STEERING PROJECTIONS IN VON NEUMANN ALGEBRAS

Adam Wegert

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Abstract. A steering projection of an arbitrary von Neumann algebra is introduced. It is shown that a steering projection always exists and is unique (up to Murray-von Neumann equivalence). A general decomposition of arbitrary projections with respect to a steering projection is established.

Keywords: Murray-von Neumann order, central projection, steering projection.

Mathematics Subject Classification: 46L10.

1. INTRODUCTION

The Murray-von Neumann order (in the set of all (equivalence classes of) projections) is a useful tool in studies of $W^*$-algebras. For example, this order is involved to define types of von Neumann algebras. In most classical textbooks on operator algebras (see e.g. [1, 4, 5, 7, 8, 10]) this order is introduced mainly for this purpose. However, even from a purely set-theoretic point of view the Murray-von Neumann order is an interesting subject of investigation. Although it was carefully studied by Tomiyama ([11]) and Griffin Jr. ([2, 3]) in general von Neumann algebras, none of the books mentioned above discusses this topic so generally. The aim of this paper is to give a new form as well as new proofs of the results by Tomiyama and Griffin Jr.

Our approach concentrates on distinguishing a certain projection, called by us steering, in an arbitrary von Neumann algebra, which turns out to be unique up to Murray-von Neumann equivalence. The key property of this projection is the following: it allows decomposing any other projection into parts each of which is a ‘cardinal multiple’ of a part of the steering projection except one part which is a refinement – see Theorem 4.4 in Section 4. This is the main difference between our approach and those of [2, 3, 11]. Moreover, for a reader familiar with Tomiyama’s results, it may be nontrivial that homogenous projections (in the terminology of [11]) of different dimensions actually come from a common (steering) projection – which is an immediate
consequence of our main result. From Theorem 4.4 one can derive that the general-
ized dimension function (see [11]) of the steering projection in a properly infinite von
Neumann algebra is equal to $\aleph_0$ on $\Pi_\infty$ and $\mathrm{III}$ summands and 1 elsewhere. So, as
a consequence – in that case the steering projection turns out to be $\aleph_0$-homogenous
which may be not transparent in the $\Pi_\infty$ case, since our definition in that case is in
a totally different spirit.

In part of the presentation we involve traces on von Neumann algebras of type $\Pi_1$.
To reduce the size of the paper, the existence and fundamental properties of them
are stated without proofs. For details the reader is referred to any of the books cited
above.

The presented material mainly comes from [6]. The idea of steering projections
was introduced therein. Also the approach to the so-called dimension theory of
$W^*$-algebras by means of steering projections comes from that paper.

2. PRELIMINARIES

Throughout this paper we will use the following notation: $\mathfrak{A}$ will (usually) stand for a
von Neumann algebra. The center of $\mathfrak{A}$ will be denoted by $\mathfrak{Z}(\mathfrak{A})$ and by $\mathfrak{E}(\mathfrak{A})$ we denote
the lattice of all projections in $\mathfrak{A}$. Typical projections in $\mathfrak{A}$ will be denoted usually by
$P, Q, R \ldots$ while projections in $\mathfrak{Z}(\mathfrak{A})$ will be denoted by $Z$. For a projection $P \in \mathfrak{A}$, $C_P$
will stand for the central carrier of $P$; that is, $C_P$ is the smallest projection $Z \in \mathfrak{Z}(\mathfrak{A})$
such that $P \leq Z$. We briefly recall the basic facts about von Neumann algebras which
will be used constantly:

- The center of any von Neumann algebra is a von Neumann algebra as well.
- Let $P \in \mathfrak{A}$ be a projection in a von Neumann algebra. Then $P\mathfrak{A}P$ is also a von
Neumann algebra.
- $\mathfrak{Z}(P\mathfrak{A}P)$, the center of $P\mathfrak{A}P$, coincides with $\mathfrak{Z}(\mathfrak{A})P$.
- Let $P \in \mathfrak{A}$ be a (nonzero) projection. Then $P$ is properly infinite if and only if
$P = \sum_{n \in \mathbb{N}} P_n$ where each $P_n \sim P$.

**Definition 2.1.** We say that two projections $P, Q \in \mathfrak{A}$ are Murray-von Neumann
equivalent if there exists $V \in \mathfrak{A}$ such that:

$$V^*V = P, \quad VV^* = Q. \quad (2.1)$$

Then we will write $P \sim Q$. Given two projections $P, Q$ in $\mathfrak{A}$ we will write $P \preceq Q$ iff
$P \sim Q_0$ for a certain subprojection $Q_0 \leq Q$ belonging to $\mathfrak{A}$.

The next theorem is one of the most important tools in dealing with Murray-von
Neumann order. Its proof may be found in [5] (Theorem 6.2.7) or [10] (Chapter V,
Theorem 1.8).

**Theorem 2.2** (Comparison Theorem). For any pair $P, Q \in \mathfrak{A}$ of projections there
exists a central projection $Z \in \mathfrak{A}$ such that

$$QZ \preceq PZ, \quad P(I - Z) \preceq Q(I - Z).$$
Definition 2.3. For a cardinal number \( \alpha \), a projection \( P \in \mathfrak{A} \) is called \( \alpha \)-decomposable (in \( \mathfrak{A} \)) if the cardinality of any family of mutually orthogonal nonzero subprojections \( Q_i \in \mathfrak{A} \) of \( P \) is at most \( \alpha \). For \( \alpha = \aleph_0 \) we call \( P \) countably decomposable.

The following result will allow us to define the so-called steering projection in a type III von Neumann algebra. For the proof see [5, Theorem 6.3.4].

Theorem 2.4. Suppose that \( P, Q \in \mathfrak{A} \) are two projections, \( P \) is properly infinite and \( Q \) is countably decomposable. If \( C_Q \leq C_P \), then \( Q \preceq P \).

Recall that a projection \( P \) in a von Neumann algebra \( \mathfrak{A} \) acting on a Hilbert space \( H \) is called cyclic when \( P(H) = \mathfrak{A}x \) for some vector \( x \in H \). Then we have the following result (Lemma 6.3.9 in [5]):

Theorem 2.5. Suppose that \( \{ P_i \}_{i \in J_1} \) and \( \{ Q_j \}_{j \in J_2} \) are two infinite families consisting of mutually orthogonal nonzero projections, each \( P_i \) is cyclic and \( Q \preceq P \) where \( Q := \sum_{j \in J_2} Q_j \), \( P := \sum_{i \in J_1} P_i \). Then \( |J_1| \leq |J_2| \). If moreover each \( Q_j \) is cyclic and \( P \sim Q \) then \( |J_1| = |J_2| \).

Remark 2.6. Theorem 2.5 is valid for infinite orthogonal families of cyclic projections as well as for families of countably decomposable projections. This follows from the fact that each nonzero projection is the sum of a family of nonzero cyclic projections (see for instance [4, Theorem 5.5.9]) – so, in case of a countably decomposable projection this family is countable. So, an infinite family of nonzero countably decomposable projections may be replaced by a family of nonzero cyclic projections without changing the sum and the cardinality.

For the proof of the next result consult [5, Theorem 6.3.11].

Theorem 2.7 (Generalized Invariance of Dimension). Suppose that \( P, R \in \mathfrak{A} \) are two projections and \( R \) is finite and nonzero. Let \( \mathfrak{P} = \{ P_i \}_{i \in J_1} \) and \( \mathfrak{Q} = \{ Q_j \}_{j \in J_2} \) be two orthogonal families of subprojections of \( P \) maximal with respect to the property:

\[ P_i \sim Q_j \sim R, \quad i \in J_1, \quad j \in J_2. \]

Then \( |J_1| = |J_2| \).

Definition 2.8. A projection \( P \in \mathfrak{A} \) is called abelian if \( P \mathfrak{A} P \) is a commutative von Neumann algebra.

We briefly recall some basic properties of abelian projections (see [5] for a detailed presentation).

Let \( P, Q, R \in \mathfrak{A} \) be three projections with \( P \) and \( R \) abelian. Then:

- if \( Q \preceq P \), then \( Q = C_QP \),
- \( P \) is finite,
- if \( Q \sim P \), then \( Q \) is also abelian,
- if \( C_R = C_P \), then \( R \sim P \).

The next result is of key importance. For the proof (of a slightly more general version than that presented below) see [5, Theorem 6.5.2].
Theorem 2.9 (Type Decomposition). Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $H$. There are mutually orthogonal central projections $Z^I, Z^{II_1}, Z^{II_\infty}, Z^{III}$ with sum $I$ and such that either $Z^\varepsilon = 0$ or $\mathfrak{A}Z^\varepsilon$ is a type $\varepsilon$ von Neumann algebra where $\varepsilon \in \{I, II_1, II_\infty, III\}$.

3. STEERING PROJECTION

The aim of this part is to define a steering projection in an arbitrary von Neumann algebra, which, in a sense, controls the Murray-von Neumann order in the algebra. Since its definition depends on the type of the algebra, it will be given separately for each type. The simplest cases include types I and II$_1$, whereas the case of type II$_\infty$ algebras is most involved.

Definition 3.1.

(a) If $\mathfrak{A}$ is a type II$_1$ von Neumann algebra, we will simply call the identity of $\mathfrak{A}$ the steering projection.

(b) If $\mathfrak{A}$ is a type I von Neumann algebra, a projection $P \in \mathfrak{A}$ is called a steering projection if $P$ is abelian and $C_P = I$.

Definition 3.2 ([6]). A von Neumann algebra $\mathfrak{A}$ is called quasi-abelian if for every projection $P \in \mathfrak{A}$ the relation $P \sim C_P$ holds. A projection $P$ is called quasi-abelian if $P = 0$ or $P\mathfrak{A}P$ is a quasi-abelian von Neumann algebra.

Definition 3.3. Assume $\mathfrak{A}$ is a type III von Neumann algebra. A projection $P \in \mathfrak{A}$ is called a steering projection if $P$ is quasi-abelian and $C_P = I$.

The existence as well as uniqueness (up to equivalence) of steering projections in type I and II$_1$ von Neumann algebras can easily be established. It appears that similar properties hold for type III algebras, but they are not so obvious. To establish them, we need the following auxiliary result.

Lemma 3.4. If $P \in \mathfrak{A}$ is a projection, then the following conditions are equivalent:

(i) $P$ is quasi-abelian,

(ii) $Q \sim PC_Q$ for every subprojection $Q \leq P$,

(iii) for a projection $Q$ we have that $P \preceq Q \iff P \leq C_Q$.

Proof. Suppose that $P$ is quasi-abelian and $Q \leq P$. Since $P\mathfrak{A}P$ is quasi-abelian, $Q \sim_{P\mathfrak{A}P} C'_Q$ where the latter projection is the central carrier of $Q$ computed with respect to $P\mathfrak{A}P$. But $C'_Q = PC_Q$, so $Q \sim PC_Q$. This shows that (ii) follows from (i). The reverse implication is proved similarly.

To show that (iii) follows from (ii), suppose that

$$P \leq C_Q.$$  \hspace{1cm} (3.1)

If $P = 0$, we are done; so, let $P \neq 0$. Let $\{P_i\}_{i \in I}$ be a maximal family of projections with the following properties:
a) each $P_i$ is a nonzero subprojection of $P$,
b) $P_i \preceq Q$ for every $i \in \mathcal{I}$,
c) for $i \neq j$ we have $PC_iP_j = 0$ (in particular, $P_i$’s are mutually orthogonal).

It follows from the maximality and (3.1) that $P = \sum_{i \in \mathcal{I}} PC_i$. Note also that $C_{PC_iP_j} = P_iC_iP_jC = 0$ (because $C_iC_jP = 0$) which means that $\{C_iC_i\}_{i \in \mathcal{I}}$ is orthogonal. From our assumption, condition $P_i \preceq P$ implies $P_i \sim C_iP_i$; thus $C_iP_i \preceq Q$ and $(C_iC_i)C_iP_i = C_iP_i \preceq C_iC_iP$. This yields

$$P = \sum_{i \in \mathcal{I}} C_iP_i \preceq \sum_{i \in \mathcal{I}} (C_iC_i)Q \leq Q.$$ 

The reverse implication in (iii) needs no additional assumptions: $P \preceq Q$ implies $P \preceq C_iQ$ and thus $P \preceq C_iQ$.

It remains to prove (ii) under the assumption of (iii): let $Q \preceq P$ and consider the projection $R := Q + (I - C_iQ)P$. Then

$$C_R = C_{Q+(I-C_i)Q}P = C_Q \vee C_{(I-C_i)Q}P = C_QCP \vee (I - C_i)CP$$

$$= \left(C_Q \vee (I - C_i)\right)CP = CP \geq P.$$ 

This yields, from (iii), that $P \preceq R$, thus $C_iP \preceq C_iQ = C_QQ + C_Q(I-C_i)P = Q \preceq C_iP$, i.e. $Q \sim C_iP$. □

**Theorem 3.5.** Suppose $\mathfrak{A}$ is a type III von Neumann algebra. Then $\mathfrak{A}$ has a steering projection. Any two steering projections are equivalent.

**Proof.** Let $\{P_i\}_{i \in \mathcal{I}}$ be a maximal family of nonzero projections with properties:

a) $C_iC_iP_i = 0$ for $i \neq j$,
b) each $P_i$ is countably decomposable.

Put $P := \sum_{i \in \mathcal{I}} P_i$ and observe that $CP = I$. This follows from the maximality of the above family and the fact that every nonzero projection contains a nonzero subprojection which is countably decomposable (recall that any cyclic projection is such). Now observe that each $P_i$ is quasi-abelian: taking into account the previous lemma, it is enough to show that for a projection $Q_i \preceq P_i$ we have $Q_i \sim C_iP_i$. Since we are working in a type III von Neumann algebra, all projections are properly infinite, so, by Theorem 2.4, it is enough to show that $C_iC_iP_i = C_iC_i$. But the latter is equal to $C_iC_iP_i = C_i$. It remains to note that $P$ is also quasi-abelian. In fact, for $Q \in P\mathfrak{A}P$ of the form $Q = \sum_{i \in \mathcal{I}} P_iQ_0P_i$ we have that:

$$C_iQ = C\sum_{i \in \mathcal{I}} P_iQ_0P_i = \sum_{i \in \mathcal{I}} C_iP_iQ_0P_i \sim \sum_{i \in \mathcal{I}} P_iQ_0P_i = Q.$$ 

For uniqueness, note that if $P, Q$ are steering, then both are quasi-abelian and $P \preceq C_i = I = C_i \preceq Q$, thus, by the previous lemma, $P \preceq Q$ and $Q \preceq P$ and consequently $P \sim Q$. □
We turn to deal with the last case – when $\mathfrak{A}$ is a type $\mathrm{II}_\infty$ von Neumann algebra. First, let us introduce the following notation: for a projection $Q \in \mathfrak{A}$ and $n \in \{1, 2, \ldots \} \cup \{\omega\}$, by $n \odot Q$ we denote any projection of the form $\sum_{k=1}^{n} Q_k$ where each $Q_k \sim Q$ (when $n = \omega$, we think of the sum $\sum_{k=1}^{\infty} Q_k$). If $n = 0$ we put simply $n \odot P := 0$. For example, any properly infinite projection $P$ could be written as $P = \omega \odot P$.

We shall first establish some properties of the trace concerning the Murray-von Neumann order. They will be applied later. (Confront Lemma 3.7 with Theorem 8.4.4 in [5].)

**Theorem 3.6.** Let $\mathfrak{A}$ be a finite von Neumann algebra and $\operatorname{tr} : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A})$ denote its (unique) trace. If $P, Q \in \mathcal{E}(\mathfrak{A})$, then:

(a) $P \leq Q \iff \operatorname{tr}(P) \leq \operatorname{tr}(Q)$,

(b) $P \sim Q \iff \operatorname{tr}(P) = \operatorname{tr}(Q)$.

**Lemma 3.7.** If $\mathfrak{A}$ is a type $\mathrm{II}_1$ von Neumann algebra, $\operatorname{tr} : \mathfrak{A} \to \mathcal{Z}(\mathfrak{A})$ is its trace, $P \in \mathfrak{A}$ is a projection and $Z \in \mathfrak{A}$ is a central element (not necessarily a projection) with $0 \leq Z \leq \operatorname{tr}(P)$, then there is a projection $Q \leq P$ such that $\operatorname{tr}(Q) = Z$.

**Proposition 3.8.** Suppose that $\mathfrak{A}$ is a type $\mathrm{II}_\infty$ von Neumann algebra and $E$ is a finite projection in $\mathfrak{A}$. Let $\operatorname{tr} : E\mathfrak{A}E \to \mathcal{Z}(E\mathfrak{A}E)$ be the trace on $E\mathfrak{A}E$. If $P$ and $Q$ are two projections in $E\mathfrak{A}E$ and $\operatorname{tr}(P) \leq n \cdot \operatorname{tr}(Q)$ (where $n$ is a positive integer), then $P \leq n \odot Q$ in $\mathfrak{A}$.

**Proof.** We will show the existence of a sequence of mutually orthogonal projections $P_1, P_2, \ldots, P_n$ such that $\sum_{i=1}^{n} P_i = P$ and $\operatorname{tr}(P_i) = \frac{1}{n} \operatorname{tr}(P)$ for $i = 1, 2, \ldots, n$. Using Lemma 3.7 we find a projection $P_1 \leq P$ with $\operatorname{tr}(P_1) = \frac{1}{n} \operatorname{tr}(P)$. Suppose that we have already constructed mutually orthogonal projections $P_1, \ldots, P_k$ ($k < n$) with the properties: $\sum_{i=1}^{k} P_i \leq P$ and $\operatorname{tr}(P_i) = \frac{1}{n} \operatorname{tr}(P)$. Then $P - \sum_{i=1}^{k} P_i$ is a projection with:

$$\operatorname{tr}(P - \sum_{i=1}^{k} P_i) = \frac{n-k}{n} \operatorname{tr}(P) \geq \frac{1}{n} \operatorname{tr}(P).$$

Again, by Lemma 3.7 we can find a projection $P_{k+1} \leq P - \sum_{i=1}^{k} P_i$ with $\operatorname{tr}(P_{k+1}) = \frac{1}{n} \operatorname{tr}(P)$. After the construction we know that $P_1 + P_2 + \ldots + P_n \leq P$ and

$$\operatorname{tr}(P_1 + P_2 + \ldots + P_n) = \operatorname{tr}(P_1) + \operatorname{tr}(P_2) + \ldots + \operatorname{tr}(P_n) = n \cdot \frac{1}{n} \operatorname{tr}(P) = \operatorname{tr}(P).$$

From the faithfulness of the trace we conclude that

$$P_1 + P_2 + \ldots + P_n = P.$$

By virtue of Theorem 3.6 we have:

(i) $P_j \sim P_k$ for $j, k \in \{1, \ldots, n\}$ and consequently $P \sim n \odot P_1$,

(ii) $\operatorname{tr}(P_1) \leq \operatorname{tr}(Q)$, thus $P_1 \leq Q$.

From (i) and (ii) we infer that $P \sim n \odot P_1 \leq n \odot Q$. \qed
Lemma 3.9. Let $\mathfrak{A}$ be a type $\Pi_\infty$ von Neumann algebra. For a projection $P \in \mathfrak{A}$ the following conditions are equivalent:

(i) $P$ is finite,

(ii) for any projection $Q \in \mathfrak{A}$, $P \leq CQ$ iff there is a sequence of central projections $
 \{Z_n\}_{n \in \mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} Z_n = I$ and $PZ_n \leq n \odot Q$ for each $n$.

Proof. Assume (ii) holds for a projection $P \in \mathfrak{A}$. Choose $Q_0 \in \mathfrak{A}$ to be a finite projection with $CQ_0 = I$. Since then $P \leq CQ_0$, (ii) allows us to find central projections $Z_1, Z_2, \ldots$ such that $PZ_n \leq n \odot Q_0$. Since $n \odot Q_0$ is finite, thus $PZ_n$ is finite as well and hence $P = \sum_{n \in \mathbb{N}} PZ_n$ is also finite. Conversely, if $P$ is finite and $P \leq CQ$, we can find a family $\{Q_i\}_{i \in \mathcal{I}}$ of mutually orthogonal projections such that $P = \sum_{i \in \mathcal{I}} Q_i$ and for all $i \in \mathcal{I}$ relation $Q_i \leq Q$ holds. Since $P$ is finite, there is a (unique) trace $\text{tr} : \mathfrak{A}P \to 3(P\mathfrak{A}P) (= 3(\mathfrak{A})P)$. For each $Q_i$ we find central (in $\mathfrak{A}$) projections $Z_{i,n,k}$, $n \in \mathbb{N}$, $k = 1, \ldots, 2^n$ such that:

$$\text{tr}(Q_i) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \frac{k}{2^n} Z_{i,n,k}P.$$

(This can simply be deduced from the commutative Gelfand-Naimark theorem and the total disconnectedness of the Gelfand spectrum of $3(\mathfrak{A})$.) From the above equality we obtain $\text{tr}(Z_{i,n,k}P) \leq 2^n \text{tr}(Q_i)$. We infer from Proposition 3.8 that $Z_{i,n,k}P \leq 2^n \odot Q_i \leq 2^n \odot Q$.

Moreover, by $P = \text{tr}(P) = \sum_{i \in \mathcal{I}} \text{tr}(Q_i) = \sum_{i \in \mathcal{I}} \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \frac{k}{2^n} Z_{i,n,k}P$ we have $PZ = P$ and thus $Z \geq P$ where $Z := \bigvee_{i,n,k} Z_{i,n,k}$ (these projections need not be mutually orthogonal!). Reindexing the family $\{Z_{i,n,k}\}_{i,n,k}$ we obtain a collection $\{C_j\}_{j \in \mathcal{J}}$ of central projections with $PC_j \leq n_j \odot Q$ and $C := \bigvee_{j \in \mathcal{J}} C_j \geq P$. Now equip $\mathcal{J}$ with a well ordering and denote by $j_0$ its first element. We put $C_{j_0}' := C_{j_0}$ and if $C_j'$’s are already defined for all $j < l$, then we put $C_l' := C_l(I - \sum_{j<l} C_j)$. Then $C_l' \leq C_l$, the family $\{C'_j\}_{j \in \mathcal{J}}$ is orthogonal and $\sum_{j \in \mathcal{J}} C'_j = C$. We define $C'_{\infty} := I - C$. Then the family $\{C_k\}_{k \in \mathcal{K}}$ where $\mathcal{K} := \mathcal{J} \cup \{\infty\}$ is still orthogonal, $\sum_{k \in \mathcal{K}} C'_k = I$ and $PC'_k \leq n_k \odot Q$ for each $k \in \mathcal{K}$ (since $PC'_{\infty} = P - PC = 0$, we may take any number for $n_{\infty}$, e.g. $n_{\infty} := 1$). Putting $Z_n := \sum_{k \in \mathcal{K}} C'_k$, we get the desired sequence. \qed

Proposition 3.10. Let $\mathfrak{A}$ be a type $\Pi_\infty$ and

$E_\omega(\mathfrak{A}) := \{Q \in \mathfrak{E}(\mathfrak{A}) : Q \sim \omega \odot P \text{ for some finite projection } P\}$.

Then:

(i) if $P \in E_\omega(\mathfrak{A})$ and $Z$ is a central projection, then $PZ \in E_\omega(\mathfrak{A})$,

(ii) for $P \in E_\omega(\mathfrak{A})$ and a properly infinite projection $Q$ we have $P \leq Q \iff P \leq C_Q$,

(iii) if $P \in E_\omega(\mathfrak{A})$ satisfies $C_P = I$, then $Q \sim C_Q P$ for every $Q \in E_\omega(\mathfrak{A})$. 

Proof. (i) Let $P \sim \omega \odot P_0$ where $P_0$ is finite. This allows us to write $P = \sum_{n=1}^{\infty} P_n$ where $P_n \sim P_0$ for $n \in \mathbb{N}$. Then $PZ = \sum_{n=1}^{\infty} P_nZ$ and $P_nZ \sim P_0Z$ for $n \in \mathbb{N}$ (obviously $P_0Z$ is still finite).

(ii) Suppose that $P \leq C_Q$. Take a finite projection $P_0$ such that $P \sim \omega \odot P_0$ and find, using Lemma 3.9, a sequence $\{Z_n\}_{n=1}^{\infty}$ of central projections such that $P_0Z_n \leq n \odot Q$. Since $Q$ is properly infinite, then $Q \sim \omega \odot Q$ and thus we get:

$$P_0Z_n \leq n \odot Q \leq \omega \odot Q \sim Q.$$  

The above holds for any $n \in \mathbb{N}$, thus $\sum_{n=1}^{\infty} P_0Z_n = P_0 \leq Q$. Consequently

$$P \sim \omega \odot P_0 \leq \omega \odot Q \sim Q.$$  

(iii) If $P, Q \in \mathcal{E}_\omega(\mathfrak{A})$, then (in particular) $P, Q$ are properly infinite and $Q \leq C_P(= I)$. So, we can use (ii) to get $Q \leq P$ and consequently $Q \leq C_QP$. As $C_QP \in \mathcal{E}_\omega(\mathfrak{A})$ by (i), in order to prove $C_QP \leq Q$ it is enough to show that $C_Q \geq C_{CQP}$. But $C_{CQP}$ is equal to $C_QC_P = C_Q$ and the assertion follows. \hfill \Box

Definition 3.11. Suppose that $\mathfrak{A}$ is a type $\Pi_\infty$ von Neumann algebra. Then $P \in \mathfrak{A}$ is called a steering projection if $P \in \mathcal{E}_\omega(\mathfrak{A})$ and $C_P = I$.

Theorem 3.12. Let $\mathfrak{A}$ be a type $\Pi_\infty$ von Neumann algebra. Then $\mathfrak{A}$ has a steering projection and any two steering projections are equivalent.

Proof. Since $I$ is properly infinite, for any $Q \in \mathcal{E}(\mathfrak{A})$, $\omega \odot Q$ makes sense. Now take a finite projection $P_0 \in \mathfrak{A}$ such that $C_{P_0} = I$ and form $P := \omega \odot P_0$. Then $P \in \mathcal{E}_\omega(\mathfrak{A})$ and still $C_P = I$. This establishes the existence. To deal with the uniqueness assume that $P, Q$ are both steering. Then they both belong to $\mathcal{E}_\omega(\mathfrak{A})$ and $P \leq C_Q(= I)$ and $Q \leq C_P(= I)$, so we can use Proposition 3.10 to get $P \leq Q$ and $Q \leq P$, i.e. $P \sim Q$. \hfill \Box

We have defined a steering projection in a von Neumann algebra of any of types: $I, \Pi_1, \Pi_\infty, \Pi_\infty$. Now, if $\mathfrak{A}$ is an arbitrary von Neumann algebra, then we define a steering projection to be the sum of steering projections of $\mathfrak{A}Z^1, \mathfrak{A}Z^{\Pi_1}, \mathfrak{A}Z^{\Pi_\infty}, \mathfrak{A}Z^{\Pi_\infty}$ where $Z^\varepsilon$ is defined as in Theorem 2.9 for $\varepsilon \in \{I, \Pi_1, \Pi_\infty, \Pi_\infty\}$.

As a consequence of the results of this section, we get

Theorem 3.13. An arbitrary von Neumann $\mathfrak{A}$ algebra has a steering projection. Any two steering projections in $\mathfrak{A}$ are equivalent. If $P$ is a steering projection in $\mathfrak{A}$, then $C_P = I$ and for any nonzero central $Z \in \mathcal{E}(\mathfrak{A})$ the projection $PZ$ is steering in $\mathfrak{A}Z$.

The proof is left to the reader.

4. GENERAL DECOMPOSITION

Before formulating the main theorem, we need some auxiliary results, which are however interesting in themselves. By Card we mean the class of all cardinal numbers, while Card$_\infty$ is its subclass of infinite cardinals. For $\alpha \in \text{Card}$ by $\alpha^+$ we denote the
immediate successor of $\alpha$. The notation of the form $Q \sim \alpha \odot P$ (where $P$ and $Q$ are projections in a common von Neumann algebra and $\alpha \in \text{Card}$) means that $Q$ may be written in the form $Q = \sum_{s \in S} Q_s$, where $Q_s \sim P$ for all $s \in S$ and $|S| = \alpha$. In particular, $Q \sim 0 \odot P$ is equivalent to $Q = 0$. We start with the following result (which may be seen as a variation of Lemma 1 in [11], Theorem 2 in [2] and Lemma 1.2 in [3]).

**Lemma 4.1.** Let $\mathfrak{A}$ be a von Neumann algebra of type $\varepsilon$ where $\varepsilon \in \{ I, II_\infty, III \}$ and $P$ be a steering projection in $\mathfrak{A}$. Suppose that the projections $Q, Q'$ satisfy: $Q \sim \alpha \odot P$ and $Q' \sim \beta \odot P$ where $\alpha, \beta \in \text{Card}_\infty \cup \{ 0 \}$. Then $Q \sim Q' \iff \alpha = \beta$. If $\mathfrak{A}$ is of type $I$, the assumption for $Q$ being properly infinite can be dropped and $\alpha$ may be taken arbitrary.

**Proof.** The ‘if’ part is obvious. Further, if, for example, $\alpha = 0$, then $0 = Q \sim Q'$, so $Q' = 0$ and $\beta = 0$. We therefore assume that $\alpha, \beta > 0$ and consider three cases, namely when $\mathfrak{A}$ is of type $\varepsilon$ with:

- $\varepsilon = I$; then a steering projection is defined to be abelian, hence finite. Since
  $$\alpha \odot P \sim Q \sim Q' \sim \beta \odot P,$$
  so, by Theorem 2.7 we get $\alpha = \beta$. (Note that we have not used the fact that $\alpha, \beta$ are infinite.)

- $\varepsilon = II_\infty$; in this case $P \sim \omega \odot P_0$ for some finite projection $P_0$. We have then, using the fact that $\alpha, \beta$ are infinite:
  $$Q \sim \alpha \odot P \sim \alpha \odot (\omega \odot P_0) = (\alpha \cdot \aleph_0) \odot P_0 = \alpha \odot P_0,$$
  and similarly $Q' \sim \beta \odot P_0$. As $P_0$ is finite, we are in the same situation as from the previous step and we obtain $\alpha = \beta$ (by Theorem 2.7).

- $\varepsilon = III$; from the proof of Theorem 3.5 we know that $P = \sum_{i \in I} P_i$ where each $P_i$ is countably decomposable and $\{ P_i \}$’s are centrally orthogonal, i.e. $C_{P_i}C_{P_j} = 0$ for $i \neq j$. Choose $i_0 \in I$ and denote for simplicity $Z := C_{P_{i_0}}$. Then
  $$PZ = \sum_{i \in I} P_i Z = P_{i_0} Z + \sum_{i \neq i_0} P_i Z = P_{i_0}$$
  and thus we obtain
  $$QZ \sim (\alpha \odot P) Z \sim \alpha \odot PZ = \alpha \odot P_{i_0} \sim$$
  $$\sim Q' Z \sim (\beta \odot P) Z \sim \beta \odot PZ = \beta \odot P_{i_0}.$$

Now, using Remark 2.6 (see also Theorem 2.5), we obtain $\alpha = \beta$. □

**Proposition 4.2.** Suppose that $\mathfrak{A}$ is a von Neumann algebra of type $\varepsilon$ where $\varepsilon \in \{ I, II_\infty, III \}$ with a steering projection $P$, $Q$ is a properly infinite projection and $\alpha \in \text{Card}_\infty \cup \{ 0 \}$. Assume that the following condition is satisfied: $\alpha \odot P \preceq Q$ and for any central $0 \neq Z \in \mathfrak{E}(\mathfrak{A})$, $QZ$ does not contain $\alpha^+$ copies of $PZ$, i.e. $\alpha^+ \odot PZ \not\preceq QZ$. Then $Q \sim \alpha \odot P$. If $\mathfrak{A}$ is of type $I$, the assumption for $Q$ being properly infinite can be dropped and $\alpha$ may be taken arbitrary.
Proof. Step 1. First we will show that there is a central projection $Z \neq 0$ such that:

$$QZ \sim \alpha \odot PZ.$$  \hfill (4.1)

Case $\alpha = 0$. With our assumptions we have that

$$P \preceq Q \iff C_Q = I. \hfill (4.2)$$

Indeed, if $\varepsilon \in \{I, III\}$, then $P$ is quasi-abelian (even abelian for $\varepsilon = I$), thus $P \preceq Q \iff P \leq C_Q$ (by Lemma 3.4), but the latter inequality is equivalent to $C_P \leq C_Q$. Since $P$ is steering, so $C_P = I$ which yields (4.2). In case $\varepsilon = II_{\infty}$ we are assuming that $Q$ is properly infinite and $P$, being a steering projection, belongs to $\mathfrak{C}_0(\mathfrak{A})$, so again we have $P \preceq Q \iff P \leq C_Q$, this time by Proposition 3.10. The argument as above gives (4.2). As we assumed that $\alpha = 0$, i.e. $Q$ contains no copy of $P$, we have $P \not\approx Q$; so, from (4.2) we obtain $C_Q \neq I$. Taking $Z := I - C_Q \neq 0$ we have

$$QZ = Q - QC_Q = 0 \sim 0 \odot (PZ).$$

Case $\alpha > 0$. From our assumption, there exists an orthogonal family $\{P_i\}_{i \in \mathcal{I}_0}$ with $|\mathcal{I}_0| = \alpha$ such that each $P_i \sim P$, $P_i \leq Q$. We extend this family to a maximal family $\{P_i\}_{i \in \mathcal{I}}$ with all the above properties – the fact that we still have $|\mathcal{I}| = \alpha$ follows from the assumption about $Q$ (in the case when $\varepsilon = I$ and $\alpha$ is finite the first family is already maximal). If it happens that $Q = \sum_{i \in \mathcal{I}} P_i(\approx \alpha \odot P)$ then putting $Z := I$ we get (4.1). Suppose now that $Q' := Q - \sum_{i \in \mathcal{I}} P_i \neq 0$. By the maximality of the family $\{P_i\}_{i \in \mathcal{I}}$ we have $P \not\approx Q'$ and by the Comparison Theorem, we find a nonzero central projection $Z$ such that $Q'Z \prec PZ$. It turns out that this $Z$ can be taken to fulfill (4.1) for $\varepsilon \in \{II_{\infty}, III\}$. Indeed, for those $\varepsilon$ the steering projection $P$ is properly infinite, so $PZ$ is also properly infinite and the same is true for $P_{i_0}Z$ (being equivalent to $PZ$; here $i_0 \in \mathcal{I}$ is fixed). Moreover, $Q'Z \leq P_{i_0}Z$ and since $Q' \perp P_{i_0}$, also $Q'Z \perp P_{i_0}Z$. Putting these facts together we obtain

$$P_{i_0}Z \leq Q'Z + P_{i_0}Z \leq 2 \odot P_{i_0}Z \sim P_{i_0}Z \sim PZ,$$

hence all of the above components are equivalent. In particular,

$$QZ = Q'Z + \sum_{i \in \mathcal{I}} P_i Z = Q'Z + P_{i_0}Z + \sum_{i \neq i_0} P_i Z \sim \alpha \odot P.$$

It remains to investigate the case when $\varepsilon = I$. Now $P$ is no longer properly infinite. As before, we have $Q'Z \preceq PZ$ and

$$QZ = Q'Z + \sum_{i \in \mathcal{I}} P_i Z.$$

If it happens that $Q'Z = 0$ then we are already done, because

$$QZ = \sum_{i \in \mathcal{I}} P_i Z \sim \alpha \odot P.$$
So, assume that $Q'Z \neq 0$ and form $0 \neq Z' := C_{Q'Z} = ZC_{Q'Z} \leq Z$. This implies that

$$Q'Z' = Q'ZZ' \leq PZZ' = PZ' \sim P_{i_0}Z'$$

for each $i_0 \in I$ meaning that there is $P'$ with

$$Q'Z' \sim P' \leq P_{i_0}Z' \leq P_{i_0}.$$ 

Recall that now $P$ is abelian (in particular quasi-abelian) so by Lemma 3.4 we have

$$P' = C_{P'}P_{i_0}.$$  \hspace{1cm} (4.3)

But observe that since $P' \sim Q'Z'$, thus we have

$$C_{P'} = C_{Q'Z'} = Z'C_{Q'} = ZC_{Q'C} = ZC_{Q'} = Z',$$

so (4.3) turns out to be $P' = P_{i_0}Z'$. This means that

$$Q'Z' \sim P' = P_{i_0}Z' \sim PZ',$$

so we have that

$$QZ' = Q'Z' + \sum_{i \in I} PZ' \sim (\alpha + 1) \odot PZ'.$$

In the case when $\alpha$ is infinite this yields $QZ' \sim \alpha \odot PZ'$, while in case of finite $\alpha$ we obtain a contradiction with the fact that $Q'Z \neq 0$. In that case we conclude that $Q'Z = 0$, which was discussed before.

**Step 2.** Let $\{Z_j\}_{j \in J}$ be a maximal family of nonzero central projections such that for each $j \in J$ we have $QZ_j \sim \alpha \odot PZ_j$. Then we obtain

$$Q\left(\sum_{j \in J} Z_j\right) = \sum_{j \in J} QZ_j \sim \sum_{j \in J} \alpha \odot PZ_j \sim \alpha \odot P\left(\sum_{j \in J} Z_j\right).$$

We will show that $\sum_{j \in J} Z_j = I$. Assume the contrary, i.e. $Z := I - \sum_{j \in J} Z_j \neq 0$. Form $\mathfrak{A}_0 := \mathfrak{A}Z'$. Note that $\mathfrak{A}_0$ is again a von Neumann algebra of type $\varepsilon$ and denote by $P' := PZ'$ a steering projection for $\mathfrak{A}_0$ (see Theorem 3.13) and $Q' := QZ'$. Take $0 \neq Z \in \mathfrak{M}(\mathfrak{A}_0)$. In particular, $Z \leq Z'$ and

$$Q'Z = QZ'Z = QZ \quad \text{and} \quad P'Z = PZ'Z = PZ.$$ \hspace{1cm} (4.4)

It follows from our assumption that $QZ$ does not contain $\alpha^+$ copies of $PZ$ so, by (4.4), $Q'Z$ does not contain $\alpha^+$ copies of $P'Z$. Moreover, $Q$ contains $\alpha$ copies of $P$ and consequently $Q' = QZ'$ contains $\alpha$ copies of $PZ' = P'$. So, all assumptions of the theorem are satisfied for $\mathfrak{A}_0$ and $Q'$, thus we can apply Step 1 for $\mathfrak{A}_0$ and find $0 \neq Z_0 \in \mathfrak{M}(\mathfrak{A}_0)$ such that

$$QZ_0 = QZ'Z_0 = QZ_0 \sim \alpha \odot PZ_0 = \alpha \odot PZ_0.$$

Since $Z_0 \leq Z' = I - \sum_{j \in J} Z_j$, thus $Z_0$ is orthogonal to all of $Z_j$'s and this contradicts the maximality of the taken family. As a consequence, $\sum_{j \in J} Z_j = I.$  \hfill \Box
Lemma 4.3. Let $\mathfrak{A}$ be a von Neumann algebra, $(X, \leq_X)$ be a well ordered set with maximal and minimal elements $x_{\text{max}}, x_{\text{min}}$ and $\{Z_x\}_{x \in X} \subset \mathfrak{Z}(\mathfrak{A})$ be a family of central projections. Suppose that this family satisfies the following conditions:

a) if $x \leq_X y$, then $Z_x \geq Z_y$,

b) for any limit element $x \in X \setminus \{x_{\text{min}}\}$, $Z_x = \bigwedge_{y < x} Z_y$.

Denote $W_x := Z_x - Z_x^+$ for $x \neq x_{\text{max}}$ (where $x^+$ is the immediate successor of $x$). Then $\sum_{x \neq x_{\text{max}}} W_x = Z_{x_{\text{min}}} - Z_{x_{\text{max}}}$.

Proof. The proof is by transfinite induction: we claim that for $x \in X$ we have

$$\sum_{y < x} W_y = Z_{x_{\text{min}}} - Z_x. \tag{4.5}$$

For $x = x_{\text{min}}$ both sides of (4.5) are equal to 0 (we use the convention that the summation over the empty set is 0).

For a successor element $x = y^+$ we are assuming that $\sum_{t < x} W_t = Z_{x_{\text{min}}} - Z_y$, so:

$$\sum_{t < x} W_t = \sum_{t < x} W_t + W_y = Z_{x_{\text{min}}} - Z_y + Z_y - Z_{y^+} = Z_{x_{\text{min}}} - Z_x.$$

Finally, let $x \neq x_{\text{min}}$ be a limit element. From the transfinite induction hypothesis we have $\sum_{t < x} W_t = Z_{x_{\text{min}}} - Z_y$ for every $y < x$. We claim that

$$\sum_{t < x} W_t = \bigvee_{y < x} \sum_{t < x} W_t. \tag{4.6}$$

To prove (4.6), observe that obviously $\sum_{t < x} W_t \leq \sum_{t < x} W_t$ and if for each $y < x$ we have $\sum_{t < x} W_t \leq W$, then in particular $W_y \leq W$ (just put $y^+$ in place of $y$), so $W_y W = W_y$ and thus $(\sum_{y < x} W_y) W = \sum_{y < x} W_y W = \sum_{y < x} W_y$ which means that $\sum_{y < x} W_y \leq W$ and proves (4.6). Now we compute:

$$\sum_{t < x} W_t = \bigvee_{y < x} (Z_{x_{\text{min}}} - Z_y) = Z_{x_{\text{min}}} - \bigwedge_{y < x} Z_y = Z_{x_{\text{min}}} - Z_x.$$

Thus, we have established that (4.5) is valid for any $x \in X$. Put $x := x_{\text{max}}$ to get the assertion. $\square$

For the purpose of the next result, let $\Lambda_1 = \text{Card}$, $\Lambda_\Pi = \{0, 1\} \cup \text{Card}_\infty$ and $\Lambda_\Pi = \{0\} \cup \text{Card}_\infty$.

The next result comes from the treatise [6], where it is one of the most important tools, but not the main aim and therefore it was stated with no proof. However, this result is interesting in its own right. Thus, in the opinion of both the authors - of [6] and of the present paper, the proof of this result should be found in the literature.

Theorem 4.4. Let $P$ be a steering projection in a von Neumann algebra $\mathfrak{A}$ and denote by $Z^I, Z^\Pi, Z^\Pi$ the central projections such that $Z^I + Z^\Pi + Z^\Pi = I$ and $\mathfrak{A}Z^i$ is of type $i$ for $i \in \{I, \Pi, \Pi\}$. Additionally, let $Z^\Pi \in Z(\mathfrak{A})$ be such that $Z^\Pi \leq Z^\Pi$, $\mathfrak{A}Z^\Pi$ is of
type II$_1$ and $Z^{II} - Z^{III}$ is properly infinite. Then for each $Q \in \mathcal{E}(\mathfrak{A})$ there is a unique system of central projections

$$\{Z^{I}_\alpha(Q)\}_{\alpha \in \Lambda_I} \cup \{Z^{II}_\alpha(Q)\}_{\alpha \in \Lambda_{II}} \cup \{Z^{III}_\alpha(Q)\}_{\alpha \in \Lambda_{III}}$$

with the following properties:

a) $Z^i = \sum_{\alpha \in \Lambda_i} Z^{i}_\alpha(Q)$ for $i = I, II, III,$

b) for $\alpha \in \Lambda_i$ with $(i, \alpha) \neq (II, 1)$: $Z^{i}_\alpha(Q)Q \sim Q \odot Z^{i}_\alpha(Q)P,$

c) (i) $Z^{II}_\alpha(Q)Q$ is finite,

(ii) if $W$ is a nonzero central projection and $W \leq Z^{II}_\alpha(Q)$, then $WQ \neq 0$,

(iii) $(Z^{II} - Z^{III})Z^{II}_1(Q)P \sim \omega \odot (Z^{II} - Z^{III})Z^{II}_1(Q)Q$.

**Proof.** *Existence. Step 1.* First we deal with the II$_1$ part. Define

$$Z^{II}_1(Q) := \bigvee \{W \leq Z^{II}C_Q : WQ \text{ is finite}\}.$$  

Then $Z^{II}_1(Q)Q$ is finite. Obviously, $Z^{II}_1(Q) \leq C_Q$ and hence if $W \leq Z^{II}_1(Q) \leq C_Q$ is central and nonzero, then $C_{WQ} = WC_Q = W \neq 0$, which yields $WQ \neq 0$. For the last property of point (c), observe that $Z^{II} - Z^{III}$ is properly infinite (or 0), so the same is true for $(Z^{II} - Z^{III})Z^{II}_1(Q)$, thus $(Z^{II} - Z^{III})Z^{II}_1(Q)P$ is a steering projection (if nonzero!) in some type II$_\infty$ von Neumann algebra (call it $\tilde{\mathfrak{A}}$). From the fact that $Z^{II}_1(Q)Q$ is finite, we conclude that $(Z^{II} - Z^{III})Z^{II}_1(Q)Q$ is finite and its central carrier is equal to

$$(Z^{II} - Z^{III})Z^{II}_1(Q)C_Q = (Z^{II} - Z^{III})Z^{II}_1(Q)$$

being the identity of $\tilde{\mathfrak{A}}$. Hence from the properties of steering projections in type II$_\infty$ von Neumann algebras we have that

$$(Z^{II} - Z^{III})Z^{II}_1(Q)P \sim \omega \odot (Z^{II} - Z^{III})Z^{II}_1(Q)Q.$$ 

We also conclude that

$$(Z^{II} - Z^{II}_1(Q))C_Q \leq (Z^{II} - Z^{III})C_Q. \quad (4.7)$$

Indeed, (4.7) is equivalent to

$$Z^{III}C_Q \leq Z^{II}_1(Q)C_Q = Z^{II}_1(Q) \quad (4.8)$$

and this follows from the definition of $Z^{II}_1(Q)$ since $Z^{III}_1Q = Z^{III}C_QQ$ is finite ($\leq Z^{III}_1$) and $Z^{III}C_Q \leq Z^{III}C_Q$.

*Step 2.* Now we set $Z^i_0(Q) := Z^i(I - C_Q)$. It is then immediate that $Z^i_0(Q)Q = 0$ i.e.

$$Z^i_0(Q)Q \sim 0 \odot Z^i_0(Q)P.$$ 

Define

$$E^i := Z^i - Z^i_0(Q), \quad i = I, III,$$

$$E^{II} := Z^{II} - Z^{II}_0(Q) - Z^{II}_1(Q),$$
(note that $E^{{II}}$ is indeed a projection, since $Z_0^{{II}}(Q) \leq I - CQ$ and $Z_1^{{II}}(Q) \leq CQ$). Direct calculations show that $E^i = Z^iCQ$ for $i = I, III$. For $i = II$ we have that

\[
E^{{II}} \leq (Z^{{II}} - Z^{{II}_1})CQ.
\]

In fact, since $Z^{{II}_1}CQ \leq Z^{{II}_1}(Q)CQ$, we have:

\[
E^{{II}} = Z^{{II}}CQ - Z^{{II}_1}(Q) = (Z^{{II}} - Z^{{II}_1}(Q))CQ = (Z^{{II}} - Z^{{II}_1})CQ.
\]

Finally, $E^{{II}}Q$ is properly infinite (or 0); to see this, take a nonzero central projection $W \leq E^{{II}}$. Obviously $W \leq Z^{{II}}$ and also (by the above argument) $W \leq CQ$, thus $W \leq Z^{{II}_1}CQ$. $S$, if $WQ$ is finite, then $W \leq Z^{{II}_1}(Q)$ and simultaneously $W \leq E^{{II}}$ yielding a contradiction. Thus $WQ$ is infinite and this means that $E^{{II}}Q$ is properly infinite.

For $0 \neq \alpha \in \Lambda_i$ such that $(i, \alpha) \neq (II, 1)$ let

\[
E^i_\alpha := \bigvee \{W \leq E^i : W \in \mathcal{Z}(\mathfrak{A}), \alpha \circ WP \preceq WQ\}.
\]

The projections $E^i_\alpha$ have the following properties:

(i) if $\alpha \geq \beta$, then $E^i_\alpha \leq E^i_\beta$ (indeed, if $\alpha \circ WP \preceq WQ$, then of course $\beta \circ WP \preceq WQ$);

(ii) $\alpha \circ E^i_\alpha P \preceq E^i_\alpha$. To see this, write $E^i_\alpha$ as $E^i_\alpha = \bigvee_{s \in S} W_s$ where for each $s \in S$ it holds

\[
\alpha \circ W_sP \preceq W_sQ. \quad (4.9)
\]

We can find an orthogonal family $\{V_s\}_{s \in S} \subset \mathcal{Z}(\mathfrak{A}) \cap \mathcal{E}(\mathfrak{A})$ such that $V_s \leq W_s$ ($s \in S$), and $E^i_\alpha = \sum_{s \in S} V_s$. From (4.9) we conclude that

\[
V_s(\alpha \circ W_sP) \sim \alpha \circ V_sP \preceq V_sW_sQ = V_sQ
\]

and therefore summing up gives

\[
\sum_{s \in S} (\alpha \circ V_sP) \sim \alpha \circ \sum_{s \in S} V_sP = \alpha \circ E^i_\alpha P \preceq \sum_{s \in S} V_sQ = E^i_\alpha Q.
\]

(iii) If $\alpha$ is a limit (and not the first) element\(^1\) in $\Lambda_1 \setminus \{0\}$, $i = I, III$ (in $\Lambda_1 \setminus \{0, 1\}$ for $i = II$), then $E^i_\alpha = \bigwedge\{E^i_\beta : \beta < \alpha, \beta \neq 0, (i, \beta) \neq (II, 1)\}$. For the proof denote by $Z$ the right-hand side of the above relation and note that $E^i_\alpha \leq Z$: indeed, from (i) we have $E^i_\alpha \leq E^i_\beta$ for $\beta < \alpha$, so $E^i_\alpha \leq Z$. For the converse, we will show that $Z$ contains $\alpha \circ PZ$. If $Z = 0$, then it is trivial. For $Z \neq 0$ we have $PZ \neq 0$ since $C_P = I$ and

\[
C_{PZ} = ZC_P = ZI = Z \neq 0.
\]

Clearly (when $\beta \prec \alpha$) $Z \leq E^i_\beta$ and, as $\beta \circ E^i_\beta P \preceq E^i_\beta$, therefore we get:

\[
Z(\beta \circ E^i_\beta P) = \beta \circ PZ \preceq E^i_\beta Z = Z, \quad (4.10)
\]

\(^1\) For example, $\aleph_0$ is a limit cardinal, but it is not a limit element in $\Lambda_{III}$.\)
in other words, $Z$ contains $\beta \circ PZ$. But $\alpha = \sum_{0 \neq \beta < \alpha} \beta$ (in case $i = \Pi$ we exclude $\beta = 1$, all considered $\beta$ will be called admissible). Denote by $\{P_t\}_{t \in T}$ a maximal orthogonal family of subprojections of $Z$ with the property that $P_t \sim PZ$ for $t \in T$ (in particular $P_t \neq 0$). We claim that $|T| \geq \alpha$. To show this, it is enough to show that $|T| \geq \beta$ for each admissible $\beta < \alpha$ (because $\alpha$ is limit). From (4.10), there exists a maximal orthogonal family $\{Q_s\}_{s \in S}$ of subprojections of $Z$ such that $Q_s \sim PZ$ for each $s \in S$ and $|S| \geq \beta$. Arguing as in the proof of Lemma 4.1 we conclude $|T| = |S| \geq \beta$ and we are done.

We have showed that the assumptions of Lemma 4.3 are satisfied. Therefore we can form $Z^i_{\alpha}(Q) := E^i_{\alpha} - E^i_{\alpha^+}$ and with the help of this lemma conclude that:

$$\sum_{\alpha \in \text{Card}_{\infty}} Z^i_{\alpha}(Q) = E^i_{\eta_0}, \; i = \Pi, \Pi$$

and

$$\sum_{\alpha > 0} Z^i_{\alpha}(Q) = E^i_{1}$$

(note that the second term is missing, because $E^i_{\alpha} = 0$ for sufficiently large $\alpha$, for example $\alpha := |E(\mathfrak{A})^+|$).

We claim that

$$Z^i_{\alpha}(Q)Q \sim \alpha \circ Z^i_{\alpha}(Q)P.$$ (4.12)

If $Z^i_{\alpha}(Q) = 0$ then (4.12) is valid, so we assume that $Z^i_{\alpha}(Q) \neq 0$. Define

$$\tilde{Q} := Z^i_{\alpha}(Q)Q \quad \text{and} \quad \tilde{P} := Z^i_{\alpha}(Q)P \neq 0.$$ Then $\tilde{P}$ becomes a steering projection in $\tilde{\mathfrak{A}} := \mathfrak{A}Z^i_{\alpha}(Q)$. As $Z^i_{\alpha}(Q) \leq E^i_{\alpha}$ and $\alpha \circ E^i_{\alpha}P \leq E^i_{\alpha}Q$, we conclude that:

$$Z^i_{\alpha}(Q)(\alpha \circ E^i_{\alpha}P) = \alpha \circ Z^i_{\alpha}(Q)P = \alpha \circ \tilde{P} \leq Z^i_{\alpha}(Q)Q = \tilde{Q}.$$ Now take $W \in \mathfrak{Z}(\tilde{\mathfrak{A}})$. Then in particular $W \in \mathfrak{Z}(\mathfrak{A})$ and $W \leq Z^i_{\alpha}(Q)$. The latter gives $WE^i_{\alpha^+} = 0$. So, if $W \neq 0$, we cannot have $W \leq E^i_{\alpha^+}$ and thus $WQ = W\tilde{Q}$ does not contain $\alpha^+ \circ WP = \alpha^+ \circ W\tilde{P}$. Thus we have shown that all assumptions of Proposition 4.2 are satisfied and therefore

$$\tilde{Q} \sim \alpha \circ \tilde{P}.$$ To complete the proof of the existence, it remains to show that

$$E^i_{1} = E^i_{\eta_0} \quad \text{and} \quad E^i_{\Pi} = E^i, \; i = \Pi, \Pi.$$ (4.13)

(Once we have (4.13) we conclude that:

$$\sum_{\alpha \in \Lambda_1} Z^i_{\alpha}(Q) = Z^i_{\Pi}(Q) + \sum_{\alpha > 0} Z^i_{\alpha}(Q) = Z^i_{\Pi}(Q) + E^i_{1} = Z^i_{\Pi} + E^i = Z^i,$$

$$\sum_{\alpha \in \Lambda_2} Z^i_{\alpha}(Q) = Z^i_{\Pi}(Q) + \sum_{\alpha \geq \eta_0} Z^i_{\alpha}(Q) = Z^i_{\Pi}(Q) + E^i_{\Pi} = Z^i_{\Pi}(Q) + Z^i_{\Pi}(Q) + E^i_{\Pi} = Z^i_{\Pi}.$$
\[
\sum_{\alpha \in \Lambda_3} Z_{\alpha}^{\text{III}}(Q) = Z_0^{\text{III}}(Q) + \sum_{\alpha \geq \aleph_0} Z_{\alpha}^{\text{III}}(Q) = Z_0^{\text{III}}(Q) + E_0^{\text{III}} = Z_0^{\text{III}}(Q) + E_0^{\text{III}} = Z_0^{\text{III}}.
\]

For the proof of (4.13) note that \( E_1^i \leq E^i \) and \( E_{\aleph_0}^i \leq E^i \) as a straightforward consequence of the definition of \( E_{\alpha}^i \). For the reverse inequality it is sufficient to show that \( E^i \) appears in the family defining \( E_{\alpha}^i \) with suitable \( i \) and \( \alpha \), i.e.

\[
E^i P \leq E^i Q, \quad \omega \circ E_{\aleph_0}^i P \leq E_{\aleph_0}^i Q.
\]

(4.14)

Note that, as \( E^i \) are central, \( E^i \leq C_Q \) and \( C_P = I \), then

\[
C_{E^i Q} = E^i C_Q = E^i = E^i C_P = C_{E^i P}
\]

and thus \( E^i P \leq C_{E^i P} = C_{E^i Q} \). For \( i = I \) we have that \( PE^i \) is a steering projection in some von Neumann algebra of type I, in particular it is quasi-abelian, so Lemma 3.4 ensures us that \( E^i P \leq E^i Q \), giving the first part of (4.14).

For \( i = \text{III} \) we have that \( E_{\text{III}}^i \) is a steering projection in some von Neumann algebra of type \( \text{III} \) and again, from Lemma 3.4 we get \( E_{\text{III}}^i P \leq E_{\text{III}}^i Q \). But \( E_{\text{III}}^i P \) is properly infinite, hence \( E_{\text{III}}^i P \sim \omega \circ E_{\text{III}}^i P \) and thus \( \omega \circ E_{\text{III}}^i P \leq E_{\text{III}}^i Q \).

Finally, for \( i = \text{II} \) recall that \( E_{\text{II}} \leq Z_{\text{II}}^i - Z_{\text{III}}^i \) thus \( E_{\text{II}}^i P \) is a steering projection in some von Neumann algebra of type \( \text{II}_{\omega} \). Recall also that \( E_{\text{II}}^i Q \) is properly infinite. This again allows us to conclude that \( E_{\text{II}}^i P \leq E_{\text{II}}^i Q \), this time from Proposition 3.10 (and the definition of a steering projection in a type \( \text{II}_{\omega} \) von Neumann algebra). Moreover, \( E_{\text{II}}^i P \sim \omega \circ E_{\text{II}}^i P \) which gives the remaining part of (4.14).

**Uniqueness.** Suppose that we have two such systems \( \{ Z_{\alpha}^i \} \) and \( \{ W_{\alpha}^i \} \) (with \( \alpha \) and \( i \) varying as before). Our task is to show

\[
Z_{\alpha}^i = W_{\alpha}^i. \quad (4.15)
\]

For the proof of (4.15) with \( \alpha = 0 \) we will show that

\[
Z_0^i = (I - C_Q)Z^i. \quad (4.16)
\]

Once (4.16) is proved, it gives (4.15) with \( \alpha = 0 \) since (4.16) gives an explicit formula for \( Z_0^i \) and the roles of \( Z_{\alpha}^i \) and \( W_{\alpha}^i \) are symmetric.

We have that

\[
Z_0^i Q \sim 0 \circ Z_0^i P = 0
\]

and therefore \( Z_0^i Q = 0 \), hence \( Z_0^i \leq I - C_Q \). Since obviously \( Z_0^i \leq Z^i \), we have

\[
Z_0^i \leq (I - C_Q)Z^i.
\]

We will now show the reverse inequality.

For \( (i, \alpha) \neq (\text{II}, 1) \) we have \( Z_{\alpha}^i Q \sim \alpha \circ Z_{\alpha}^i P \) and if \( W \leq Z_{\alpha}^i \) is central then

\[
WQ = W(Z_{\alpha}^i Q) \sim W(\alpha \circ Z_{\alpha}^i P) = \alpha \circ WP.
\]

\[2\) Here \( \alpha = 1 \) for \( i = I \) and \( \alpha = \aleph_0 \) for \( i = \text{II, III} \).

\[3\) From the definition of the central carrier, or from the fact that \( C_{Z_0^i Q} = Z_0^i C_Q = 0 \).
and this means that \( WQ = 0 \) implies \( WP = 0 \), and as \( CP = I \), it also implies that \( W = 0 \). In particular, for \( W := (I - C)Z\alpha \) (being central) we have \( WQ = 0 \), thus \( W = 0 \) so \( Z\alpha = Z\alpha CQ \) and obviously \( Z\alpha \leq Zi \) which together yields \( Z\alpha = CQZi \).

For \((i,\alpha) = (II, 1)\) the projection \( Z_{II}^1 \) has the property that for every central projection \( 0 \neq W \leq Z_{II}^1 \), \( WQ \neq 0 \). Thus for \( W := (I - C)Z_{II}^1 \) we obtain as before \( W = 0 \) and finally \( Z\alpha \leq CQZ_{II} \). Thus we have proved that \( Z\alpha \leq CQZi \) for \( i = I, II, III \) and \( \alpha \neq 0 \). Summing up over all \( \alpha \neq 0 \) yields:

\[
\sum_{\alpha > 0} Z\alpha = Zi - Z0 \leq CQZi
\]

which means that \( Zi(I - C) \leq Z0 \) and proves (4.16).

Now we will prove (4.15) for \((i,\alpha) = (II, 1)\). For \( \alpha \geq R_0 \) we have \( Z\alpha^II \sim \alpha \circ Z\alpha^II P \) and the latter projection is properly infinite (or \( 0 \))4, so \( Z\alpha^II Q \) is properly infinite or \( 0 \) and the same is true for \( \sum_{\alpha \geq R_0} Z\alpha^II Q \). But the sum appearing in this expression is equal to \( Z\alpha^II - Z0^II - Z1^II \) - this means that \( (Z\alpha^II - Z0^II - Z1^II)Q \) is properly infinite or \( 0 \). Since \( W1^II Q \) is finite, then \( W1^II(Z\alpha^II - Z0^II - Z1^II) \) is also finite. But at the same time, since \( W1^II \) is central, we have that \( W1^II(Z\alpha^II - Z0^II - Z1^II) \) is properly infinite or \( 0 \). Therefore this second possibility takes place:

\[
W1^II(Z\alpha^II - Z0^II - Z1^II)Q = 0.
\]

By taking the central carrier we get

\[
W1^II(Z\alpha^II - Z0^II - Z1^II)CQ = 0
\]

but from the above discussion we know that \( W1^II \leq CQZ_{II} \leq CQ \), hence we get:

\[
W1^II(Z\alpha^II - Z0^II - Z1^II) = 0
\]

therefore

\[
\underbrace{W1^II Z\alpha^II - W1^II(I - C)Q}_{=W1^II} \underbrace{Z\alpha^II - W1^II Z1^II}_{=0} = 0.
\]

This means that \( W1^II = W1^II Z1^II \leq Z1^II \), and by repeating this argument (the roles of \( Z1^II \) and \( W1^II \) are symmetric) we obtain \( W1^II = Z1^II \).

Now we deal with the case \( \alpha \neq 0 \) and \((i,\alpha) \neq (II, 1)\). We claim that

\[
Z\alpha W\beta = 0 \text{ whenever } \beta \in \Lambda_i, \beta \neq \alpha.
\]

For \( \beta = 0 \) or \((i,\beta) = (II, 1)\) it has been already proved, as in these cases \( Z\beta = W\beta \) and \( Z\alpha \)'s are all mutually orthogonal. For \( \beta > 0 \) (in the case \( i = I \)) and \( \beta \in \text{Card}_\infty \) (in the case \( i \neq I \)) as \( Z\alpha Q \sim \alpha \circ Z\alpha P \) we have

\[
W\beta Z\alpha Q \sim \alpha \circ W\beta Z\alpha P.
\]

4) Later we will not always underline this fact, although it should be kept in mind.
But at the same time, from $W^i_\beta Q \sim \beta \odot W^i_\beta P$ we get

$$W^i_\beta Z^i_\alpha Q \sim \beta \odot W^i_\beta Z^i_\alpha P.$$ 

These together yield

$$\alpha \odot W^i_\beta Z^i_\alpha P \sim \beta \odot W^i_\beta Z^i_\alpha P.$$ 

Now $W^i_\beta Z^i_\alpha \neq 0$ implies $W^i_\beta Z^i_\alpha P \neq 0$ (recall that $C_P = I$) and then $W^i_\beta Z^i_\alpha P$ is a steering projection in some von Neumann algebra of an appropriate type. By virtue of Proposition 4.1 we obtain $\alpha = \beta$, thus proving (4.17). From (4.17) we have

$$Z^i_\alpha \sum_{\beta \in \Lambda_i \setminus \{\alpha\}} W^i_\alpha = Z^i_\alpha (Z^i - W^i_\alpha) = Z^i_\alpha - Z^i_\alpha W^i_\alpha = 0$$

and thus $Z^i_\alpha \leq W^i_\alpha$. Changing the role of $Z^i_\alpha$ and $W^i_\alpha$ we get (4.15). The whole proof is complete. 

The above theorem may be seen as a variation of a result of Tomiyama (Theorem 1 in [11]). A careful reader may notice that our construction is essentially in the same spirit as there.

**Corollary 4.5.** Let $Q, Q' \in \mathfrak{A}$ be two projections. Then the following conditions are equivalent:

a) $Q \preceq Q'$,

b) $Z^i_\alpha(Q)Z^i_\beta(Q') = 0$ whenever $\alpha > \beta$; and $Z^i_1(Q)Z^i_1(Q')Q \leq Z^i_1(Q)Z^i_1(Q')Q'$.

**Proof.** Step 1. First we show the following criterion for the Murray-von Neumann equivalence: for two projections $Q, Q' \in \mathfrak{A}$ we have

$$Q \sim Q' \iff Z^i_\alpha(Q) = Z^i_\alpha(Q') \ (i \in \{I, II, III\}, \ \alpha \in \Lambda_i) \text{ and } Z^i_1(Q)Q \sim Z^i_1(Q')Q'.$$

For the proof assume first that $Q \sim Q'$. Then $Z^i_\alpha(Q)Q \sim Z^i_\alpha(Q')Q' \sim \alpha \odot Z^i_\alpha(Q')P$ for $(i, \alpha) \neq (II, 1)$. For $(i, \alpha) = (II, 1)$ we have $Z^i_1(Q)Q \sim Z^i_1(Q')Q'$; since the latter projection is finite, so is $Z^i_1(Q')Q$. Further, $Q \sim Q'$ implies $C_Q = C_{Q'}$ and thus if $0 \neq W \leq Z^i_1(Q')C_Q$ is a central projection, then $0 \neq W \leq Z^i_1(Q')C_{Q'}$ and thus $WQ \neq 0$. Finally, observe that

$$(Z^{II} - Z^{II1})Z^i_1(Q')P \sim \omega \odot (Z^{II} - Z^{II1})Z^i_1(Q')Q' \sim \omega \odot (Z^{II} - Z^{II1})Z^i_1(Q')Q.$$ 

So, we have proved that the system $\{Z^i_\alpha\}_{i, \alpha}(Q')$ all have the desired properties (for $Q$) – from its uniqueness we conclude that $Z^i_\alpha(Q) = Z^i_\alpha(Q')$ for all $i \in \{I, II, III\}$ and $\alpha \in \Lambda_i$ – then also $Z^i_1(Q)Q \sim Z^i_1(Q')Q'$.

For the proof of the converse implication, denote for simplicity $Z^i_\alpha := Z^i_\alpha(Q) = Z^i_\alpha(Q')$. We have then in particular $Z^i_1Q \sim Z^i_1Q'$ and for $(i, \alpha) \neq (II, 1)$:

$$Z^i_\alpha Q \sim \alpha \odot Z^i_\alpha P \sim Z^i_\alpha Q'.$$
Since \( \sum_{i \in \{I, II, III\}} \sum_{\alpha \in \Lambda_i} Z^i_{\alpha} = I \), then

\[
Q = \sum_{i, \alpha} Z^i_{\alpha} Q \sim \sum_{i, \alpha} Z^i_{\alpha} Q' = Q'.
\]

**Step 2.** Now we shall check that (b) is implied by (a). The second part of our statement is immediate. First let \((i, \alpha), (i, \beta) \neq (II, 1)\). Then, as \(Z^i_{\alpha}(Q) \sim \alpha \odot Z^i_{\alpha}(P)\) and \(Z^i_{\beta}(Q') \sim \beta \odot Z^i_{\beta}(Q')P\) and \(Q \preceq Q'\), we have:

\[
\alpha \odot Z^i_{\alpha}(Q)Z^i_{\beta}(Q')P \sim Z^i_{\alpha}(Q)Z^i_{\beta}(Q')Q \preceq Z^i_{\alpha}(Q)Z^i_{\beta}(Q')Q',
\]

but, as \(Z^i_{\beta}(Q')Q' \sim \beta \odot Z^i_{\beta}(Q')P\), we have

\[
Z^i_{\alpha}(Q)Z^i_{\beta}(Q')Q' \sim \beta \odot Z^i_{\alpha}(Q)Z^i_{\beta}(Q')P.
\]

Therefore we have (if we denote, to simplify, \(Z := Z^i_{\alpha}(Q)Z^i_{\beta}(Q')\)) that \(\alpha \odot ZP \preceq \beta \odot ZP\). Now if we assume \(\beta < \alpha\), then the converse inequality is also true, namely \(\beta \odot ZP \preceq \alpha \odot ZP\), hence \(\alpha \odot ZP \sim \beta \odot ZP\). Now suppose, on the contrary, that \(Z \neq 0\). Then \(ZP \neq 0\) too and \(ZP\) is then a steering projection in some von Neumann algebra, and then from Lemma 4.1 we conclude \(\alpha = \beta\) which is a contradiction.

Further, we see that \(Z^I_{\alpha}(Q)Z^I_{\beta}(Q')Q \preceq Z^I_{\alpha}(Q)Z^I_{\beta}(Q')Q' = 0\) and hence, by point (c) of Theorem 4.4 for \(Q, Z^I_{\alpha}(Q)\), \(Z^I_{\beta}(Q') = 0\). Finally, if \(\alpha \in \text{Card}_{\infty}\), then \(Z^I_{\alpha}(Q)Z^I_{\alpha}(Q')Q \preceq Z^I_{\alpha}(Q')Q'\), which implies that \(Z^I_{\alpha}(Q)Z^I_{\alpha}(Q')Q\) is both properly infinite (since \(Z^I_{\alpha}(Q)Q\) is such) and finite (since \(Z^I_{\alpha}(Q')Q\) is finite). We infer that \(Z^I_{\alpha}(Q)Z^I_{\alpha}(Q')Q = 0\) and consequently \(Z^I_{\alpha}(Q)Z^I_{\alpha}(Q') = 0\), because \(Z^I_{\alpha}(Q) \leq C_Q\). This finishes the proof of (b).

**Step 3.** For the proof that (a) follows from (b), note that, since

\[
\sum_{i} \sum_{\alpha, \beta \in \Lambda_i} Z^i_{\alpha}(Q)Z^i_{\beta}(Q') = I,
\]

so it is enough to show that

\[
Z^i_{\alpha}(Q)Z^i_{\beta}(Q')Q \preceq Z^i_{\alpha}(Q)Z^i_{\beta}(Q')Q', \quad \alpha, \beta \in \Lambda_i.
\]

Fix \(i \in \{I, II, III\}\) and \(\alpha, \beta \in \Lambda_i\) and consider the cases:

1°) \(\alpha > \beta\); then from our assumptions \(Z^i_{\alpha}(Q)Z^i_{\beta}(Q') = 0\) and (4.18) is satisfied.
2°) \((i, \alpha) = (i, \beta) = (II, 1)\); then (4.18) is exactly the second part of our assumption.
Adam Wegert

(3°) \( \alpha \leq \beta \) and \( (i, \beta) = (\Pi, 1), (i, \alpha) \neq (\Pi, 1) \). Then, in particular, \( i = \Pi \) and the only possibility is that \( \alpha = 0 \). In this case \( Z^i_\alpha(Q)Q \sim 0 \circ Z^i_\alpha(Q)P = 0 \), so \( Z^i_\alpha(Q)Z^i_\beta(Q')Q = 0 \) and (4.18) is satisfied.

(4°) Both \((i, \alpha)\) and \((i, \beta)\) are different from \((\Pi, 1)\) and \( \alpha \leq \beta \). Then, as before, we conclude from the relations \( Z^i_\alpha(Q)Q \sim \alpha \circ Z^i_\alpha(Q)P \) and \( Z^i_\beta(Q')Q' \sim \beta \circ Z^i_\beta(Q')P \) that

\[
Z^i_\alpha(Q)Z^i_\beta(Q')Q \sim \alpha \circ Z^i_\alpha(Q)Z^i_\beta(Q')P,
\]

\[
Z^i_\alpha(Q)Z^i_\beta(Q')Q' \sim \beta \circ Z^i_\alpha(Q)Z^i_\beta(Q')P.
\]

Since \( \alpha \leq \beta \) then:

\[
\alpha \circ Z^i_\alpha(Q)Z^i_\beta(Q')P \leq \beta \circ Z^i_\alpha(Q)Z^i_\beta(Q')P
\]

and (4.18) is satisfied.

(5°) The remaining case is \( \alpha \leq \beta, (i, \alpha) = (\Pi, 1), (i, \beta) \neq (\Pi, 1) \). In this case \( i = \Pi \) and hence \( \beta > \alpha \) means that \( \beta \geq \mathfrak{h}_0 \). As we have \( Z^\Pi_\beta(Q')Q' \sim \beta \circ Z^\Pi_\beta(Q')P \) with infinite \( \beta \), we conclude that \( Z^\Pi_\beta(Q')Q' \) is properly infinite (or 0). But \( Z^\Pi_\Pi \) is central and finite, thus \( Z^\Pi_\Pi Z^\Pi_\beta(Q')Q' = 0 \). Taking the central carrier gives

\[
Z^\Pi_\Pi Z^\Pi_\beta(Q')C_{Q'} = 0.
\]

But recall\(^5\) that \( Z^\Pi_\Pi(Q') \leq C_{Q'} \). Thus (4.19) transforms into \( Z^\Pi_

\Pi Z^\Pi_\beta(Q') = 0 \).

We conclude that \( Z^\Pi_\beta(Q') \leq Z^\Pi - Z^\Pi_\Pi \) and then (by point (c) of Theorem 4.4 for \( Q \)) \( Z^\Pi_\beta(Q')Z^\Pi_1(Q)P \sim \omega \circ Z^\Pi_\beta(Q')Z^\Pi_1(Q)Q \). So:

\[
Z^\Pi_\beta(Q')Z^\Pi_1(Q)Q \leq Z^\Pi_\beta(Q')Z^\Pi_1(Q)P \leq \beta \circ Z^\Pi_\beta(Q')Z^\Pi_1(Q)P \sim Z^\Pi_\beta(Q')Z^\Pi_1(Q)Q'
\]

and we are done.

\[\square\]

Remark 4.6. Recently Sherman \([9]\) proved (using Lemma 3.7 and a variation of Theorem 4.4) that the Murray-von Neumann order in an arbitrary \( W^* \)-algebra is complete. That is, every set of projections has the g.l.b. as well as the l.u.b. with respect to ‘\( \preceq \)’.

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\(^5\) It follows from the proof of the previous theorem: \( Z^\Pi_\beta(Q') \leq E^\Pi \leq (Z^\Pi - Z^\Pi_\Pi)C_{Q'} \leq C_{Q'} \).
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Adam Wegert  
Adam.Wegert@im.uj.edu.pl

Jagiellonian University  
Faculty of Mathematics and Computer Science  
ul. Łojasiewicza 6, 30-348 Krakow, Poland

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