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Dedicated to the Memory of Professor Zdzisław Kamont

STABILITY OF FINITE DIFFERENCE SCHEMES FOR GENERALIZED VON FOERSTER EQUATIONS WITH RENEWAL

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Abstract. We consider a von Foerster-type equation describing the dynamics of a population with the production of offsprings given by the renewal condition. We construct a finite difference scheme for this problem and give sufficient conditions for its stability with respect to l^1 and l^∞ norms.

Keywords: structured model, renewal, finite differences, stability.

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1. INTRODUCTION

Enhanced or generalized von Foerster-McKendrick models (originated in [5]) describe populations with some structure given by age [6], size [1] or the maturity of individuals [7]. In the literature there are discrete models of that type with finite [9] or infinite matrices [15].

We consider a population with some structure given by its members size or the maturity level and with the birth process expressed by a renewal condition. An elementary outline of such equations, together with their biological interpretation, is provided in [14], see also [2]. In [1] there is a system of equations describing subpopulations existing in a common niche and competing for the same resources with the closed reproduction (members of the same subpopulation) or the open reproduction (members of different subpopulations).

The existence of solutions for generalized von Foerster equations with renewal conditions and with the functional dependence is proved in [13], which continues the sequence of results [3, 4] and [10, 12], focused on integral fixed-point equations, generated by differential-functional problems. As a main tool in the existence theory there are constructed integral fixed-point equations and functional spaces, invariant with respect to these equations. Numerical methods for PDE's are based on similar discrete equations and suitable functional spaces. Having in mind $L^1 \cap L^\infty$-dynamics of the continuous problem, we keep our discrete constructions in $l^1 \cap l^\infty$-spaces.

1.1. FORMULATION OF THE DIFFERENTIAL PROBLEM.

Let $a > 0$ and denote $E = [0,a] \times \mathbb{R}_+$, $\Omega_0 = E \times \mathbb{R}_+$, $\Omega = E \times \mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, +\infty)$. Suppose that

$$\lambda\colon \Omega \to \mathbb{R}, \quad c\colon \Omega_0 \to \mathbb{R}_+.$$

Consider the differential equation

$$\frac{\partial u}{\partial t} + c\,(t,x,z(t))\,\frac{\partial u}{\partial x} = u(t,x)\,\lambda\,(t,x,u(t,x),z(t))\,, \tag{1.1}$$

where

$$z(t) := \int_0^{+\infty} u(t,y)\,dy, \quad t \in [0,a], \tag{1.2}$$

with the initial condition

$$u(t,x) = v(x), \quad x \in \mathbb{R}_+ \tag{1.3}$$

and with the renewal condition

$$u(t,0) = \int_0^{+\infty} k(t,y)u(t,y)\,dy. \tag{1.4}$$

The well posedness of problem (1.1)–(1.4) requires the following consistency condition

$$u(0,0) = \int_0^{+\infty} k(0,y)u(0,y)\,dy, \tag{1.5}$$

which is valid throughout the paper.

Problem (1.1)–(1.4) is nonlocal because there are two nonlocal terms $z(t) = \int u dx$ and $u(t,0) = \int k\,u\,dx$, see (1.2) and (1.4). In [13] one can find additional sources of nonlocal (causal) dependence z_t and $u_{(t,x)}$, but renewal was replaced there by the usual boundary condition. Since the numerical analysis of the highly nonlocal differential problem is very technical, we simplify the PDE presented in [13], namely: functions describing growth and mortality rates have classical arguments for quantities representing the population density u and the total number z of its members, but we deal with the most demanding nonlocal term, i.e. renewal. The present paper extends our previous results concerning finite difference approximations, see [11], to the problem with renewal.

The terms $z(t)$ and $u(t,0)$ are defined by some integrals over $\mathbb{R}_+$, which causes the main difficulty in a stable discretization of the differential problem. Introducing a new class of initial functions we make sure that the quadratures approximating these quantities are well defined. Since the differential-functional problem considered here has a concrete biological interpretation, we prove nonnegativity of discrete approximations of its solutions. Applying discrete comparison functions, we demonstrate that solutions of our scheme are bounded in l^∞ and l^1 norms. Stability criteria for a finite difference scheme are also derived by means of discrete comparison functions. It is clear that practical computations cannot be performed on unbounded meshes. Therefore, in practical numerical experiments, we replace an infinite mesh by its sufficiently large finite restriction. It is possible to write a natural generalization of our scheme to the case of many species with many features.

The main ideas of the paper come from Z. Kamont's monograph [8], however there is a significant difference: we study the dynamics of discrete solutions in $l^\infty \cap l^1$ norms, whereas the cited monograph is focussed on l^∞ estimates. Our task is much harder, thus our recurrence estimates are subtle. It is possible to extend our stability results to the case of Hale-type functional dependence for the unknown density and total size of population.

2. FINITE DIFFERENCE SCHEME

For a given number $N_0 \in \mathbb{N}$ introduce discretization parameters $h_0 = \frac{a}{N_0}$ and $h_1 > 0$. Infinite regular meshes on the sets E_0, E are given as follows. Denote by $\left(t^{(i)}, x^{(j)}\right)$, $i = 0, \ldots, N_0, j = 0, \ldots,$ where $t^{(i)} = ih_0, x^{(j)} = jh_1$, knots of the mesh. The mesh in E will be denoted by E_h, and define the initial mesh $E_{0.h} = \left\{x^{(j)} \colon j = 0, \ldots\right\}$. The values of any discrete function $u \colon E_h \to \mathbb{R}_+$ at the knots are denoted by $u^{(i,j)} = u\left(t^{(i)}, x^{(j)}\right)$. It follows from the biological interpretation that characteristics are non decreasing in the interior of the set E, hence we define the respective discrete operators δ_0, δ_-

$$\delta_0 u^{(i,j)} = \frac{u^{(i+1,j)} - u^{(i,j)}}{h_0}, \quad \delta_- u^{(i,j)} = \frac{u^{(i,j)} - u^{(i,j-1)}}{h_1},$$

which approximate the partial derivatives $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$ at the knots. Let

$$c^{(i,j)}[z] = c(t^{(i)}, x^{(j)}, z^{(i)}), \quad \lambda^{(i,j)}[u,z] = \lambda(t^{(i)}, x^{(j)}, u^{(i,j)}, z^{(i)}).$$

Consider the following finite difference scheme

$$\delta_0 u^{(i,j)} + c^{(i,j)}[z]\delta_- u^{(i,j)} = u^{(i,j)}\lambda^{(i,j)}[u,z] \tag{2.1}$$

for $i = 0, \ldots, N_0 - 1$, $j = 1, \ldots,$ with the initial condition $u^{(0,j)} = v^{(j)}, j = 0, \ldots,$ where

$$z^{(i)} = h_1 \sum_{j=0}^{+\infty} u^{(i,j)}, \qquad i = 0, \ldots, N_0, \tag{2.2}$$

and with the renewal

$$u^{(i,0)} = h_1 \sum_{j=0}^{+\infty} k^{(i,j)} u^{(i,j)}, \qquad i = 0, \ldots, N_0. \tag{2.3}$$

Discrete consistency condition is given by

$$v^{(0)} = h_1 \sum_{j=0}^{+\infty} k^{(0,j)} v^{(j)}.$$

Introduce the following normed spaces. In the space l^∞, of all bounded sequences $\psi = (\psi^{(j)})_{j\in\mathbb{N}}$, we have natural supremum norm $\|\psi\|$. The space l^1, of all summable sequences $\psi = (\psi^{(j)})_{j\in\mathbb{N}}$, is equipped with the norm

$$\|\psi\|_1 = h_1 \sum_{j=0}^{+\infty} |\psi^{(j)}| \quad \text{for} \quad (\psi^{(j)}) \in l^1.$$

For a given discretization parameter $h_1 > 0$, denote by R_h the infinite regular mesh on the set $\mathbb{R}_+$, i.e. $R_h = \{x^{(j)} \colon j = 0, 1, \ldots\}$, where $x^{(j)} = jh_1$. Given any function $y \colon \mathbb{R}_+ \to \mathbb{R}_+$ we denote by y_h its restriction to the mesh R_h. If the function y is bounded, then $\|y_h\| < +\infty$. If $y \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ (summable functions), then it is possible that $\|y_h\|_1 = +\infty$. To exclude this case we introduce a new class of functions.

Definition 2.1. A function $f \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ is of class $L^1_{\mathcal{M}}$ iff there is a decreasing function $g \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $|f(x)| \le g(x)$, $x \in \mathbb{R}_+$.

If $f \in L^1_{\mathcal{M}}$, then $\|f_h\|_1 < +\infty$.

We formulate the following assumptions on given functions and discretization parameters:

(KH) There is $\hat{k} \in \mathbb{R}_+$ such that $h_1\hat{k} < 1$ and $0 \le k^{(i,j)} \le \hat{k}$ on E_h.

(V) $v \in CB(\mathbb{R}_+, \mathbb{R}_+)$ (nonnegative, bounded and continuous functions) and $v \in L^1(\mathbb{R}_+, \mathbb{R}_+)$.

(V1) There is $V \in L^1_{\mathcal{M}}$ such that

$$v^{(j)} \le V^{(j)}, \quad V^{(0)} \ge h_1 \hat{k} \sum_{j=0}^{+\infty} V^{(j)} + 1.$$

(C) $c : \Omega_0 \to \mathbb{R}_+$ is bounded, continuous in (t, x, q) and there is a constant $L_c^* \in \mathbb{R}_+$ such that

$$|c(t, x, q) - c(t, x, \bar{q})| \le L_c^* V(x)\, |q - \bar{q}|,$$

where a function V is given in Assumption (V1).

(C1) There are $\varepsilon_0, \hat{c} > 0$ such that $c : \Omega_0 \to \mathbb{R}_+$ satisfies

$$c(t, 0, q) \ge \varepsilon_0 \hat{c} > 0, \quad \hat{c} \ge c(t, x, q) \ge 0 \quad \text{on} \quad \Omega_0.$$

(Λ) $\lambda\colon \Omega \to \mathbb{R}$ is continuous in (t,x,w,q) and

$$|\lambda(t,x,w,q) - \lambda(t,x,\bar{w},\bar{q})| \le L_\lambda(|w-\bar{w}| + |q-\bar{q}|),$$

where $L_\lambda > 0$.

($\Lambda 1$) There is a constant $M_\lambda \in \mathbb{R}_+$ such that

$$\lambda(t,x,w,q) \le M_\lambda$$

for $(t,x) \in E,\ w,q \in \mathbb{R}_+$.

(N) The initial function is nonnegative and

$$1 - \frac{h_0}{h_1} c(t,x,q) + h_0 \lambda(t,x,p,q) \ge 0 \quad \text{on} \quad E_h.$$

We give some properties of the scheme. It follows from (2.1) that values $u^{(i+1,j)}$ can be explicitly computed for all $j = 1, \ldots$

$$u^{(i+1,j)} = u^{(i,j)} \left(1 - \frac{h_0}{h_1} c^{(i,j)}[z] + h_0 \lambda^{(i,j)}[u,z]\right) + \frac{h_0}{h_1} c^{(i,j)}[z] u^{(i,j-1)}. \tag{2.4}$$

By the renewal equation (2.3), we have

$$u^{(i+1,0)} = \frac{h_1}{1 - h_1 k^{(i+1,0)}} \sum_{j=1}^{+\infty} k^{(i+1,j)} u^{(i+1,j)}. \tag{2.5}$$

Given a discrete function $u\colon E_h \to \mathbb{R}$, we have $u^{(i,\cdot)}\colon R_h \to \mathbb{R}$, $i = 0, \ldots, N_0$.

Lemma 2.2. *If Assumption* (C1), (N), (KH) *are satisfied, then any solution of* (2.1)–(2.3) *is nonnegative.*

Proof. An elementary proof goes by induction on i. It uses (2.4) and the explicit representation (2.5). □

Lemma 2.3. *If Assumptions* (C), (C1), (N), (KH), (Λ1), (V1) *are satisfied, then*

$$u^{(i,j)} \le V^{(j-i)} (1 + h_0 M_\lambda)^i \qquad \textit{for} \quad j \ge i,$$

$$u^{(i,j)} \le V^{(0)} \frac{(1 + h_0 M_\lambda)^i}{(1 - h_1 \hat{k})^{i-j}} \qquad \textit{for} \quad j \le i.$$

Proof. The proof is by induction on i. The estimate is obvious for $i = 0$. Assume the assertion holds for arbitrary $i = 0, \ldots, N_0 - 1$. We prove it for $i+1$ by virtue of (2.4) and (2.5). Since the function V is decreasing, it is easy to show the first assertion for $j \ge i$. Indeed, by (2.4) we have

$$u^{(i+1,j)} \le V^{(j-i)} (1 + h_0 M_\lambda)^i (1 - \theta + h_0 M_\lambda) + \theta V^{(j-1-i)} (1 + h_0 M_\lambda)^i$$

where $\theta = (h_0/h_1) c^{(i,j)}[z]$. Hence

$$u^{(i+1,j)} \le V^{(j-1-i)} (1 + h_0 M_\lambda)^i (1 + h_0 M_\lambda) \quad \text{for} \quad j \ge i.$$

For $1 \le j \le i$, denote $\theta = (h_0/h_1)c^{(i,j)}[z]$. Since $\theta \in [0,1]$, then we have

$$\begin{aligned} u^{(i+1,j)} &\le V^{(0)} \frac{(1+h_0M_\lambda)^i}{(1-h_1\hat{k})^{i-j}} \left(1 - \frac{h_0}{h_1}c^{(i,j)}[z] + h_0\lambda^{(i,j)}[u,z]\right) + \\ &\quad + V^{(0)} \frac{(1+h_0M_\lambda)^i}{(1-h_1\hat{k})^{i-(j-1)}} \frac{h_0}{h_1}c^{(i,j)}[z] \le \\ &\le V^{(0)} \frac{(1+h_0M_\lambda)^i}{(1-h_1\hat{k})^{i+1-j}} \left[(1-h_1\hat{k})\,(1-\theta+h_0M_\lambda)+\theta\right] \le \\ &\le V^{(0)} \frac{(1+h_0M_\lambda)^i}{(1-h_1\hat{k})^{i+1-j}} (1+h_0M_\lambda). \end{aligned}$$

Let $j = 0$. Then we insert (2.4) to (2.5), apply the inductive assertion and we have

$$\begin{aligned} u^{(i+1,0)} &\le \frac{h_1\hat{k}}{1-h_1\hat{k}} V^{(0)}(1+h_0M_\lambda)^i \times \\ &\quad \times \sum_{j=1}^{i} \left\{ \frac{1 - \frac{h_0}{h_1}c^{(i,j)}[z] + h_0\lambda^{(i,j)}[u,z]}{(1-h_1\hat{k})^{i-j}} - \frac{\frac{h_0}{h_1}c^{(i,j)}[z]}{(1-h_1\hat{k})^{i-j+1}} \right\} + \\ &\quad + \frac{h_1\hat{k}}{1-h_1\hat{k}}(1+h_0M_\lambda)^{i+1} \sum_{j=i+1}^{+\infty} V^{(j-i-1)} \le \\ &\le \frac{h_1\hat{k}}{1-h_1\hat{k}}(1+h_0M_\lambda)^{i+1} \left\{ \sum_{j=1}^{i} \frac{V^{(0)}}{(1-h_1\hat{k})^{i-j+1}} + \sum_{j=0}^{+\infty} V^{(j)} \right\} \le \\ &\le V^{(0)} \frac{(1+h_0M_\lambda)^{i+1}}{1-h_1\hat{k}} \left\{ \frac{h_1\hat{k}}{(1-h_1\hat{k})^{i+1}} \sum_{j=1}^{i} (1-h_1\hat{k})^j + 1 \right\} = \\ &= V^{(0)} \frac{(1+h_0M_\lambda)^{i+1}}{(1-h_1\hat{k})^{i+1}}. \end{aligned}$$

This completes the proof. □

Corollary 2.4. *Under the assumptions of Lemma 2.3 the solution of* (2.1)–(2.2) *is bounded in* l^∞ *and* l^1.

3. STABILITY OF FINITE DIFFERENCE SCHEME

Consider the finite difference scheme (2.1)–(2.3) with perturbed right-hand sides:

$$\bar{u}^{(i+1,j)} = \bar{u}^{(i,j)} \left(1 - \frac{h_0}{h_1}c^{(i,j)}[\bar{z}] + h_0\lambda^{(i,j)}[\bar{u},\bar{z}]\right) + \frac{h_0}{h_1}c^{(i,j)}[\bar{z}]\bar{u}^{(i,j-1)} + h_0\xi^{(i,j)} \quad (3.1)$$

for $i = 0, \dots, N_0 - 1$, $j = 1, \dots$, the perturbed initial condition $\bar u^{(0,j)} = v^{(j)} + \bar\xi^{(j)}$, $j = 0, \dots$, and

$$\bar z^{(i)} = h_1 \sum_{j=0}^{+\infty} \bar u^{(i,j)} + \tilde\xi^{(i)}, \quad \bar u^{(i,0)} = h_1 \sum_{j=0}^{+\infty} k^{(i,j)} \bar u^{(i,j)} + \hat\xi^{(i)}, \quad i = 0, \dots, N_0. \tag{3.2}$$

($\bar{\mathrm{U}}$) The solution of (3.1), (3.2) satisfies:

(i) $|\bar u^{(i,j)} - \bar u^{(i,j-1)}| \le h_1 L_u$ on E_h with a constant $L_u > 0$,
(ii) $\bar u^{(i,j)} \le V^{(i,j)}$, where

$$V^{(i,j)} := V^{(j-i)}(1 + h_0 M_\lambda)^i \qquad \text{for} \quad j \ge i,$$
$$V^{(i,j)} := V^{(0)} \frac{(1 + h_0 M_\lambda)^i}{(1 - h_1 \hat k)^{i-j}} \qquad \text{for} \quad j \le i.$$

(Ξ) The perturbations $\xi^{(i,j)}$, $\hat\xi^{(i)}$, $\bar\xi^{(j)}$, $\tilde\xi^{(i)}$ satisfy

$$|\xi^{(i,j)}| \le \delta_h V^{(j)}, \quad |\bar\xi^{(j)}| \le \delta_h V^{(j)}, \quad |\hat\xi^{(i)}| \le \hat\delta_h, \quad |\tilde\xi^{(i)}| \le \tilde\delta_h$$

and $\tilde\delta_h = \hat\delta_h / \hat k$.

Denote $\varepsilon_u^{(i,j)} = \bar u^{(i,j)} - u^{(i,j)}$ and $\varepsilon_z^{(i)} = \bar z^{(i)} - z^{(i)}$. To get l^∞ and l^1 error estimates, we intend to estimate $|\varepsilon_u^{(i,j)}|$ by some $W_h^{(i,j)}$ which is summable, increasing in i, decreasing in j.

Remark 3.1. The function $V^{(i,j)}$ satisfies the inequalities

$$V^{(i,0)} \ge h_1 \hat k \sum_{j=0}^{+\infty} V^{(i,j)}, \quad V^{(i+1,j)} \ge (1 + h_0 M_\lambda) V^{(i,j-1)}.$$

Define $W_h^{(i,j)}$ by means of $V^{(i,j)}$:

$$W_h^{(i,j)} := \beta_h V^{(j-i)}(1 + h_0 Q)^i \qquad \text{for} \quad j \ge i,$$
$$W_h^{(i,j)} := \beta_h V^{(0)} \frac{(1 + h_0 Q)^i}{(1 - h_1 \hat k)^{i-j}} \qquad \text{for} \quad j \le i$$

with some constants β_h, Q to be specified in terms of the data.

Lemma 3.2. *If* $|\varepsilon_u^{(i,j)}| \le W_h^{(i,j)}$ *and Assumptions* (Ξ), (KH) *are satisfied, then*

$$|\varepsilon_z^{(i)}| \le h_1 \sum_{j=0}^{+\infty} W_h^{(i,j)} + \tilde\delta_h, \quad |\varepsilon_u^{(i,0)}| \le h_1 \hat k \sum_{j=0}^{+\infty} W_h^{(i,j)} + \hat\delta_h. \tag{3.3}$$

Proof. By the definition of the error $\varepsilon_z^{(i)}$, we have

$$|\varepsilon_z^{(i)}| \le h_1 \sum_{j=0}^{+\infty} |\varepsilon_u^{(i,j)}| + |\tilde\xi^{(i)}| \le h_1 \sum_{j=0}^{+\infty} W_h^{(i,j)} + \tilde\delta_h.$$

The second assertion is analogous. □

Lemma 3.3. *If Assumptions* (C), (C1), (N), (Ξ), ($\bar{\mathrm{U}}$), (Λ), (Λ1) *are satisfied and* $|\varepsilon_u^{(i,j)}| \le W_h^{(i,j)}$, *then*

$$|\varepsilon_u^{(i+1,j)}| \le (1+h_0 M_\lambda) W_h^{(i,j-1)} + \\ + h_0 L_u L_c^* V^{(j)} |\varepsilon_z^{(i)}| + h_0 L_\lambda V^{(i,j)} \left[W_h^{(i,j)} + |\varepsilon_z^{(i)}| \right] + h_0 \delta_h V^{(j)}. \tag{3.4}$$

Proof. Subtracting the both sides of (3.1), (2.4) and applying Assumptions (C1), (N), we get

$$|\varepsilon_u^{(i+1,j)}| \le |\varepsilon_u^{(i,j)}| \left(1 - \frac{h_0}{h_1} c^{(i,j)}[z] + h_0 \lambda^{(i,j)}[u,z] \right) + \frac{h_0}{h_1} c^{(i,j)}[z] |\varepsilon_u^{(i,j-1)}| + \\ + |\bar{u}^{(i,j)} - \bar{u}^{(i,j-1)}| \frac{h_0}{h_1} |\Delta c^{(i,j)}| + h_0 \bar{u}^{(i,j)} |\Delta \lambda^{(i,j)}| + h_0 |\xi^{(i,j)}|,$$

where

$$\Delta c^{(i,j)} = c^{(i,j)}[\bar{z}] - c^{(i,j)}[z], \quad \Delta \lambda^{(i,j)} = \lambda^{(i,j)}[\bar{u}, \bar{z}] - \lambda^{(i,j)}[u,z].$$

The remaining part of the proof is a consequence of Assumptions (Ξ), ($\bar{\mathrm{U}}$), (C), (Λ) and the definition of $W_h^{(i,j)}$. □

Lemma 3.4. *If* $\beta_h = \max\left\{ \delta_h, \hat{\delta}_h \right\}$, *Assumptions* (C), (C1), (Ξ), ($\bar{\mathrm{U}}$), (Λ), (Λ1), (KH), (V1) *are satisfied and*

$$Q \ge M_\lambda + L_u L_c^* \frac{V^{(0)}}{\hat{k}(1-h_1\hat{k})^{N_0}} + L_\lambda \left(V^{(0)} + \frac{V^{(0)}}{\hat{k}(1-h_1\hat{k})^{N_0}} \right) + 1,$$

then

$$W_h^{(i,0)} \ge h_1 \hat{k} \sum_{j=0}^{+\infty} W_h^{(i,j)} + \hat{\delta}_h$$

and

$$W_h^{(i+1,j)} \ge (1+h_0 M_\lambda) W_h^{(i,j-1)} + h_0 L_u L_c^* V^{(j)} \frac{W_h^{(i,0)}}{\hat{k}} + \\ + h_0 L_\lambda V^{(i,j)} \left(W_h^{(i,j)} + \frac{W_h^{(i,0)}}{\hat{k}} \right) + h_0 \delta_h V^{(j)}.$$

Proof. By the definition of the comparison function W_h, we obtain

$$
\begin{aligned}
&h_1\hat{K}\sum_{j=0}^{+\infty} W_h^{(i,j)} + \hat{\delta}_h \leq \\
&\leq h_1\hat{k}\beta_h(1+h_0Q)^i\left(\sum_{j=0}^{i-1} V^{(0)}\frac{(1-h_1\hat{k})^j}{(1-h_1\hat{k})^i} + \sum_{j=i}^{+\infty} V^{(j-i)}\right) + \beta_h \leq \\
&\leq \beta_h\left(V^{(0)}\frac{(1+h_0Q)^i}{(1-h_1\hat{k})^i}\left(1-(1-h_1\hat{k})^{i-1}\right) + \left(V^{(0)}-1\right)(1+h_0Q)^i + 1\right) \leq \\
&\leq \ \beta_h V^{(0)}\frac{(1+h_0Q)^i}{(1-h_1\hat{k})^i} = W_h^{(i,0)}.
\end{aligned}
$$

Denote by RHS the right-hand side of the last assertion. For $j \geq i$, we have the estimates

$$
\begin{aligned}
RHS &\leq \beta_h(1+h_0Q)^i\left((1+h_0M_\lambda)V^{(j-i-1)} + h_0V^{(j)}\frac{L_uL_c^*V^{(0)}}{\hat{K}(1-h_1\hat{k})^i}\right) + \\
&\quad + h_0L_\lambda V^{(j-i)}(1+h_0Q)^i\left(\beta_hV^{(j-i)} + \frac{\beta_hV^{(0)}}{\hat{k}(1-h_1\hat{k})^i}\right) + h_0\beta_hV^{(j)} \leq \\
&\leq \beta_hV^{(j-i-1)}(1+h_0Q)^i\left(1+h_0M_\lambda + h_0\frac{L_uL_c^*V^{(0)}}{\hat{k}(1-h_1\hat{k})^i} + \right. \\
&\quad \left. + h_0L_\lambda V^{(0)}\left[1+\frac{1}{\hat{k}(1-h_1\hat{k})^i}\right] + h_0\right) \leq \\
&\leq \beta_hV^{(j-i-1)}(1+h_0Q)^{i+1} = W_h^{(i+1,j)}.
\end{aligned}
$$

In the similar way we proceed for $1 \leq j \leq i$:

$$
\begin{aligned}
RHS &\leq \beta_hV^{(0)}\frac{(1+h_0Q)^i}{(1-h_1\hat{k})^{i-j+1}}(1+h_0M_\lambda + h_0L_\lambda V^{(0)}) + \\
&\quad + h_0\beta_hV^{(0)}\frac{(1+h_0Q)^i}{(1-h_1\hat{k})^{i-j+1}}\left(\frac{L_uL_c^*V^{(0)}}{\hat{k}(1-h\hat{k})^i} + \frac{L_\lambda V^{(0)}}{\hat{k}(1-h\hat{k})^i} + 1\right) \leq \\
&\leq \beta_hV^{(0)}\frac{(1+h_0Q)^{i+1}}{(1-h_1\hat{k})^{i+1-j}} = W_h^{(i+1,j)}.
\end{aligned}
$$

The proof is complete. □

Theorem 3.5. *If Assumptions* (C), (C1), (N), (KH), (Ξ), ($\bar{\mathrm{U}}$), (Λ), (Λ1), (V), (V1) *are satisfied, then* $\left\|\bar{u}^{(i,\cdot)} - u^{(i,\cdot)}\right\|$, $\left\|\bar{u}^{(i,\cdot)} - u^{(i,\cdot)}\right\|_1 \to 0$, *as* $h \to 0$.

Proof. By Lemma 3.2, we have

$$|\varepsilon_z^{(i)}| \le h_1 \sum_{j=1}^{+\infty} W_h^{(i,j)} + \hat{\delta}_h/\hat{k} \le W_h^{(i,0)}/\hat{k}.$$

Hence $W_h^{(i,j)}$ satisfies comparison inequalities (3.3), (3.4) with respect to $\varepsilon_u^{(i,j)}$ with β_h and Q given in Lemma 3.4. Thus we have the pointwise estimates

$$|\varepsilon_u^{(i,j)}| \le W_h^{(i,j)} \quad \text{on} \quad E_h.$$

From this inequality we get the l^∞ and l^1 estimates:

$$\|\varepsilon_u^{(i,\cdot)}\| \le \beta_h V^{(0)} \frac{(1+h_0Q)^i}{(1-h_1\hat{k})^i},$$
$$\|\varepsilon_u^{(i,\cdot)}\|_1 \le \beta_h (1+h_0Q)^i \left(V^{(0)} \frac{1-(1-h_1\hat{k})^{i+1}}{\hat{k}(1-h_1\hat{k})^i} + \|V\|_1 \right).$$

Since β_h tends to 0, as $h \to 0$, it is seen that the error $\varepsilon_u^{(i,j)}$ does so in l^∞ and l^1. □

4. NUMERICAL EXPERIMENTS

Since practical computations cannot be performed in unbounded domains, we truncate the area to a sufficiently large bounded region. The number of knots for each time-layer is equal to N_h, depending on the discretization parameter $h = (h_0, h_1)$. Define the discretization errors in the following way

$$\Delta u = \max_{\substack{i=0,\dots,N_0 \\ j=0,\dots,N_h}} \left| u^{(i,j)} - \tilde{u}\big(t^{(i)}, x^{(j)}\big) \right|, \quad \Delta z = \max_{i=0,\dots,N_0} \left| z^{(i)} - \tilde{z}\big(t^{(i)}\big) \right|,$$

where the discrete functions u, z, obtained by the difference scheme (2.1)–(2.3), approximate the solution $\tilde{u}$ and the function $\tilde{z}$. The results of experiments are presented in the tables. In both examples we assume that $h_1 = 2h_0$.

Example 4.1. Let

$$\lambda(t,x,u,z) = 1/(1+t) - 2ux(\sin x + 1)\sin^2 z/(1+x^2),$$
$$c(t,x,z) = (t+1)(\sin x + 1)\sin^2 z/(1+x^2),$$
$$k(t,x) = 2e(\cos x + 1)/(\pi(e+1)),$$
$$v(x) = 1/(1+x^2).$$

The solution of problem (1.1)–(1.4) with the above functions is equal to $\tilde{u}(t,x) = \dfrac{t+1}{1+x^2}$. Easy calculation gives $\tilde{z}(t) = \dfrac{\pi}{2}(t+1)$.

Table 1

h_0	$h_1 N_h$	Δu	Δz
0.01	50	0.012012	0.025081
	100	0.009782	0.002658
0.005	100	0.006021	0.012510
	200	0.004989	0.001289

Example 4.2. Consider problem (1.1)-(1.4) with the functions

$$\begin{aligned} \lambda(t,x,u,z) &= -2(1+t)u - 2ux(1+t)(\sin x+1)\sin^2 z/(1+x^2), \\ c(t,x,z) &= (t+1)(\sin x+1)\sin^2 z/(1+x^2), \\ k(t,x) &= \frac{2(\cos x+1)}{\pi(1+t)(\exp(-1-t)+1)}, \\ v(x) &= 1/(1+x^2), \end{aligned}$$

whose solution is equal to $u(t,x) = \dfrac{1}{(1+t)^2+x^2}$ and $z(t) = \dfrac{\pi}{2(t+1)}$.

Table 2

h_0	$h_1 N_h$	Δu	Δz
0.01	50	0.006444	0.021513
	100	0.004207	0.009745
	200	0.004537	0.004752
0.005	50	0.007555	0.022450
	100	0.003249	0.010740
	200	0.002160	0.004867
	250	0.002154	0.003708

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