1 - 1)(n_2 - 1)$;
(b) $\pi(G_1 \Box G_2) \leq \max\{n_1\pi_2,\ n_2\pi_1\}$.

The inequalities are also valid for minimal routings. Moreover, the equality in (a) *holds if both G_1 and G_2 are Cayley digraphs.*

Xu *et al.* [62] considered the Cartesian product of k optimal graphs and obtained the following results.

Theorem 2.16 (Xu *et al.* [62], 2006)**.** *For each $i = 1, 2, \ldots, k$, let G_i be a connected graph of order n_i, $\xi_i = \xi(G_i)$ and $\pi_i = \pi(G_i)$, and let $G = G_1 \Box G_2 \Box \ldots \Box G_k$. For each $i = 1, 2, \ldots, k$,*

(a) *if G_i is vertex-optimal then G is vertex-optimal and*

$$\xi(G) = \sum_{i=1}^{k} n_1 n_2 \ldots n_{i-1}(\xi_i - 1)n_{i+1} \ldots n_k + (k-1)n_1 n_2 \ldots n_k + 1;$$

(b) *if G_i is edge-optimal then G is edge-optimal and*

$$\pi(G) = \max_{1 \leq i \leq k} \{n_1 n_2 \ldots n_{i-1}\pi_i n_{i+1} \ldots n_k\}.$$

As applications of Theorem 2.16, vertex-forwarding indices and edge-forwarding indices of many well-known graphs can be determined, see Section 7 in this paper for details.

By Theorem 2.9 and Theorem 2.16, the Cartesian product of quasi-Cayley graphs is vertex-optimal and the Cartesian product of orbital regular graphs is edge-optimal.

Qian and Zhang [45] also obtained some results on forwarding indices of Cartesian products. As a generalization of the forwarding indices, they introduced a new concept, called *t-forwarding indices* by considering exactly t paths instead of only one path between two vertices in a routing. The interested reader should refer to their original paper.

3. CONNECTIVITY CONSTRAINT

Note that a complete graph K_n has $\xi(K_n) = 0$ and a star $K_{1,\, n-1}$ has $\xi(K_{1,\, n-1}) = (n-1)(n-2)$, which reach the lower bound and the upper bound in Theorem 2.3, respectively. It seems true that the more high the connectivity is, the more small the vertex-forwarding index is. However, as the reader can verify through results stated in Section 7, the connectivity does not play an important role for the forwarding indices. Nevertheless, several authors paid their attention to the forwarding indices of k-connected or k-edge-connected graphs. The purpose of this section is to survey some upper bounds and research problems on the forwarding indices of k-connected or k-edge-connected graphs.

3.1. κ-CONNECTED GRAPHS

We start with some results on 2-connected graphs, obtained by Heydemann *et al.*

Theorem 3.1. *If G is a 2-connected graph of order n, then:*

(a) $\xi(G) \leq \frac{1}{2}(n-2)(n-3)$, *this bound is best possible in view of* $K_{2,n-2}$ (Heydemann *et al.* [32], 1989)*;*
(b) $\xi_m(G) \leq n^2 - 7n + 12$ *for* $n \geq 6$ *and diameter* 2*, this bound is the best possible since it is reached for a wheel of order n minus one edge with both ends of degree* 3 (Heydemann *et al.* [32], 1989)*;*
(c) $\xi_m(G) \leq n^2 - 7n + 12$ *for* $n \geq 7$*, this bound is the best possible since it is reached for a fan of order n (that is, the join of a vertex and a path of order $n-1$)* (Heydemann *et al.* [33], 1992)*;*
(d) $\pi(G) \leq \lfloor \frac{1}{4} n^2 \rfloor$ *and this bound is the best possible in view of the cycle* C_n (Heydemann *et al.* [33], 1992).

For k-connected graphs with $k \geq 3$, we can state the following result.

Theorem 3.2 (Heydemann *et al.* [33], 1992)**.** $\xi(G) \leq n^2 - (2k+1)n + 2k$ *for any k-connected graph G of order n with $k \geq 3$ and $n \geq 8k - 10$.*

Heydemann *et al.* gave a graph to show that this upper bound is the best possible for all odd k. At the same time, they proposed the following extremum problem.

Problem 3.3 (Heydemann *et al.* [33], 1992)**.** *Find the best upper bound $f(n,k)$, $g(n,k)$, $h(n,k)$ and $s(n,k)$ such that for any k-connected graph G of order n with $k \geq 3$,*

$$\xi(G) \leq f(n,k),\ \xi_m(G) \leq g(n,k),\ \pi(G) \leq h(n,k) \text{ and } \pi_m(G) \leq s(n,k)$$

for n large enough compared to k.

Quickly, the upper bounds are established.

Theorem 3.4 (Fernandez de la Vega and Manoussakis [20], 1992)**.** *For any integer $k \geq 1$,*

(a) $f(n,k) \leq (n-1)\lceil (n-k-1)/k \rceil$;
(b) $g(n,k) \leq \frac{1}{2}n^2 - (k-1)n + \frac{3}{8}(k-1)^2$ *if n is substantially larger than k;*
(c) $h(n,k) \leq n\lceil (n-k-1)/k \rceil$.

Considering the maximum degree constraint, Zhou *et al.* [71] improved the upper bounds of $f(n,k)$ and $h(n,k)$ in Theorem 3.4 as follows.

Theorem 3.5 (Zhou *et al.* [71], 2008)**.** *If G is a k-connected graph of order n with the maximum degree Δ, then:*

(a) $\xi(G) \leq (n-1)\lceil (n-k-1)/k \rceil - (n - \Delta - 1)$ *and*
(b) $\pi(G) \leq n\lceil (n-k-1)/k \rceil - (n - \Delta)$.

Conjecture 3.6 (Fernandez de la Vega and Manoussakis [20], 1992). *For any positive integer k,*

(a) $f(n,k) \le \lceil \frac{1}{k}(n-k)(n-k-1) \rceil$ *for $n \ge 2k \ge 2$, which would be the best possible in view of the complete bipartite graph $K_{k,n-k}$;*
(b) *there exists a function $q(k)$ such that if $n \ge q(k)$, then $g(n,k) \le \frac{1}{2}n^2 - (k-1)n - \frac{3}{2}k^2 + k + \frac{7}{2}$;*
(c) $h(n,k) \le \lceil \frac{n^2}{2k} \rceil$ *for $n \ge 2k \ge 2$, which would be the best possible in view of the graph obtained from two complete graphs K_m plus a matching $e_1, e_2, \ldots, e_k$ between them, $m \ge k$.*

It can be easily verified that the conjecture (a) and (c) are true for $k = 1$ and $k = 2$. Zhou *et al.* [71] proved that Conjecture 3.6 (a) is true for a 3-regular graph, that is, the following theorem holds.

Theorem 3.7 (Zhou *et al.* [71], 2008). *If G is a 3-regular and 3-connected graph of order $n \ge 4$, then $\xi(G) \le \lceil (n-3)(n-4)/3 \rceil$.*

3.2. λ-EDGE-CONNECTED GRAPHS

Theorem 3.8. *If G is a 2-edge-connected graph of order n, then:*

(a) $\pi_m(G) \le \lfloor \frac{1}{2}n^2 - n + \frac{1}{2} \rfloor$ (Heydemann *et al.* [32,33], 1989);
(b) $\pi(G) \le \lfloor \frac{1}{4}n^2 \rfloor$ (Cai [11], 1990).

Heydemann *et al.* [32] conjectured that for any λ-edge-connected graph G of order n, $\pi(G) \le \lfloor \frac{1}{2}n^2 - (\lambda-1)n + \frac{1}{2}(\lambda-1)^2 \rfloor$. Later, Heydemann *et al.* [33] gave a counterexample to disprove this conjecture and proposed the following conjecture.

Conjecture 3.9 (Heydemann *et al.* [33], 1992). *For any λ-edge-connected graph G of order n with $\lambda \ge 3$ and $n \ge 3\lambda + 3$,*

$$\pi_m(G) = \max\left\{ \left\lceil \frac{n^2}{2} \right\rceil - n - 2(\lambda-1)^2,\ \left\lfloor \frac{n^2}{2} \right\rfloor - 2n + 5\lambda - \frac{3}{2}(\lambda^2+1) \right\}.$$

The same problem as the ones in Problem 3.3 can be considered for λ-edge-connected graphs.

Problem 3.10. *Find the best upper bound $f'(n,\lambda), g'(n,\lambda), h'(n,\lambda)$ and $s'(n,\lambda)$ such that for any λ-connected graph G of order n with $\lambda \ge 2$,*

$$\xi(G) \le f'(n,\lambda),\ \xi_m(G) \le g'(n,\lambda),\ \pi(G) \le h'(n,\lambda) \text{ and } \pi_m(G) \le s'(n,\lambda)$$

for n large enough compared to λ.

The following theorem is the only known result so far on this problem.

Theorem 3.11 (Fernandez de la Vega and Manoussakis [20], 1992). *For any integer $\lambda \ge 3$,*

$$g'(n,\lambda) = \left\lceil \frac{n^2}{2} \right\rceil - n - 2(\lambda-1)^2 \text{ for } n \ge \max\left\{3\lambda+3,\ \frac{1}{2}(\lambda+1)^2\right\}.$$

3.3. STRONGLY CONNECTED DIGRAPHS

It is clear that the concept of the forwarding indices can be similarly defined for digraphs. Many general results, such as Theorem 2.3 and Theorem 2.4 are valid for digraphs. Manoussakis and Tuza [42] consider the forwarding index of strongly k-connected digraphs and obtained the following result similar to Theorem 2.4.

Theorem 3.12 (Manoussakis and Tuza [42], 1996)**.** *Let D be a strongly connected digraph of order n. Then:*

(a) $B(D) \leq \pi(D) \leq \pi_m(D) \leq (n-1)(n-2)+1$, *and*
(b) *The equalities $\pi(D) = \pi_m(D) = B(D)$ are true if and only if there exists a minimal routing in D which induces the same load on every edge.*

In addition to the validity of Theorem 3.2 for digraphs, the following results are also obtained.

Theorem 3.13 (Manoussakis and Tuza [42], 1996)**.** *Let D be a k-connected digraph of order $n \geq 3$, and $k \geq 1$. Then:*

(a) $\pi(D) \leq (n-1)\lceil \frac{1}{k}(n-k-1)\rceil + 1$;
(b) $\xi_m(D) \leq n^2 - (2k+1)n + 2k$ *for* $n \geq 2k+1$;
(c) $\pi_m(D) \leq n^2 - (3K+2)n + 4k + 3$ *for* $n \geq 4k-1$.

4. DEGREE CONSTRAINT

Although Saad [47] showed that for general graphs determining the forwarding index problem is NP-complete, yet many authors are interested in the forwarding indices of a graph. Specially, it is still of interest to determine the exact value of the forwarding index with some graph-theoretical parameters. For example, Chung *et al.* [16], Bouabdallah and Sotteau [10] proposed to determine the minimum forwarding indices of (n, Δ)-graphs that has order n and maximum degree Δ. Given Δ and n, let

$$\xi_{\Delta,n} = \min\{\xi(G) : \ |V(G)| = n, \Delta(G) = \Delta\},$$
$$\pi_{\Delta,n} = \min\{\pi(G) : \ |V(G)| = n, \Delta(G) = \Delta\}.$$

4.1. PROBLEMS AND TRIVIAL CASES

Problem 4.1 (Chung *et al.* [16], 1987)**.** *Given $\Delta \geq 2$ and $n \geq 4$, determine $\xi_{\Delta,n}$, and exhibit an (n, Δ)-graph G and a routing R of G for which $\xi(G, R) = \xi_{\Delta,n}$.*

Problem 4.2 (Bouabdallah and Sotteau [10], 1993)**.** *Given $\Delta \geq 2$ and $n \geq 4$, determine $\pi_{\Delta,n}$, and exhibit an (n, Δ)-graph G and a routing R of G for which $\pi(G, R) = \pi_{\Delta,n}$.*

For $\Delta \geq n-1$, G is a complete graph. In this case any routing R can be composed only of single-edge paths so that the minimum $\xi = 0$ and $\pi = 2$ is achieved, that is, $\xi_{\Delta,n} = \xi(G, R) = 0$ and $\pi(G, R) = \pi_{\Delta,n} = 2$ for $\Delta \geq n-1$.

For $\Delta = 2$ the only connected graph fully utilizing the degree constraint is easily seen to be a cycle. Because of the simplicity of cycles, the vertex-forwarding index problem can be solved completely for $\Delta = 2$.

Theorem 4.3. *For any $(n, 2)$-graph with $n \geq 3$,*

(a) $\xi_{2,n} = \xi(C_n, R_m) = \left\lfloor \frac{1}{4}(n-1)^2 \right\rfloor$ (Chung *et al.* [16], 1987)*;*
(b) $\pi_{2,n} = \pi(C_n, R_m) = \left\lfloor \frac{1}{4}n^2 \right\rfloor$ (Bouabdallah and Sotteau [10], 1993).

For $3 \leq \Delta < n-1$, an approximation lower bound of $\xi_{\Delta,n}$ is established as follows.

Theorem 4.4 (Chung *et al.* [16], 1987)**.** *For any given $\Delta \geq 3$ and $n \geq 4$,*

$$\xi_{\Delta,n} \geq \lfloor 1 + o(1) \rfloor n \log_{\Delta-1} n.$$

Solé [52] gave a lower bound for $\pi_{\Delta,n}$, in terms of the genus g,

$$\pi_{\Delta,n} \geq \frac{n^{\frac{3}{2}}\sqrt{2} - 6n(g+2)}{6\Delta(g+2)} \quad \text{for } n > 18(g+2)^2.$$

Vrt'o [56] improved this bound as follows.

Theorem 4.5 (Vrt'o [56], 1995)**.** *For any given genus g,*

$$\pi_{\Delta,n} \geq \frac{n^{\frac{3}{2}}}{15\sqrt{3\Delta(g+1)}}.$$

Vrt'o gave examples to show that this bound is the best possible up to a constant factor for all n and Δ. For example, if $G = C_p \Box C_p$, then $n = p^2, \Delta = 4, g = 1$ and $\pi(G) = p\lfloor \frac{p^2}{4} \rfloor \leq \frac{p^3}{4}$; if $G = K_{p,p}$, then $n = 2p$, $\Delta = p$, $g = \lceil \frac{(p-2)^2}{4} \rceil$ and $\pi(G) < 6$.

4.2. RESULTS ON $\xi_{\Delta,\, N}$ WITH $\Delta \geq 3$

Problem 4.1 was solved for $n \leq 15$ or any n and Δ with $\frac{1}{3}(n+4) \leq \Delta \leq n-1$ by Heydemann *et al.* [31].

Theorem 4.6 (Heydemann *et al.* [31], 1988)**.** *For any (n, Δ)-graph,*

(a) *if n is even or n odd and Δ even, $\xi_{\Delta,\, n} = n - 1 - \Delta$ for $\Delta \geq \frac{1}{3}(n+1)$ or for $n = 12$ or 13 and $\Delta = 4$;*
(b) *if n and Δ are odd, $\xi_{\Delta,\, n} = n - \Delta$ for $\Delta \geq \frac{1}{3}(n+4)$ or for $n = 13$ and $\Delta = 5$.*

Problem 4.1 has not been completely solved for $\Delta < \frac{1}{3}(n+4)$.

In general, determining the exact value of $\xi_{\Delta,\, n}$ is quite difficult for any Δ with $\Delta \geq 3$. For some special value of Δ, Heydemann *et al.* [31] obtained some results.

Theorem 4.7 (Heydemann *et al.* [31], 1988)**.** *For an (n, Δ)-graph,*

(a) $\xi_{n-2p-1,\, n} = 2p$ *for any n and p with $n \geq 3p+2$;*
(b) $\xi_{2p+1,\, n} = n - 2p - 1$ *for any odd n with $2p+1 \leq n \leq 6p-1$;*
(c) $\xi_{2p,\, n} = n - 2p - 1$ *for any n and p with $2p+1 \leq n \leq 6p-1$ and $p \geq 3$;*

(d) $\xi_{\Delta,n} \geq n - 1 - \Delta$;
(e) $\xi_{\Delta,n} = n - 1 - \Delta \Rightarrow$ *every* (n,Δ)*-graph* G *with* $\xi(G) = \xi_{\Delta,n}$ *is* Δ*-regular and diameter* 2;
(f) $\xi_{\Delta,n} \geq n - \Delta$ *if* n *and* Δ *are odd.*

An asymptotic result on $\xi_{\Delta,n}$ has been given by Chung *et al.* [16].

Theorem 4.8 (Chung *et al.* [16], 1987)**.** *For any given* $\Delta \geq 3$,

$$[1 + o(1)]n \log_{\Delta-1} n \leq \xi_{\Delta,n} \leq \left[3 + O\left(\frac{1}{\log \Delta}\right)\right] n \log_{\Delta} n,$$

where the upper bound holds for $\Delta \geq 6$.

4.3. RESULTS ON $\pi_{\Delta,N}$ WITH $\Delta \geq 3$

Similar to Theorem 4.7, the result on $\pi_{\Delta,n}$ can be stated as follows.

Theorem 4.9 (Bouabdallah and Sotteau [10], 1993)**.** *For any* n *and* $\Delta \geq 3$,

(a) $\pi_{\Delta,n} \geq \lceil \frac{4(n-1)}{\Delta} \rceil - 2$;
(b) $\pi_{\Delta,n} \geq \lceil \frac{4(n-1)}{\Delta} \rceil - 2 \Rightarrow$ *every* (n,Δ)*-graph* G *with* $\pi(G) = \pi_{\Delta,n}$ *is* Δ*-regular and diameter* 2 *and* G *has a minimal routing for which the load of all edges is the same;*
(c) $\pi_{\Delta,n} \geq \lceil \frac{4n-2)}{\Delta} \rceil - 2$ *if* n *and* Δ *are odd;*
(d) $\pi_{\Delta,n} \leq \pi_{\Delta',n}$ *for any* n *and* $\Delta' \leq n-1$ *with* $\Delta' < \Delta$.

Problem 4.2 was solved for $n \leq 15$ by Bouabdallah and Sotteau [10], who also obtained $\pi_{n-2,n} = 3$ for any $n \geq 6$, $n \neq 7$ and $\pi_{n-2,n} = 4$ for any $n = 4, 5, 7$. Xu *et al.* [65] determined $\pi_{n-2p-1,n} = 8$ if $3p + \lceil \frac{1}{3}p \rceil + 1 \leq n \leq 3p + \lceil \frac{3}{5}p \rceil$ and ≥ 2. The authors in [10] and [65] determined $\pi_{n-2p-1,n}$ for $n \geq 4p$ and $p \geq 1$ except a little gap.

Theorem 4.10 (Bouabdallah and Sotteau [10], 1993; Xu *et al.* [65], 2004)**.** *For any* $p \geq 1$,

$$\pi_{n-2p-1,n} = \begin{cases} 3, & \textit{if} \quad n \geq 10p + 1, \\ 4, & \textit{if} \quad 6p + 1 \leq n < 10p + 1, \\ 5, & \textit{if} \quad 4p + 2\lceil \frac{1}{3}p \rceil + 1 \leq n \leq 6p, \\ 6, & \textit{if} \quad 4p + 1 \leq n \leq 4p + \lceil \frac{2}{3}p \rceil. \end{cases}$$

Note the value of $\pi_{n-2p-1,n}$ has not been determined for $4p + \lceil \frac{2}{3}p \rceil + 1 \leq n \leq 4p + 2\lceil \frac{1}{3}p \rceil$. However, these two numbers are different only when $p = 3k + 1$. Thus, we proposed the following conjecture.

Conjecture 4.11. *For any* $p \geq 1$, $\pi_{n-2p-1,n} = 5$ *if* $4p + \lceil \frac{2}{3}p \rceil + 1 \leq n \leq 4p + 2\lceil \frac{1}{3}p \rceil$.

Theorem 4.12 (Xu *et al.* [67], 2005)**.** *For any* $p \geq 1$*,*

$$\pi_{n-2p,n} = \begin{cases} 3, & \text{if} \quad n \geq 10p-2 \text{ or } n = 10p-4, \\ 4, & \text{if} \quad 6p+1 \leq n < 10p-4 \text{ or } n = 10p-3, \\ 6, & \text{if} \quad 4p+1 \leq n \leq 4p + \lceil \frac{1}{3}(2p-1) \rceil - 2. \end{cases}$$

An asymptotic result on $\pi_{\Delta,\,n}$ has been given by Heydemann *et al.* [32].

Theorem 4.13 (Heydemann *et al.* [32], 1989)**.** *For any given* $\Delta \geq 3$*,*

$$\left[\frac{2}{\Delta} + o(1)\right] \; n \log_{\Delta-1} n \leq \pi_{\Delta,\,n} \leq 24 \frac{\log_2(\Delta-1)}{\Delta} \, n \log_{\Delta-1} n,$$

where the upper bound holds for $\Delta \geq 6$*.*

4.4. GENERAL RESULTS SUBJECT TO DEGREE AND DIAMETER

Theorem 4.14 (Xu *et al.* [63], 2007)**.** *For any connected graph* G *of order* n *and maximum degree* Δ*,*

$$\xi(G) \leq (n-1)(n-2) - \left(2n - 2 - \Delta \left\lfloor 1 + \frac{n-1}{\Delta} \right\rfloor \right) \left\lfloor \frac{n-1}{\Delta} \right\rfloor .$$

Considering a special case of $\Delta = n-1$ in Theorem 4.14, we obtain the upper bound in (2.1) immediately.

Theorem 4.15 (Heydemann *et al.* [32], 1989)**.** *For a graph* G *of order* n*, maximum degree* Δ *and diameter* d*,*

(a) $\xi(G) \leq \xi_m(G) \leq (n-1)(n-2) - 2(\varepsilon(G) - \Delta)$*;*
(b) $\xi(G) \leq \xi_m(G) \leq n^2 - 3n - \lfloor \frac{1}{2}d \rfloor^2 - \lceil \frac{1}{2}d \rceil^2 + d + 2.$

Theorem 4.16 (Heydemann *et al.* [32], 1989)**.** *If* G *is a graph of order* n *and diameter* 2 *with no vertex of degree one, then* $\pi_m(G) \leq 2n - 4$*.*

This upper is the best possible for π_m by considering the graph union of a complete K_{n-2} with a path (x, u, v, y) joining two different vertices x and y of K_{n-2}. The condition "no vertex of degree one" is necessary. Consider a graph G contained from a complete graph K_{n-1} plus one vertex x joined to one vertex y of K_{n-1}. It is clear that $\pi(G) = \pi_m(G) = 2(n-1)$.

Some upper bounds on the forwarding indies for digraphs subject to minimum degree constraints are obtained.

Theorem 4.17 (Manoussakis and Tuza [42], 1996)**.** *Let* D *be a strongly connected digraph of order* n *and minimum degree* δ*. Then:*

(a) $\xi_m(D) \leq n^2 - (\delta+2)n + \delta + 1$*;*
(b) $\pi_m(D) \leq \max\{n^2 - 3n\delta + 2\delta^2 + \delta,\; n^2 - (2\delta+3)n + \delta^2 + 4\delta + 3\}$ *if* n *is sufficient large compared to* δ*.*

Considering the minimum degree δ rather then the maximum degree Δ, we can propose an analogy of $\xi_{\Delta,n}$ and $\pi_{\Delta,n}$ as follows. Given Δ and n, let

$$\begin{aligned}\xi_{\delta,n} &= \min\{\xi(G):\ |V(G)| = n, \delta(G) = \delta\},\\ \pi_{\delta,n} &= \min\{\pi(G):\ |V(G)| = n, \delta(G) = \delta\}.\end{aligned}$$

However, the problem determining $\xi_{\delta,n}$ and $\pi_{\delta,n}$ is simple.

Theorem 4.18 (Xu *et al.* [63], 2007)**.** *For any n and δ with $n > \delta \geq 1$,*

$$\xi_{\delta,n} = \left\lceil \frac{2(n-1-\delta)}{\delta} \right\rceil \quad \text{and } \pi_{\delta,n} = \left\lceil \frac{2(n-1)}{\delta} \right\rceil.$$

5. DIFFICULTY AND SOME KNOWN METHODS

5.1. DIFFICULTY

As we have seen in Subsection 2.1, the problem of computing forwarding indices for general graphs is NP-complete. Also, for a given graph G, determining its forwarding indices $\xi(G)$ and $\pi(G)$ is also very difficult.

The first difficulty is designing a routing R such that $\xi_x(G,R)$ for any $x \in V(G)$ or $\pi_x(G,R)$ for any $e \in E(G)$ can be conveniently computed. An ideal routing is a minimal routing since it can be found by the current algorithms for finding shortest paths. However, in general, it is not always the case that forwarding indices of a graph can be obtained by a minimal routing.

For example, consider the wheel W_7. The hub x, other vertices $0, 1, \ldots, 5$.

A minimal and bidirectional routing R_m is defined as follows:

$$\begin{cases} R_m(i, i+2) = R(i+2, i) = (i, i+1, i+2) (\text{mod } 6), & i = 0, 1, \ldots, 5,\\ R_m(i, i+3) = R(i+3, i) = (i, x, i+3) (\text{mod } 6), & i = 0, 1, 2,\\ \text{direct edge}, & \text{otherwise.}\end{cases}$$

Then,

$$\xi_x(W_7, R_m) = 6,\ \xi_i(W_7, R_m) = 2,\ i = 0, 1, \ldots, 5.$$

Thus, we have $\xi(W_7, R_m) = 6$.

However, if we define a routing R that is the same as the minimal routing R_m except for $R(2,5) = (2,1,0,5)$, $R(5,2) = (5,4,3,2)$. Then the routing R is not minimal.

$$\xi_x(W_7, R) = 4,\ \tau_2(W_7, R) = \tau_5(W_7, R) = 2, \quad \xi_i(W_7, R) = 3,\ i = 0, 1, 3, 4.$$

Thus, $\xi(W_7, R) = 4 < 6 = \xi(W_7, R_m)$.

The second difficulty is that the forwarding indices are not attained by a bidirectional routing, in general. For example, for the hypercube Q_n $(n \geq 2)$, $\xi(Q_n) = 2^{n-1}(n-2)+1$. Since $2^{n-1}(n-2)+1$ is odd, $\xi(Q_n)$ can not be attained by a bidirectional routing.

5.2. SOME KNOWN METHODS

Given the knowledge of the authors, one of the actual methods of determining the forwarding index is to compute the sum of distance over all pairs of vertices according to (2.4) for some Cayley graphs. In fact, the forwarding indices of many Cayley graphs are determined by using (2.4), for example, the n-cube Q_n [32], the folded cube [36], the augmented cube [66] and so on, list in Section 7.

Although Cayley graphs, one class of vertex-transitive graphs, are of high symmetry, it is not always easy to compute the distance from a fixed vertex to all other vertices for some Cayley graphs. For example, the n-dimensional cube-connected cycle CCC_n, constructed from Q_n by replacing each of its vertices with a cycle $C_n = (0, 1, \ldots, n-1)$ of length n, is a Cayley graph proved by Carlsson *et al.* [12]. Until now, one has not yet determined exactly its sum of distance over all pairs of vertices, and so only can give its forwarding indices asymptotically (see, Shahrokhi and Székely [50], and Yan *et al.* [69]).

Unfortunately, for the edge-forwarding index, there is no analogy of (2.4). But the lower bound of $\pi(G)$ given in (2.2) is useful. One may design a routing R such that $\pi(G,R)$ attains this lower bound. For example, the edge-forwarding indices of the folded cube [36] and the augmented cube [66] are determined by this method.

Making use of results on the Cartesian product is one of methods determining forwarding indices. Using Theorem 2.16, Xu *et al.* [62] determined the vertex-forwarding indices and the edge-forwarding indices for the generalized hypercube $Q(d_1, d_2, \ldots, d_n)$, the undirected toroidal mesh $C(d_1, d_2, \ldots, d_n)$, the directed toroidal mesh $\overrightarrow{C}(d_1, d_2, \ldots, d_n)$, all of which can be regarded as the Cartesian products.

To the knowledge of the authors, until now one has not yet found an approximation algorithm with a good performance ratio for finding routings of general graphs such that their forwarding indices are as small as possible. However, Fernandez de le Vega and Manoussakis [21] showed that the problem of determining the value of the forwarding index (respectively, the forwarding index of minimal routings) is an instance of the multicommodity flow problem (respectively, flow with multipliers). Since many very good heuristics or approximation algorithms are known for these flow problems [3, 38, 40], it follows from these results that all of these algorithms can be used for calculating the forwarding index. In fact, using this method, approximation values of edge-forwarding indices for some well-known networks are computed, such as star graphs by Gauyacq [26], cube-connected cycles and butterfly networks by Shahrokhi and Székely [50] (see Section 7 for details).

6. RELATIONS TO OTHER TOPICS

6.1. TO LAPLACIAN EIGENVALUES

The Laplacian eigenvalues of a graph are defined to be the eigenvalues of its Laplacian matrix. The edge-forwarding index is closely related to Laplacian eigenvalues, as shown first by the following result of Mohar [44].

Theorem 6.1 (Mohar [44], 1989)**.** *If the smallest nonzero Laplacian eigenvalue of a graph G with maximum degree Δ is λ, then the edge-forwarding index of G satisfies the inequality*

$$\pi(G) \geq \sqrt{\frac{n}{2\Delta - \lambda}}.$$

As we know that the Laplacian eigenvalues of the product graph $G_1 \Box G_2$ are equal to all the possible sums of the eigenvalues of G_1 and G_2, the smallest nonzero Laplacian eigenvalue of $G_1 \Box G_2$ is $\min\{\lambda_1, \lambda_2\}$, where λ_i is the smallest nonzero Laplacian eigenvalue of G_i for each $i = 1, 2$. From Theorem 6.1, we immediately obtain the following corollary, observed first by Qian and Zhang [45].

Corollary 6.2 (Qian and Zhang [45], 2004)**.** *If the smallest nonzero Laplacian eigenvalues of G_i of order n_i is λ_i and the maximum degree of G_i is Δ_i for each $i = 1, 2$, then*

$$\pi(G_1 \Box G_2) \geq \sqrt{\frac{n_1 n_2}{2(\Delta_1 + \Delta_2) - \min\{\lambda_1, \lambda_2\}}}.$$

Teranishi [53] established another lower bound on the edge-forwarding index of a graph in terms of its Laplacian eigenvalues.

Theorem 6.3 (Teranishi [53], 2002)**.** *If G has Laplacian eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$, then the edge-forwarding index of G satisfies the inequality*

$$\pi(G) \geq \frac{2n}{n-1}\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \ldots + \frac{1}{\lambda_n}\right),$$

where n is the order of G.

6.2. TO EXPANDING FACTORS

The *edge-expanding factor*, an important measurement of the edge-connectivity, of a graph G of order n is defined as

$$\beta(G) = \min\left\{\frac{d(X)}{|X||\bar{X}|} : X \subset V, 1 \leq |X| \leq n-1\right\},$$

where $\bar{X}$ denotes the complement of X in V, and $d(X)$ denotes the number of edges between X and $\bar{X}$ in G. Similarly, the *vertex-expanding factor* γ is defined as

$$\gamma(G) = \min\left\{\frac{|N(X)|}{|X||X^+|} : X \subset V, 1 \leq |X| \leq n-1, |X^+| \geq 1\right\},$$

where $N(X)$ is the set neighbors of X not in X, and $|X^+|$ denotes the complement of $X \cup N(X)$ in V. A path is called *transversal* if the one of its endpoints is in X and the other in X^+.

Solé [52] first investigated the relations between expanding factors and forwarding indices, obtained the following result.

Theorem 6.4 (Solé [52], 1995)**.** *Let G be a connected graph. Then:*

(a) $\xi(G)\gamma(G) \geq 2$ *with equality only if there is an optimal uniform routing which loads all the vertices of some vertex cut with load ξ, by using transversal paths only;*

(b) $\pi(G)\beta(G) \geq 2$ *with equality only if there is an optimal routing ρ which loads uniformly the edges of some cut with load π.*

Qian and Zhang (2004) revealed some further properties concerning the forwarding indices of product graphs. In particular, they showed the following result.

Theorem 6.5 (Qian and Zhang [45], 2004)**.** *Let G_i be a connected graph of order n_i with $\pi_i\beta_i = 2$, there $\beta_i = \beta(G_i)$ and $\pi_i = \pi(G_i)$ for $i = 1, 2$. Then*

$$\pi(G_1 \Box G_2) = \max\{\pi_1 n_2, \pi_2 n_1\}.$$

6.3. TO OPTICAL INDICES

Many aspects of wavelength routing in optical networks can be modelled in terms of graph problems (see Bermond *et al.* [5]).

For a given routing R, a pair of paths $\{P_1, P_2\}$ in R is called *conflicting* if there exists an edge e such that it is contained in both P_1 and P_2. The *conflict graph* G_R of a graph G with a given routing R is defined as an undirected graph with vertex-set R and edges corresponding to pairs of conflicting paths in R. The *wavelength count* $w(G, R)$ is defined as the smallest number of wavelengths that must be assigned to the paths of R so that no two paths that share an edge receiving the same wavelength, which is equal to the chromatic number of the conflict graph G_R, and the *optical index* is defined as $w(G) = \min\{w(G, R) : \ \forall\ R\}$.

Some results concerning optical indices and their relation to forwarding indices are collected in the survey papers [5, 9, 24]. An elementary relation is as follows.

Theorem 6.6. *$\pi(G) \leq w(G)$ for any (di)graph G.*

There are some (di)graphs with equality. In particular, the equality is known to hold for all trees [23], cycles [7] and trees of cycles [6], hypercubes [7], some families of recursive circulant graphs [1], Cartesian sums of complete graphs [4, 55], tori (Cartesian products of cycles of the same length) of even [4] and odd [48] order, as well as grids (Cartesian products of paths of the same length) of even order [4].

However, the equality is not always true even for a class of trees known as subdivided stars by Bermond *et al.* [5], and for a class of digraphs by Kosowski [39], who proved that the problem of deciding whether $w(G) \leq 3$ is NP-complete.

Problem 6.7. *Study the difference between $w(G)$ and $\pi(G)$.*

6.4. FAULT-TOLERANT FORWARDING AND OPTICAL INDICES

Considering that edges and/or vertices may fail in a network, Maňuch and Stacho [43] proposed concepts: fault-tolerant forwarding and optical indices.

Suppose that f is the number of faults that are tolerated in the optical network G. The *f-fault-tolerant routing* of G is defined as

$$R_f(G) = \{P_i(x,y) : \ x \neq y, 0 \leq i \leq f\},$$

where, for each pair of distinct vertices $x, y \in V$, the paths $P_0(x,y), P_1(x,y), \ldots,$ $P_f(x,y)$ are pairwise-internally vertex-disjoint or edge-disjoint. Let $\mathscr{R}_f(G)$ be the set of f-fault-tolerant routings in a graph G.

The parameter $\pi(R_f(G))$ denotes the maximum number of times an edge of G appears in paths of $R_f(G)$. The *f-fault-tolerant edge-forwarding index* of G is defined as

$$\pi_f(G) = \min\{\pi(R_f(G)) : \ R_f(G) \in \mathscr{R}_f(G)\}.$$

Clearly, $\pi_0(G) = \pi(G)$. Thus, $\pi_f(G)$ is a generalization of $\pi(G)$. Gupta *et al.* [28, 29] determined $\pi_f(G)$ for the complete symmetric digraph and the complete symmetric bipartite digraph.

Theorem 6.8 (Gupta *et al.* [28], 2004 or [29], 2006)**.** *Let K_n^* and $K_{n,n}^*$ be a complete symmetric digraph and a complete bipartite symmetric digraph, respectively. Then*

$$\pi_f(K_n^*) = 2f + 1 \ for\ all\ n\ and\ f \leq n-2,$$

and

$$\pi_f(K_{n,n}^*) = \begin{cases} 5f+3 & if\ f < \frac{n}{2} - 1, \\ 5f+2 & if\ \frac{n}{2} - 1 \leq f < n-1, \\ 5f+1 & if\ f = n-1. \end{cases}$$

For any $i = 0, 1, \ldots, f$, *level i* of the routing R_f is the set of paths $P_i(x,y) \in R_f(G)$, for all $x \neq y$. In 2009, Chen and Qian [14] constructed a leveled f-fault tolerant routing for the folded hypercube, and so determined its f-fault-tolerant edge-forwarding index.

In general, determining $\pi_f(G)$ is quite difficult. Use $d_f(G; x, y)$ to denote the minimum sum of lengths of f internally vertex-disjoint paths connecting x and y in G. Gupta *et al.* [29] established a lower bound of $\pi_f(G)$ in terms of $d_f(G; x, y)$.

Theorem 6.9 (Gupta *et al.* [29], 2006)**.** *For any digraph G of order n and size ε, if $f \leq n-2$, then*

$$\pi_f(G) \geq \left\lceil \frac{1}{\varepsilon} \sum_{x,y \in V} d_{f+1}(G; x, y) \right\rceil.$$

Gupta *et al.* [29] established a close connection between leveled f-fault tolerant routings in the complete symmetric digraph K_n^* and the existence of f disjoint idempotent Latin squares.

Theorem 6.10 (Gupta *et al.* [29], 2006). *There exists a large set of disjoint idempotent Latin squares of order n if and only if there exists a symmetric leveled* $(n-2)$*-fault tolerant routing for a complete digraph* K_n^*.

Dinitz *et al.* [17] constructed leveled $(n-2)$-tolerant routings of the complete symmetric digraph K_n^* that have minimum or close to minimum optical indices.

Use $w(R_f(G))$ to denote the smallest number of wavelengths that must be assigned to the paths of $R_f(G)$ so that no two paths that share an edge receive the same wavelength. The *f-fault-tolerant optical index* of G is

$$w_f(G) = \min\{w(R_f(G)) : R_f(G) \in \mathscr{R}_f(G)\}.$$

Clearly, $w_0(G) = w(G)$. Thus, $w_f(G)$ is a generalization of $w(G)$. The following result is an immediate consequence of the definitions.

Theorem 6.11. *For any symmetric digraph G with connectivity k, and any* $0 \le f \le k$, $\pi_f(G) \le w_f(G)$.

Conjecture 6.12 (Maňuch and Stacho [43], 2003). *For any symmetric digraph G with connectivity k, and any* $0 \le f \le k$, $\pi_f(G) = w_f(G)$.

Bessy and Lepelletier [8] showed that this conjecture is true for a complete symmetric digraph K_n^* and a complete balanced bipartite symmetric digraph $K_{n,n}^*$.

6.5. RESTRICTED FORWARDING INDICES

We conclude this section with new concepts: restricted forwarding indices, proposed by Xu *et al.* [64].

It is well known that the study on the forwarding indices was motivated by the problem of maximizing network capacity. However, in practice, the vertex (resp. edge) load capacity is associated with the network hardware, which could not be changed. When the loads of vertices or edges are restricted, the number of pairs of vertices that can communicate at the same time is limited. A natural optimization problem is: what is the maximum number of ordered vertex pairs that can communicate synchronously in a graph with a given vertex or edge load? To this aim, Xu *et al.* [64] proposed the concepts of the *vertex load restricted forwarding index* and the *edge load restricted forwarding index*, defined as follows, respectively. Given nonnegative integer ℓ,

$$\xi^\ell(G) = \max\{|\mathscr{P}| : \ \xi(G, \mathscr{P}) \le \ell\} \text{ and}$$
$$\pi^\ell(G) = \max\{|\mathscr{P}| : \ \pi(G, \mathscr{P}) \le \ell\},$$

where $\mathscr{P}$ is a set of paths joining different ordered pairs of vertices in G.

Xu *et al.* [64] established several general bounds on $\xi^\ell(G)$ and $\pi^\ell(G)$ for simple undirected graph G, proved that the problems determining $\xi^\ell(G)$ and $\pi^\ell(G)$ under a fixed routing are NP-complete, and presented two approximation algorithms for computing $\xi^\ell(G)$ and $\pi^\ell(G)$.

7. FORWARDING INDICES OF SOME GRAPHS

The forwarding indices of some particular graphs are determined. We list all main results that we are interested in and have known, all of which that are not noted can be found in Heydemann *et al.* [32] or determined easily.

1. For a complete graph K_n, $\xi(K_n) = 0$ and $\pi(K_n) = 2$.

2. For a star $K_{1,\,n-1}$, $\xi(K_{1,\,n-1}) = (n-1)(n-2)$ and $\pi(K_{1,\,n-1}) = 2(n-1)$.

3. For a path P_n, $\xi(P_n) = 2\left\lfloor \frac{1}{2}n \right\rfloor \left(\left\lceil \frac{1}{2}n \right\rceil - 1\right)$ and $\pi(P_n) = 2\left\lfloor \frac{1}{2}n \right\rfloor \left\lceil \frac{1}{2}n \right\rceil$.

4. For the complete bipartite $K_{m,\,n}$ $(m \geq n)$, $\xi_m(K_{m,\,n}) = \xi(K_{m,\,n}) = \lceil \frac{m(m-1)}{n} \rceil$ and $\pi_m(K_{m,1}) = 2m$ and if $2 \leq n \leq m$,

$$\left\lceil \frac{2m(m-1) + 2n(n-1)}{mn} \right\rceil + 2 \leq \pi_m(K_{m,\,n}) \leq \left\lceil \frac{m-1}{n} \right\rceil.$$

In particular,

$$\pi_m(K_{n,\,n}) = \pi_m(K_{n,\,n}) = \begin{cases} 4, & \text{for } n = 2, \\ 5, & \text{for } n = 3, 4, \\ 6, & \text{for } n \geq 5. \end{cases}$$

5. For a directed cycle C_d $(d \geq 3)$, $\xi(C_d) = \frac{1}{2}(d-1)(d-2)$. For an undirected cycle C_d $(d \geq 3)$, $\xi_m = \xi(C_d) = \left\lfloor \frac{1}{4}(d-1)^2 \right\rfloor$ and $\pi_m = \pi(C_d) = \left\lfloor \frac{1}{4}d^2 \right\rfloor$.

6. The n-dimensional undirected toroidal mesh $C(d_1, d_2, \ldots, d_n)$ is defined as the Cartesian product $C_{d_1} \square C_{d_2} \square \ldots \square C_{d_n}$ of n undirected cycles $C_{d_1}, C_{d_2}, \ldots, C_{d_n}$ of order $d_1, d_2, \ldots, d_n$, $d_i \geq 3$ for $i = 1, 2, \ldots, n$. The $C(d, d, \ldots, d)$, denoted by $C_n(d)$, is called a d-ary n-cube a generalized n-cube. Xu *et al.* [62] determined that

$$\begin{aligned} \xi(C(d_1, d_2, \ldots, d_n)) &= \sum_{i=1}^{n} d_1 d_2 \ldots d_{i-1} \left\lfloor \frac{d_i^2}{4} \right\rfloor d_{i+1} \ldots d_n - d_1 d_2 \ldots d_n + 1, \\ \pi(C(d_1, d_2, \ldots, d_n)) &= \max_{1 \leq i \leq n} \left\{ d_1 d_2 \ldots d_{i-1} \left\lfloor \frac{d_i^2}{4} \right\rfloor d_{i+1} \ldots d_n \right\}. \end{aligned}$$

In particular,

$$\xi(C_n(d)) = nd^{n-1} \left\lfloor \frac{1}{4} d^2 \right\rfloor - (d^n - 1), \quad \text{and} \quad \pi(C_n(d)) = d^{n-1} \left\lfloor \frac{1}{4} d^2 \right\rfloor.$$

The last result was obtained by Heydemann *et al.* [32].

7. For the circulant digraph $G(d^n; S)$ with $S = \{1, d, \ldots, d^{n-1}\}$, $d \geq 2$ and $n \geq 2$, Xu *et al.* [68] determined

$$\xi(G(d^n; S)) = \frac{1}{2}(d-1)d^n n - (d^n - 1) \quad \text{and} \quad \pi(G(d^n; S)) = \frac{1}{2}(d-1)d^n.$$

For the circulant graph $TL_d = G(3d^2 + 3d + 1; \pm\{1, 3d+1, 3d^2 - 1\})$, Thomson and Zhou [54] determined $\pi(TL_d) = \frac{1}{3}\, d(d+1)(2d+1)$ for $d \geq 2$.

8. The n-dimensional directed toroidal mesh $\overrightarrow{C}(d_1, d_2, \ldots, d_n)$ is defined as the Cartesian product $\overrightarrow{C}_{d_1} \Box \overrightarrow{C}_{d_2} \Box \ldots \Box \overrightarrow{C}_{d_n}$ of n directed cycles $\overrightarrow{C}_{d_1}, \overrightarrow{C}_{d_2}, \ldots, \overrightarrow{C}_{d_n}$ of order $d_1, d_2, \ldots, d_n$, $d_i \geq 3$ for each $i = 1, 2, \ldots, n$. Set $\overrightarrow{C}_n(d) = \overrightarrow{C}(d, d, \ldots, d)$. Xu *et al.* [62] determined that

$$\xi(\overrightarrow{C}(d_1, d_2, \ldots, d_n)) = \frac{1}{2}\left(\sum_{i=1}^{n}(d_i - 3)\right) d_1 d_2 \ldots d_n + (n-1) d_1 d_2 \ldots d_n + 1,$$

$$\pi(\overrightarrow{C}(d_1, d_2, \ldots, d_n)) = \frac{1}{2} \max_{1 \leq i \leq n} \{d_1 \ldots d_{i-1} d_i (d_i - 1) d_{i+1} \ldots d_n\}.$$

In particular,

$$\xi(\overrightarrow{C}_n(d)) = \frac{n}{2}\, d^n (d-1) - d^n + 1, \quad \text{and} \quad \pi(\overrightarrow{C}_n(d)) = \frac{1}{2}\, d^n (d-1).$$

9. The n-dimensional generalized hypercube, denoted by $Q(d_1, d_2, \ldots, d_n)$, where $d_i \geq 2$ is an integer for each $i = 1, 2, \ldots, n$, is defined as the Cartesian products $K_{d_1} \Box K_{d_2} \Box \ldots \Box K_{d_n}$. If $d_1 = d_2 = \ldots = d_n = d \geq 2$, then $Q(d, d, \ldots, d)$ is called the d-ary n-dimensional cube, denoted by $Q_n(d)$. It is clear that $Q_n(2)$ is Q_n. Xu *et al.* [62] determined that

$$\begin{aligned} \xi(Q(d_1, d_2, \ldots, d_n)) &= -\sum_{i=1}^{n} d_1 d_2 \ldots d_{i-1} d_{i+1} \ldots d_n + (n-1) d_1 d_2 \ldots d_n + 1, \\ \pi(Q(d_1, d_2, \ldots, d_n)) &= \max_{1 \leq i \leq n} \{d_1 d_2 \ldots d_{i-1} 2 d_{i+1} \ldots d_n\}. \end{aligned}$$

In particular,

$$\xi(Q_n(d)) = ((d-1)n - d) d^{n-1} + 1, \quad \text{and} \quad \pi(Q_n(d)) = 2d^{n-1}.$$

For the n-dimensional hypercube Q_n,

$$\xi(Q_n) = (n-2)2^{n-1} + 1 \quad \text{and} \quad \pi(Q_n) = 2^n.$$

The last result was obtained by Heydemann *et al.* [32].

10. For the crossed cube CQ_n ($n \geq 2$), $\pi(CQ_n) = \pi_m(CQ_n) = 2^n$ decided by Chang *et al.* [13]. However, $\xi(CQ_n)$ has not been determined so far.

11. For the folded cube FQ_n, decided by Hou *et al.* [36],

$$\xi(FQ_n) = \xi_m(FQ_n) = (n-1)2^{n-1} + 1 - \frac{n+1}{2}\binom{n}{\lceil \frac{n}{2} \rceil},$$

$$\pi(FQ_n) = \pi_m(FQ_n) = 2^n - \binom{n}{\lceil \frac{n}{2} \rceil}.$$

12. For the augmented cube AQ_n proposed by Choudum and Sunitha [15], Xu and Xu [66] showed that

$$\xi(AQ_n) = \frac{2^n}{9} + \frac{(-1)^{n+1}}{9} + \frac{n2^n}{3} - 2^n + 1,$$

and

$$\pi(AQ_n) = 2^{n-1}.$$

13. For the cube-connected cycle $CCC(n)$ and the k-dimensional wrapped butterfly $WBF_k(n)$, Yan, Xu, and Yang [69], Shahrokhi and Székely [50], determined

$$\xi(CCC_n) = \frac{7}{4}n^2 2^n(1 - o(1)),$$

$$\pi(CCC(n)) = \pi_m(CCC(n)) = \frac{5}{4}n^2 2^n(1 - o(1)),$$

$$\pi(WBF(n)) = \pi_m(WBF_2(n)) = \frac{5}{4}n^2 2^{n-1}(1 + o(1)).$$

Hou, Xu and Xu [37] determined

$$\xi(WB_k(n)) = \frac{3n(n-1)}{2}k^n - \frac{n(k^n - 1)}{k-1} + 1.$$

14. For the star graph S_n, Gauyacq [26] obtained that

$$2(n-1)!(n-1) + \lceil 2\alpha \rceil \le \pi(S_n) \le 2(n-1)!(n-1) + 2\lceil \alpha \rceil,$$

where $\alpha = (n-2)! \sum_{i=2}^{n-1} \frac{n-i}{i}$.

15. For the complete-transposition graph CT_n, Gauyacq [26] obtained that

$$2(n-2)!(2n-3) - \lfloor 2\beta \rfloor \le \pi(CT_n) \le 2(n-2)!(2n-3) - 2\lfloor \beta \rfloor,$$

where $\beta = 2(n-2)! \sum_{i=3}^{n} \frac{1}{i}$.

16. For the undirected de Bruijn graph $UB(d, n)$ and Kautz graph $UK(d, n)$, the upper bounds of their vertex-forwarding indices and edge-forwarding indices are, respectively, given as follows (see [16, 32]):

$$\xi(UB(d, n)) \le (n-1)d^n, \qquad \xi(UK(d, n)) \le (n-1)d^n,$$
$$\pi(UB(d, n)) \le 2nd^{n-1}, \qquad \pi(UK(d, n)) \le 2(n-1)d^{n-2}(d+1).$$

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