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ON THE LONGEST PATH IN A RECURSIVELY PARTITIONABLE GRAPH

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Abstract. A connected graph G with order $n \geq 1$ is said to be recursively arbitrarily partitionable (R-AP for short) if either it is isomorphic to K_1, or for every sequence $(n_1, \ldots, n_p)$ of positive integers summing up to n there exists a partition $(V_1, \ldots, V_p)$ of $V(G)$ such that each V_i induces a connected R-AP subgraph of G on n_i vertices. Since previous investigations, it is believed that a R-AP graph should be "almost traceable" somehow. We first show that the longest path of a R-AP graph on n vertices is not constantly lower than n for every n. This is done by exhibiting a graph family $\mathcal{C}$ such that, for every positive constant $c \geq 1$, there is a R-AP graph in $\mathcal{C}$ that has arbitrary order n and whose longest path has order $n - c$. We then investigate the largest positive constant $c' < 1$ such that every R-AP graph on n vertices has its longest path passing through $n \cdot c'$ vertices. In particular, we show that $c' \leq \frac{2}{3}$. This result holds for R-AP graphs with arbitrary connectivity.

Keywords: recursively partitionable graph, longest path.

Mathematics Subject Classification: 05C99, 68R10.

1. INTRODUCTION

Let $n \geq 1$ be a positive integer. A *n-graph* is a graph whose *order*, i.e. its number of vertices, is n. Throughout this paper, we denote by $LP(G)$ the order of the longest path in a given connected graph G. We say that G is *recursively arbitrarily partitionable* (R-AP for short) if and only if one of the following two conditions hold.

— The graph G is an isolated vertex.
— For every sequence $(n_1, \ldots, n_p)$ of positive integers that performs a partition of n, there exists a partition $(V_1, \ldots, V_p)$ of $V(G)$ such that $G[V_i]$ is a connected R-AP subgraph of G on n_i vertices for all $i \in \{1, \ldots, p\}$.

The property of being R-AP was introduced in [7] as a strengthened version of the property of being *arbitrarily partitionable*. The property of being AP was itself introduced to deal with a problem of resource sharing among an arbitrary number of users (see [1, 2, 5, 8] for further details).

R-AP graphs have been mainly studied in the context of some simple classes of graphs like trees [7], a family of unicyclic 1-connected graphs called *suns* [6], and a class of 2-connected graphs called *balloons* [4, 7]. Although these works did not lead to numerous general properties of R-AP graphs, they however suggest that the property of being R-AP is even closer to *traceability*[1)] than the one of being AP. For instance, we know that if T is a R-AP n-tree, then $LP(T) \geq n-2$. It was also empirically observed[2)] that if B is a R-AP n-balloon, then $LP(B) \geq n-4$. Such bounds do not exist regarding AP trees and AP balloons since the structure of these graphs is much less predictable (see [3] and [4], respectively).

Regarding these observations, one could naively think that there should exist a small positive constant $c \geq 1$ such that $LP(G) \geq n-c$ for every R-AP n-graph G. In this work, we first show, in Section 3, that such a constant does not exist by exhibiting a class $\mathcal{C}$ of R-AP graphs such that for every c there exists a n-graph C in $\mathcal{C}$ such that $LP(C) = n-c$ for some n. The graphs of $\mathcal{C}$ are 1-connected, but an equivalent result regarding 2-connected graphs is derived by slightly modifying our construction. We then investigate, in concluding Section 4, the greatest constant $c' \leq 1$ such that every R-AP n-graph has its longest path passing through $n \cdot c'$ of its vertices. In particular, we exhibit another family of graphs showing that $c' \leq \frac{2}{3}$. This upper bound also holds regarding ℓ-connected R-AP graphs, no matter what is the value of ℓ.

2. DEFINITIONS AND PRELIMINARY RESULTS

First observe that adding edges to a R-AP graph does not make it loose its property of being R-AP.

Remark 2.1. If G is spanned by a R-AP subgraph, then G is R-AP.

Because every path is clearly R-AP, the next result follows by Remark 2.1.

Remark 2.2. Every traceable graph is R-AP.

Determining whether a n-graph G is R-AP is laborious since, according to the original definition, one has to check whether G can be partitioned following every partition of n. We thus usually prefer to check the following equivalent condition which derives from the fact that a R-AP graph is partitionable into R-AP subgraphs at will.

Remark 2.3 ([7])**.** A connected n-graph G is R-AP if and only if for every $\lambda \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ there exists a partition $(V_\lambda, V_{n-\lambda})$ of $V(G)$ such that $G[V_\lambda]$ and $G[V_{n-\lambda}]$ are connected R-AP subgraphs of G on λ and $n-\lambda$ vertices, respectively.

Let us now introduce the following subclass of *caterpillar graphs*.

1) A *traceable* graph is a graph that has a Hamiltonian path.

2) Private communication.

Definition 2.4. Let $a, b \geq 2$ be two positive integers and consider three vertex-disjoint paths u_1u_2, $v_1, \ldots, v_a$ and $w_1, \ldots, w_b$ of order 2, a and b, respectively. The *caterpillar* $Cat(a, b)$ is the tree obtained by identifying the vertices u_1, v_1 and w_1.

Throughout this paper, every mention to caterpillar graphs actually refers to caterpillars of the form $Cat(a, b)$. Two examples of such caterpillars are given in Figure 1. This family of caterpillars is important regarding R-AP graphs since it was proven in [7] that most of R-AP trees are caterpillars of this kind. The authors of [7] also gave a complete characterization of R-AP caterpillars.

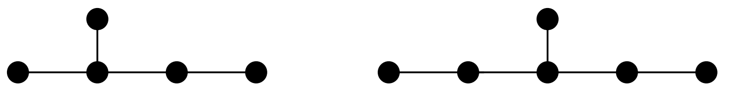

Fig. 1. The caterpillars $Cat(2, 3)$ and $Cat(3, 3)$

Theorem 2.5 ([7]). *A caterpillar $Cat(a, b)$ is R-AP if and only if a and b take values in Table 1.*

Table 1. Values a and b ($a \leq b$) such that $Cat(a, b)$ is R-AP

a	b
$2, 4$	$\equiv 1 \mod 2$
3	$\equiv 1, 2 \mod 3$
5	$6, 7, 9, 11, 14, 19$
6	7
7	$8, 9, 11, 13, 15$

3. LONGEST PATH AND ADDITIVE FACTOR

In this section, we prove the following result.

Theorem 3.1. *There does not exist a positive constant $c \geq 1$ such that we have $LP(G) \geq n - c$ for every R-AP n-graph G.*

This is proved by exhibiting a counterexample for every possible value of c. For this purpose, we introduce the family of *connected cycles* graphs.

Definition 3.2. Let $k \geq 1$ and $x, y \geq 0$ be three positive integers. The *connected cycles* graph $CC_k(x, y)$ is the graph with the following vertices:

— Let $u_1 \ldots u_x$ and $v_1 \ldots v_y$ be paths with order x and y, respectively.
— For every $i \in \{1, \ldots, k\}$, let $a_ib_ie_id_ic_ia_i$ be a cycle with length 5.
— For every $i \in \{1, \ldots, k-1\}$, let $w_{i,i+1}$ be a vertex.

These vertices are linked in $CC_k(x,y)$ in the following way: $u_x a_1, v_y e_k \in E(CC_k(x,y))$ and we have $w_{i,i+1}e_i, w_{i,i+1}a_{i+1} \in E(CC_k(x,y))$ for every $i \in \{1,\dots,k-1\}$.

An example of a connected cycles graph is depicted in Figure 2. Notice that $LP(CC_k(1,1)) = |V(CC_k(1,1))| - k$. Thus, by showing that all graphs $CC_k(1,1)$ are R-AP, we can contradict the existence of the constant c mentioned in Theorem 3.1.

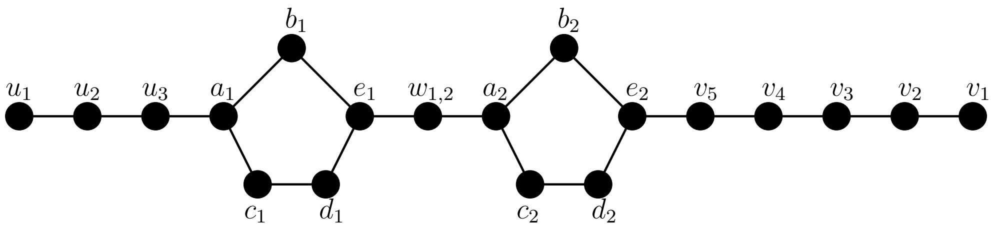

Fig. 2. The connected cycles graph $CC_2(3,5)$

Before proving that $CC_k(1,1)$ is R-AP for every k, we first introduce another graph structure we encounter while partitioning a connected cycles graph.

Definition 3.3. Let $k \geq 1$ and $x \geq 0$ be two positive integers. The *partial connected cycles* graph $PCC_k(x)$ is the graph obtained by removing the vertex e_k from $CC_k(x,0)$.

We are now ready to prove the main result of this section.

Lemma 3.4. *The graph $PCC_k(x)$ is R-AP for every $k \geq 1$ and $x \geq 1$ such that $x \not\equiv 2 \mod 3$. The graph $CC_k(x,y)$ is R-AP for every $k \geq 1$ and $x,y \geq 1$ whenever $x \not\equiv 2 \mod 3$ or $y \not\equiv 2 \mod 3$.*

Proof. The proof is by induction on k and uses the terminology introduced in Definition 3.2. For each value of k, we prove that the result is true for all possible values of x and (possibly) y which satisfy the claim. Recall that, according to Remark 2.3, a connected n-graph G is R-AP if and only if for every $\lambda \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$ we can partition $V(G)$ into two parts V_λ and $V_{n-\lambda}$ inducing connected R-AP subgraphs of G with order λ and $n-\lambda$, respectively.

Case 1. $k = 1$.
First, every graph $PCC_1(x)$ is R-AP since it is spanned by $Cat(3, x+1)$, which is R-AP according to the assumption on x.

We now prove that every graph $C = CC_1(x,y)$ is R-AP whenever the conditions of the claim are fulfilled. This is proved by induction on $x+y$ by showing that there is a partition of $V(C)$ into two parts V_λ and $V_{n-\lambda}$ satisfying the conditions above for every $\lambda \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$ where $n = 5+x+y$. For each value of λ, we give a satisfying subset V_λ, and it is understood that $V_{n-\lambda} = V(C) - V_\lambda$. We further assume $x \not\equiv 2 \mod 3$ since the graphs $CC_1(x,y)$ and $CC_1(y,x)$ are isomorphic.

First, when dealing with $\lambda \geq x+5$, we can pick up, as V_λ, the λ first vertices of the ordering $\{u_1, \dots, u_x, a_1, b_1, c_1, d_1, e_1, v_y, \dots, v_1\}$ of $V(C)$ to get a partition of C into a traceable graph or $CC_1(x, y-(\lambda-(x+6)))$ which is R-AP by the induction hypothesis, and a path. For $\lambda = x$, one can consider $V_\lambda = \{u_1, \dots, u_x\}$ so that the two induced graphs are traceable. Now, if $\lambda = x+2$ or $\lambda = x+3$, then we can choose $\{u_1, \dots, u_x, a_1, b_1\}$ or $\{u_1, \dots, u_x, a_1, c_1, d_1\}$, respectively, as V_λ, so that the two induced subgraphs are paths. Next, consider $\lambda = x+4$. Then $V_\lambda = \{u_1, \dots, u_x, a_1, b_1, c_1, d_1\}$ yields a correct partition of C. Indeed, on the one hand, $C[V_\lambda]$ is a caterpillar $Cat(3, x+1)$ which is R-AP since otherwise it would mean that $x \equiv 2 \mod 3$, a contradiction. On the other hand, the graph $C[V_{n-\lambda}]$ is a path.

Now consider $\lambda = x+1$. If $V_\lambda = \{u_1, \dots, u_x, a_1\}$ does not provide a satisfying partition of C, then $y \equiv 2 \mod 3$ since $C[V_{n-\lambda}]$ is $Cat(3, y+1)$ and is not R-AP. Consider now, as V_λ, the λ first vertices of the ordering $(v_1, \dots, v_y, e_1, b_1, d_1, c_1, a_1, u_x, \dots, u_1)$ of $V(C)$. If this choice of V_λ does not yield a correct partition of C once again, then it means that either $C[V_\lambda]$ is the caterpillar $Cat(3, y+1)$, or a connected cycles graph $CC_1(x', y)$ with $x' \equiv 2 \mod 3$. But then we get that either $x+1 = y+4$ or $x+1 = y+5+x'$, respectively, which both imply that $x \equiv 2 \mod 3$, a contradiction.

Finally consider every value $\lambda \in \{1, \dots, x-1\}$. On the one hand, if $x - \lambda \not\equiv 2 \mod 3$, then choose $V_\lambda = \{u_1, \dots, u_\lambda\}$ so that $C[V_\lambda]$ and $C[V_{n-\lambda}]$ are a path, and $CC_1(x-\lambda, y)$ which is R-AP by the induction hypothesis. On the other hand, i.e. $x - \lambda \equiv 2 \mod 3$, we have $\lambda \not\equiv 0 \mod 3$ since otherwise we would have $x \equiv 2 \mod 3$. We can assume that $\lambda \notin \{y, y+2, y+3\}$, since otherwise we could deduce a correct partition of C as in the cases $\lambda \in \{x, x+2, x+3\}$, respectively. Then consider, as V_λ, the λ first vertices of $(v_1, \dots, v_y, e_1, b_1, d_1, c_1, a_1, u_x, \dots, u_1)$. If this choice of V_λ does not yield a correct partition of C, then $C[V_\lambda]$ is either a caterpillar $Cat(3, y+1)$ which is not R-AP, or a graph $CC_1(x', y)$ with $x' \equiv 2 \mod 3$. But note then that the first situation cannot occur because $\lambda \not\equiv 0 \mod 3$. For the second situation, note that, because $\lambda \not\equiv 0 \mod 3$, we have $y \not\equiv 2 \mod 3$. Since we have $x', y < x$, the graph $CC_1(y, x')$ is actually R-AP by the induction hypothesis.

Case 2. Arbitrary k.
Let us now suppose that the result is true for every i up to $k-1$, and let us prove it for k. Consider first $C = PCC_k(x)$ for consecutive values of $x \not\equiv 2 \mod 3$. As we did before, to prove that C is R-AP we show that there exists a partition of $V(C)$ satisfying our conditions for every possible value of λ. One may choose V_λ as follows.

— If $\lambda \equiv 1 \mod 3$, then one may consider, as V_λ, the first λ vertices of the ordering $(b_k, d_k, c_k, a_k, w_{k-1,k}, e_{k-1}, b_{k-1}, d_{k-1}, c_{k-1}, a_{k-1}, \dots, w_{1,2}, e_1, b_1, d_1, c_1, a_1, u_x, \dots, u_1)$ of $V(C)$. On the one hand, notice that $C[V_\lambda]$ is either a path, or covered by a R-AP caterpillar or a partial connected cycles graph $PCC_{k'}(x')$ with $k' \leq k-1$ and $x' \not\equiv 2 \mod 3$, which is R-AP by the induction hypothesis. On the other hand, observe that $C[V_{n-\lambda}]$ is either traceable, or spanned by a connected cycles graph $CC_{k''}(x, y)$ for some $k'' \leq k-1$ and y, which is R-AP according to the induction hypothesis.

— If $\lambda \equiv 2 \mod 3$, then one can obtain similar partitions of C from the ordering $(d_k, c_k, b_k, a_k, w_{k-1,k}, e_{k-1}, d_{k-1}, c_{k-1}, b_{k-1}, a_{k-1}, \dots, w_{1,2}, e_1, d_1, c_1, b_1, a_1, u_x, \dots, u_1)$ of $V(C)$.

— Otherwise, if $\lambda \equiv 0 \mod 3$, then one has to consider as V_λ the first λ vertices of the ordering $(u_1, \dots, u_x, a_1, b_1, c_1, d_1, e_1, w_{1,2}, \dots, a_{k-1}, b_{k-1}, c_{k-1}, d_{k-1}, e_{k-1}, w_{k-1,k}, a_k, b_k, c_k, d_k)$ of $V(C)$ when $x \equiv 1 \mod 3$, or the ordering $(u_1, \dots, u_x, a_1, c_1, d_1, b_1, e_1, w_{1,2}, \dots, a_{k-1}, c_{k-1}, d_{k-1}, b_{k-1}, e_{k-1}, w_{k-1,k}, a_k, c_k, d_k, b_k)$ otherwise, i.e. when $x \equiv 0 \mod 3$. The two induced subgraphs $C[V_\lambda]$ and $C[V_{n-\lambda}]$ are then R-AP. Indeed, on the one hand, $C[V_\lambda]$ is either isomorphic to a path or spanned by a connected cycles graph $CC_{k'}(x, y)$ for $k' \le k-1$ and some y. On the other hand, the subgraph $C[V_{n-\lambda}]$ is spanned by some $PCC_{k''}(x')$ graph with $k'' \le k$ and $x' \not\equiv 2 \mod 3$.

To end up proving the claim, we have to show that $CC_k(x, y)$ is R-AP whenever $x \not\equiv 2 \mod 3$ or $y \not\equiv 2 \mod 3$. As for the base case, we show this by induction on $x + y$. Once again, we assume that $x \not\equiv 2 \mod 3$ for a given graph $C = CC_k(x, y)$.

For some $\lambda \in \{1, \dots, y\}$, one can consider $V_\lambda = \{v_1, \dots, v_\lambda\}$ so that C is partitioned into a path and $CC_k(x, y - \lambda)$ which is R-AP according to the induction hypothesis. When $\lambda = y + 1$, one can choose $V_\lambda = \{v_1, \dots, v_y, e_k\}$ so that C is partitioned into a path and a partial connected cycles graph which is R-AP by the induction hypothesis since $x \not\equiv 2 \mod 3$. For other values of λ, one may choose V_λ as follows.

— If $\lambda \equiv 0 \mod 3$, one can consider, as V_λ, the λ first vertices from the ordering $(u_1, \dots, u_x, a_1, b_1, c_1, d_1, e_1, w_{1,2}, \dots, w_{k-1,k}, a_k, b_k, c_k, d_k\ e_k, v_y, \dots, v_1)$ of $V(C)$ when $x \equiv 1 \mod 3$, from $(u_1, \dots, u_x, a_1, c_1, d_1, b_1, e_1, w_{1,2}, \dots, w_{k-1,k}, a_k, c_k, d_k, b_k, e_k, v_y, \dots, v_1)$ otherwise, i.e. when $x \equiv 0 \mod 3$. The two induced subgraphs are then R-AP since they are traceable or isomorphic to connected cycles graphs which are R-AP according to the induction hypotheses.

— If $\lambda \not\equiv 0 \mod 3$ and $\lambda - (y + 1) \equiv 0 \mod 3$, then one can consider the λ first vertices of the ordering $(v_1, \dots, v_y, e_k, b_k, d_k, c_k, a_k, w_{k-1,k} \dots, e_1, b_1, d_1, c_1, a_1, u_x, \dots, u_1)$ of $V(C)$. For each such partition, we get, on the one hand, that $C[V_\lambda]$ is either a path, a R-AP caterpillar, or a R-AP (partial) connected cycles graph. In particular, note that when $C[V_\lambda]$ is a caterpillar or partial connected cycles graph, then this graph is R-AP since $y \not\equiv 2 \mod 3$ because of the assumptions on λ. On the other hand, the graph $C[V_{n-\lambda}]$ is either a path, or a (partial) connected cycles graph which is R-AP by the induction hypothesis.

— If $\lambda \not\equiv 0 \mod 3$ and $\lambda - (y + 1) \equiv 1 \mod 3$, then one may pick up, as V_λ, the λ first vertices from the ordering given to deal with the previous case. This choice of V_λ makes, on the one hand, $C[V_\lambda]$ being spanned by either a path, or $CC_{k'}(x', y)$ where $k' \le k - 1$ and $x' \not\equiv 2 \mod 3$ which is R-AP by the induction hypothesis. On the other hand, $C[V_{n-\lambda}]$ is a path, or is spanned by some graph $CC_{k''}(x, y')$ for $k'' \le k - 1$ and some y' which is R-AP, again by the induction hypothesis.

— Otherwise, if $\lambda \not\equiv 0 \mod 3$ and $\lambda - (y+1) \equiv 2 \mod 3$, then some similar partitions of C may be obtained from the ordering $(v_1, \dots, v_y, e_k, d_k, c_k, b_k, a_k, w_{k-1,k} \dots, w_{1,2}, e_1, d_1, c_1, b_1, a_1, u_x, \dots, u_1)$ of $V(C)$. □

Note that Lemma 3.4 provides a full characterization of R-AP (partial) connected cycles graphs since every such graph whose parameters do not satisfy this lemma is not R-AP. To be convinced of that fact, one just has to consider successive partitions of such a graph for $\lambda = 3$.

We finally deduce Theorem 3.1 as a corollary of Lemma 3.4.

Proof of Theorem 3.1. We have $LP(CC_{c+1}(1,1)) = |V(CC_{c+1}(1,1))| - (c+1)$ for every $c \geq 1$. Therefore, for every possible value of c, we have a graph showing that c does not contradict the claim. □

Finally notice that by adding the edge u_1v_1 to any connected cycles graph $CC_k(1,1)$, we get a 2-connected graph which is R-AP according to Remark 2.1 and whose longest path has order $LP(CC_k(1,1)) + 1$. Therefore, Theorem 3.1 is also true when restricted to 2-connected graphs.

4. LONGEST PATH AND MULTIPLICATIVE FACTOR

The graph $CC_k(1,1)$ has order $n = 6k+1$ while its longest path has order $n - k$ for every $k \geq 1$. Thus, even if the connected cycles graphs confirm that the order of the longest path in a R-AP n-graph is not constantly lower than n up to an additive factor, they do not reject the strong relationship between the properties of being R-AP and traceable. We now discuss how to create this relationship thanks to a multiplicative factor.

Question 4.1. *What is the biggest $c < 1$ such that $LP(G) \geq n \cdot c$ for every R-AP n-graph G?*

Regarding the connected cycles graphs, we get that $c \leq \frac{5}{6}$. In this section, we deduce a better upper bound on c thanks to the following graph construction.

Definition 4.2. Let $k, k' \geq 1$ be two positive integers. The *urchin* $W(k,k')$ is the graph obtained as follows.

— Let A, B, C be three sets of k, k and k' distinct vertices, respectively.
— Add a perfect matching between the vertices of A and B.
— Add all possible edges between distinct vertices in $B \cup C$.

This construction is illustrated in Figure 3. Note that the urchin $W(k,k)$ has order $3k$ while its longest path has order $2k+2$. We then get that $LP(W(k,k))/n$ tends to $\frac{2}{3}$ as k grows to infinity. In what follows, we show that any urchin $W(k,k)$ is R-AP, and thus that the following holds regarding Question 4.1.

Theorem 4.3. *Regarding Question 4.1, we have $c \leq \frac{2}{3}$.*

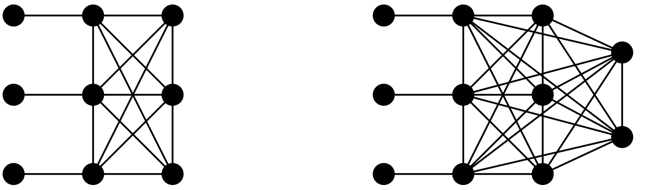

Fig. 3. The urchins $W(3,3)$ and $W(3,5)$

We prove that an urchin $W(k,k')$ is R-AP for some values of k and k'.

Lemma 4.4. *The urchin $W(k,k')$ is R-AP for every $k \geq 2$ and $k' \geq k-2$.*

Proof. We introduce some terminology to deal with the vertices of any urchin $W(k,k')$. The vertices of A are denoted $u_1,\dots,u_k$, and those of B are denoted $v_1,\dots,v_k$ in such a way that u_iv_i is an edge for every $i \in \{1,\dots,k\}$. The vertices of C are denoted $w_1,\dots,w_{k'}$ arbitrarily.

The claim is proved by induction on both k and k'. As a base case, note that every urchin $W(2,k')$ is traceable, and thus R-AP by Remark 2.2. Suppose now that $W(i,i')$ is R-AP for every i up to $k-1$ and $i' \geq i-2$. We now prove that the urchin n-graph $W = W(k,k')$ is R-AP for every $k' \geq k-2$. For this purpose, we show, for every value of $\lambda \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$, that $V(W)$ can be partitioned into two parts V_λ and $V_{n-\lambda}$ inducing R-AP graphs on λ and $n-\lambda$ vertices, respectively.

We first deal with the easy cases, i.e. $\lambda \in \{1,2,3\}$. For $\lambda = 1$, consider $V_\lambda = \{u_1\}$ so that the two induced subgraphs are K_1 and $W(k-1,k'+1)$. Since $k' \geq k-2$, this subgraph is R-AP by the induction hypothesis. For $\lambda = 2$, let $V_\lambda = \{u_1,v_1\}$. The two induced subgraphs then are K_2 and $W(k-1,k')$, which is R-AP for the same reason as the previous case. Now, for $\lambda = 3$, choose $V_\lambda = \{u_1,v_1,w_1\}$. The two induced subgraphs then are a path, and the urchin $W(k-1,k'-1)$ which is R-AP, again by the induction hypothesis.

We now deal with the remaining values of λ, i.e. $\lambda \geq 4$. The part V_λ is obtained by choosing two disjoint sets V'_λ and V''_λ, and then considering their union. On the one hand, in the case where $\lambda \equiv 1 \mod 3$, let $x = \lfloor \frac{\lambda-4}{3} \rfloor$. Clearly, x is an integer. First, let $V'_\lambda = \emptyset$ if $x = 0$, or $V'_\lambda = \bigcup_{i=1}^{x} \{u_i,v_i,w_i\}$ otherwise. Then set $V''_\lambda = \{v_{x+1},u_{x+1},v_{x+2},u_{x+2}\}$. The two induced subgraphs then are a path or $W(x+2,x)$, and $W(k-(x+2),k'-(x-2))$, which are R-AP by the induction hypothesis since $k' \geq k-2$.

On the other hand, i.e. $\lambda \not\equiv 1 \mod 3$, let $x = \lfloor \frac{\lambda}{3} \rfloor$ and $y \equiv \lambda \mod 3$ with $y \in \{0,2\}$. Then, let $V'_\lambda = \bigcup_{i=1}^{x} \{u_i,v_i,w_i\}$. The strategy for choosing V''_λ depends on whether $y = 0$ or $y = 2$.

— $y = 0$. Choose $V''_\lambda = \emptyset$. In this situation, the two induced subgraphs are $W(x,x)$ and $W(k-x,k'-x)$ which are R-AP by the induction hypothesis since $k' \geq k-2$.
— $y = 2$. Let $V''_\lambda = \{v_{x+1},u_{x+1}\}$. The two induced subgraphs then are $W(x+1,x)$ and $W(k-(x+1),k'-x)$, which are R-AP according to the induction hypothesis. □

Theorem 4.3 follows as a corollary of Lemma 4.4. Note that Lemma 4.4 is tight in the sense that urchins $W(k,k-x)$ with $x \geq 3$ are not R-AP since such a graph W

cannot be partitioned as requested for $\lambda = 3$. Indeed, as a set V_λ with size 3 inducing a R-AP subgraph of W, one has to consider, following the terminology introduced in the proof of Lemma 4.4, a part of the form $\{u_i, v_i, w_j\}$ or $\{w_i, w_j, w_\ell\}$. After having successively picked several sets with size 3 off W, one necessarily gets an urchin $W(k', 0)$ with $k' \geq 3$. Such a graph is clearly not partitionable for $\lambda = 3$ once again.

We can strengthen Theorem 4.3 as follows. Let $W = W(k, k')$ be a R-AP urchin. Observe that by adding the edges $u_1u_2, \ldots, u_1u_k$ to W, we get a 2-connected graph W_2 which is R-AP by Remark 2.1. By then adding the edges $u_2u_3, \ldots, u_2u_k$ to W_2, we get another R-AP graph W_3 which is 3-connected. By repeating this procedure as many times as needed, we get an ℓ-connected R-AP graph W_ℓ for any value of ℓ assuming k and k' are big enough. Note that we have $LP(W_i) = LP(W) + 2i$, and thus that $LP(W_i)/LP(W)$ tends to 1 as k grows to infinity. Therefore, the statement of Theorem 4.3 is also true when restricted to ℓ-connected R-AP graphs, no matter what is the value ℓ.

Theorem 4.5. *Theorem 4.3 is also true when Question 4.1 is restricted to R-AP graphs of arbitrary connectivity.*

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