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ON DYNAMICAL SYSTEMS INDUCED BY p-ADIC NUMBER FIELDS

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Abstract. In this paper, we construct dynamical systems induced by p-adic number fields $\mathbb{Q}_p$. We study the corresponding crossed product operator algebras induced by such dynamical systems. In particular, we are interested in structure theorems, and free distributional data of elements in the operator algebras.

Keywords: prime fields, p-adic number fields, the Adele ring, p-adic von Neumann algebras, p-adic dynamical systems.

Mathematics Subject Classification: 05E15, 11R47, 46L54, 47L15, 47L55.

1. INTRODUCTION

The relations between primes and operator algebra theory have been studied in various different approaches. The study of such relations is (i) to provide new tools in operator algebra, (ii) to apply operator-algebraic techniques (e.g., free probability) to number theory, and (iii) to establish bridges between number theory and operator algebra theory.

For instance, in [4], we studied how primes act "on" certain von Neumann algebras, which are von Neumann algebras generated by Adelic measure spaces. Also, the primes as operators in certain von Neumann algebras have been studied in [5] and [8]. In [6] and [7], we studied primes as linear functionals acting on arithmetic functions, i.e., each prime induces a free-probabilistic structure on arithmetic functions. In such a case, one can understand arithmetic functions as Krein-space operators via Krein-space representations (see [12] and [13]). These studies are all motivated by number-theoretic results (e.g., [3, 14] and [15]), under free probability techniques (e.g., [16, 17] and [19]).

Independently, Arveson studied *histories* as a group of actions induced by *real numbers* $\mathbb{R}$ on (type I subfactors of) $B(H)$, satisfying certain additional conditions, where H is an infinite-dimensional separable *Hilbert space*. By understanding the field

$\mathbb{R}$ as an additive group $(\mathbb{R}, +)$, he defined an E_0-*group* $\Gamma_{\mathbb{R}}$ of $*$-homomorphisms acting on $B(H)$ indexed by $\mathbb{R}$. By putting additional conditions on $\Gamma_{\mathbb{R}}$, he defined a *history* Γ acting on $B(H)$ (e.g., [1, 2] and [8]). In [9], by *framing* a group Γ to groupoids generated by partial isometries, we studied possible *distortions* Γ_G *of a history* Γ. It shows that whenever a history Γ acts on H, a family of partial isometries distorts (or reduces, or restricts) the "original" historical property (in the sense of Arveson) of Γ. And such distortions are completely characterized by *groupoid actions*, sometimes called the E_0-*groupoid actions* induced by partial isometries on $B(H)$. The above framed (E_0-)groupoids Γ_G induce corresponding C^*-subalgebras $C^*(\Gamma_G)$ of $B(H)$, investigated by *dynamical system theory* and *free probability* (e.g., [16, 17] and [19]).

p-adic analysis provides a important tool for studying *non-Archimedean geometry at small distance* (e.g., [18]). It is not only interested in various mathematical fields but also in physics and related scientific areas (e.g., [3–5, 12, 13] and [18]). The *p-adic number fields* (or p-adic number fields) $\mathbb{Q}_p$ and the *Adele ring* $\mathbb{A}_{\mathbb{Q}}$ play key roles in modern *number theory*, *analytic number theory*, *L-function theory*, and *algebraic geometry* (e.g., [3, 14] and [15]).

Like in [1, 2, 8] and [9], providing Archimedean dynamical systems obtained from $\mathbb{R}$, or discrete groups, or discrete groupoids, one may construct non-Archimedean dynamical systems induced by p-adic number fields $\mathbb{Q}_p$, or by certain algebraic structures induced by $\mathbb{Q}_p$, for primes p. This idea motivates our works here.

Especially, we construct a monoid $\sigma(\mathbb{Q}_p) = (\sigma(\mathbb{Q}_p), \cap)$, where $\sigma(\mathbb{Q}_p)$ is the σ-algebra consisting of all Haar-measurable subsets of $\mathbb{Q}_p$, and act it on an arbitrary von Neumann algebra M, to construct a *semigroup* W^**-dynamical system* $(M, \sigma(\mathbb{Q}_p), \alpha)$, where α is a suitable monoidal action of $\sigma(\mathbb{Q}_p)$ acting on M. The construction of such dynamical systems is motivated by those of [1, 2, 8] and [9].

From a semigroup W^*-dynamical system $(M, \sigma(\mathbb{Q}_p), \alpha)$, the corresponding crossed product W^*-algebra,

$$M \times_\alpha \sigma(G_p)$$

is well-defined, as a von Neumann algebra generated by M and $\alpha\left(\sigma(\mathbb{Q}_p)\right)$ satisfying α-relations. We study free probability on such von Neumann algebras and consider connection with number-theoretic results and free-probabilistic data.

2. DEFINITIONS AND BACKGROUND

In this section, we introduce basic definitions and backgrounds of the paper.

2.1. p-ADIC NUMBER FIELDS $\mathbb{Q}_p$ AND THE ADELE RING $\mathbb{A}_{\mathbb{Q}}$

Throughout this paper, we denote the set of all natural numbers (or positive integers) by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, and the set of all rational numbers by $\mathbb{Q}$.

Let us fix a prime p. Define *the p-norm* $|\cdot|_p$ on the rational numbers $\mathbb{Q}$ by

$$|q|_p = \left| p^r \frac{a}{b} \right|_p \overset{def}{=} \frac{1}{p^r},$$

whenever $q = p^r \frac{a}{b} \in \mathbb{Q}^{\times} = \mathbb{Q} \setminus \{0\}$ for some $r \in \mathbb{Z}$ with the convention that $|0|_p \overset{def}{=} 0$ for all primes p.

It is easy to check that:

(i) $|q|_p \geq 0$ for all $q \in \mathbb{Q}$,
(ii) $|q_1 q_2|_p = |q_1|_p \cdot |q_2|_p$ for all $q_1, q_2 \in \mathbb{Q}$,
(iii) $|q_1 + q_2|_p \leq \max\{|q_1|_p, |q_2|_p\}$ for all $q_1, q_2 \in \mathbb{Q}$.

In particular, by (iii), we verify that

(iii)′ $|q_1 + q_2|_p \leq |q_1|_p + |q_2|_p$ for all $q_1, q_2 \in \mathbb{Q}$.

Thus, by (i), (ii) and (iii)′, the p-norm $|\cdot|_p$ is indeed a norm. However, by (iii), this norm is "*non-Archimedean*".

Definition 2.1. We define a set $\mathbb{Q}_p$ by the norm-closure of the normed space $(\mathbb{Q}, |\cdot|_p)$, for all primes p. We call $\mathbb{Q}_p$, the p-adic number field.

For a fixed prime p, all elements of the p-adic number field $\mathbb{Q}_p$ are formed by

$$p^r \left(\sum_{k=0}^{\infty} a_k p^k \right) \quad \text{with} \quad a_k \in \{0, 1, \ldots, p-1\} \tag{2.1}$$

for all $r \in \mathbb{Z}$. For example,

$$-1 = (p-1)p^0 + (p-1)p + (p-1)p^2 + \ldots.$$

The subset of $\mathbb{Q}_p$, consisting of all elements formed by

$$\sum_{k=0}^{\infty} a_k p^k \quad \text{for} \quad a_k \in \{0, 1, \ldots, p-1\},$$

is denoted by $\mathbb{Z}_p$. One can easily verify that, for any $x \in \mathbb{Q}_p$, there exist $r \in \mathbb{Z}$, and $x_0 \in \mathbb{Z}_p$, such that

$$x = p^r x_0,$$

by (2.1). Notice that if $x \in \mathbb{Z}_p$, then $|x|_p \leq 1$, and vice versa, i.e., the subset $\mathbb{Z}_p$ can be re-defined by a subset of $\mathbb{Q}_p$, such that

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}. \tag{2.2}$$

So, the subset $\mathbb{Z}_p$ of (2.2) is said to be the *unit disk of* $\mathbb{Q}_p$. Remark that

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset p^3\mathbb{Z}_p \supset \ldots.$$

It is not difficult to verify that

$$\mathbb{Z}_p \subset p^{-1}\mathbb{Z}_p \subset p^{-2}\mathbb{Z}_p \subset p^{-3}\mathbb{Z}_p \subset \ldots,$$

and hence,

$$\mathbb{Q}_p = \bigcup_{k=-\infty}^{\infty} p^k \mathbb{Z}_p, \quad \text{set-theoretically.} \tag{2.3}$$

Consider the boundary U_p of $\mathbb{Z}_p$. By construction, the boundary U_p of $\mathbb{Z}_p$ is identical to $\mathbb{Z}_p \setminus p\mathbb{Z}_p$, i.e.,

$$U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Z}_p : |x|_p = 1\}. \tag{2.4}$$

We call the subset U_p (2.4) of $\mathbb{Z}_p$ the *unit circle of* $\mathbb{Q}_p$, and all elements of U_p are said to be *units of* $\mathbb{Q}_p$. Similarly, the subsets $p^k U_p$ are the boundaries of $p^k \mathbb{Z}_p$ satisfying

$$p^k U_p = p^k \mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p \quad \text{for all} \quad k \in \mathbb{Z},$$

where $xX \overset{def}{=} \{xy : y \in X\}$, for all $x \in \mathbb{Q}_p$ and $X \subseteq \mathbb{Q}_p$. Therefore, by (2.3) and (2.4), we obtain that

$$\mathbb{Q}_p = \bigsqcup_{k=-\infty}^{\infty} p^k U_p, \quad \text{set-theoretically,} \tag{2.5}$$

where $\bigsqcup$ means the disjoint union. By [18], whenever $q \in \mathbb{Q}_p$ is given, there always exist $a \in \mathbb{Q}_p$, $k \in \mathbb{Z}$, such that

$$q \in a + p^k \mathbb{Z}_p.$$

Fact 2.2 ([18]). *The p-adic number field $\mathbb{Q}_p$ is a locally compact Banach space. In particular, the unit disk $\mathbb{Z}_p$ is compact in $\mathbb{Q}_p$.*

Define now the addition on $\mathbb{Q}_p$ by

$$\left(\sum_{n=-N_1}^{\infty} a_n p^n\right) + \left(\sum_{n=-N_2}^{\infty} b_n p^n\right) = \sum_{n=-\max\{N_1,N_2\}}^{\infty} c_n p^n, \tag{2.6}$$

for $N_1, N_2 \in \mathbb{N}$, where the summands $c_n p^n$ satisfies that

$$c_n p^n \overset{def}{=} \begin{cases} (a_n + b_n)p^n & \text{if } a_n + b_n < p, \\ p^{n+1} & \text{if } a_n + b_n = p, \\ s_n p^{n+1} + r_n p^n & \text{if } a_n + b_n = s_n p + r_n \end{cases}$$

for all $n \in \{-\max\{N_1, N_2\}, \ldots, 0, 1, 2, \ldots\}$. Clearly, if $N_1 > N_2$ (resp. $N_1 < N_2$), then, for all $j = -N_1, \ldots, -(N_1 - N_2 + 1)$, (resp. $j = -N_2, \ldots, -(N_2 - N_1 + 1)$),

$$c_j = a_j \quad (\text{resp.}\, c_j = b_j).$$

And define the multiplication from $\mathbb{Z}_p \times \mathbb{Z}_p$ into $\mathbb{Q}_p$ by

$$\left(\sum_{k_1=0}^{\infty} a_{k_1} p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_2} p^{k_2}\right) = \sum_{n=-N}^{\infty} c_n p^n, \tag{2.7}$$

where

$$c_n = \sum_{k_1+k_2=n} \left(r_{k_1,k_2} i_{k_1,k_2} + s_{k_1-1,k_2} i^c_{k_1-1,k_2} + s_{k_1,k_2-1} i^c_{k_1,k_2-1} + s_{k_1-1,k_2-1} i^c_{k_1-1,k_2-1}\right),$$

where

$$a_{k_1} b_{k_2} = s_{k_1,k_2} p + r_{k_1,k_2},$$

by the division algorithm, and

$$i_{k_1,k_2} = \begin{cases} 1 & \text{if } a_{k_1} b_{k_2} < p, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$i^c_{k_1,k_2} = 1 - i_{k_1,k_2},$$

for all $k_1, k_2 \in \mathbb{N}$, and hence, "on $\mathbb{Q}_p$", the multiplication (2.7) is extended to

$$\begin{aligned} &\left(\sum_{k_1=-N_1}^{\infty} a_{k_1} p^{k_1}\right)\left(\sum_{k_2=-N_2}^{\infty} b_{k_2} p^{k_2}\right) \\ &= \left(p^{-N_1}\right)\left(p^{-N_2}\right)\left(\sum_{k_1=0}^{\infty} a_{k_1-N_1} p^{k_1}\right)\left(\sum_{k_2=0}^{\infty} b_{k_1-N_2} p^{k_2}\right). \end{aligned} \tag{2.8}$$

Then, under the addition (2.6) and the multiplication (2.8), the algebraic triple $(\mathbb{Q}_p, +, \cdot)$ becomes a field, for all primes p. Thus the p-adic number fields $\mathbb{Q}_p$ are algebraically fields.

Fact 2.3. *Every p-adic number field $\mathbb{Q}_p$, with the binary operations* (2.6) *and* (2.8) *is indeed a field.*

Moreover, the Banach filed $\mathbb{Q}_p$ is also a (unbounded) Haar-measure space $(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \rho_p)$, for all primes p, where $\sigma(\mathbb{Q}_p)$ means the σ-algebra of $\mathbb{Q}_p$, consisting of all ρ_p-measurable subsets of $\mathbb{Q}_p$, where ρ_p is a Haar measure, satisfying

$$\rho_p\left(a + p^k \mathbb{Z}_p\right) = \rho_p\left(p^k \mathbb{Z}_p\right) = \frac{1}{p^k} = \rho\left(p^k \mathbb{Z}_p^{\times}\right) = \rho\left(a + p^k \mathbb{Z}_p^{\times}\right) \tag{2.9}$$

for all $a \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$, where $\mathbb{Z}_p^{\times} = \mathbb{Z}_p \setminus \{0\}$. Also, one has

$$\begin{aligned} \rho_p(a + p^k U_p) &= \rho_p\left(p^k U_p\right) = \rho_p\left(p^k \mathbb{Z}_p \ \setminus \ p^{k+1} \mathbb{Z}_p\right) \\ &= \rho_p\left(p^k \mathbb{Z}_p\right) - \rho_p\left(p^{k+1} \mathbb{Z}_p\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}} \end{aligned}$$

for all $a \in \mathbb{Q}_p$. By (2.3) and (2.5), the above measure ρ_p satisfying (2.9) is nicely extended fully on subsets of $\mathbb{Q}_p$, and hence, the collection $\sigma(\mathbb{Q}_p)$ of ρ_p-measurable subsets is well-defined, i.e., $\mathbb{Q}_p$ is understood as a measure space $(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \rho_p)$.

Note that, by (2.9), one obtains

$$\rho_p\left(a+p^kU_p\right)=\rho\left(p^kU_p\right)=\frac{1}{p^k}-\frac{1}{p^{k+1}} \tag{2.10}$$

for all $a\in\mathbb{Q}_p$ and $k\in\mathbb{Z}$ (also see Chapter IV of [18]).

Fact 2.4. *The Banach field* $\mathbb{Q}_p$ *is an unbounded Haar-measure space, where* ρ_p *satisfies* (2.9) *and* (2.10), *for all primes* p.

The above three facts show that $\mathbb{Q}_p$ is a unbounded Haar-measured, locally compact Banach field.

Definition 2.5. Let $\mathcal{P}=\{\text{all primes}\}\cup\{\infty\}$. The Adele ring $\mathbb{A}_{\mathbb{Q}}=(\mathbb{A}_{\mathbb{Q}},+,\cdot)$ is defined by the set

$$\begin{aligned}\Big\{(x_p)_{p\in\mathcal{P}}\,\Big|\,&\text{only finitely many } x_p\in\mathbb{Q}_p,\\ &\text{almost all other entries } x_p\in\mathbb{Z}_p \text{ for primes } p \text{ and } x_\infty\in\mathbb{Q}_\infty\Big\}\end{aligned} \tag{2.11}$$

with identification: $\mathbb{Q}_\infty=\mathbb{R}$, and $\mathbb{Z}_\infty=[0,1]$, the closed interval in $\mathbb{R}$, equipped with

$$(x_p)_{p\in\mathcal{P}}+(y_p)_{p\in\mathcal{P}}=(x_p+y_p)_{p\in\mathcal{P}}, \tag{2.12}$$

and

$$\left((x_p)_{p\in\mathcal{P}}\right)\left((y_p)_{p\in\mathcal{P}}\right)=(x_py_p)_{p\in\mathcal{P}} \tag{2.13}$$

for all $(x_p)_{p\in\mathcal{P}},(y_p)_{p\in\mathcal{P}}\in\mathbb{A}_{\mathbb{Q}}$, where the additions x_p+y_p in the right-hand side of (2.12) are in the sense of (2.6), and the multiplications x_py_p in the right-hand side of (2.13) are in the sense of (2.8).

Indeed, this algebraic structure $\mathbb{A}_{\mathbb{Q}}$ forms a ring. Also, by construction, the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is also a locally compact Banach space under product topology, equipped with the product measure. Set-theoretically,

$$\mathbb{A}_{\mathbb{Q}}\subseteq\prod_{p\in\mathcal{P}}\mathbb{Q}_p=\mathbb{R}\times\left(\prod_{p:prime}\mathbb{Q}_p\right).$$

In fact, by the very definition (2.11) of the Adele ring $\mathbb{A}_{\mathbb{Q}}$, it is a *weak direct product* $\prod'_{p\in\mathcal{P}}\mathbb{Q}_p$ of $\{\mathbb{Q}_p\}_{p\in\mathcal{P}}$, i.e.,

$$\mathbb{A}_{\mathbb{Q}}=\prod_{p\in\mathcal{P}}{}'\,\mathbb{Q}_p.$$

The product measure

$$\rho=\underset{p\in\mathcal{P}}{\times}\rho_p$$

of the Adele ring $\mathbb{A}_{\mathbb{Q}}$ is well-determined, with identification $\rho_\infty=\rho_{\mathbb{R}}$, the usual distance-measure (induced by $|\cdot|_\infty$) on $\mathbb{R}$.

Fact 2.6. *The Adele ring* $\mathbb{A}_{\mathbb{Q}}$ *is a unbounded-measure locally compact Banach ring.*

2.2. DYNAMICAL SYSTEMS INDUCED BY ALGEBRAIC STRUCTURES

In this section, we briefly discuss about dynamical systems induced by algebraic structures. Let X be an arbitrary algebraic structures, i.e., X is a semigroup, or a group, or a groupoid, or an algebra, etc. (possibly equipped with topology). Let us restrict our interests to the case where X is an algebraic structures equipped with a single operation $(\cdot)$, i.e., it is a semigroup or a monoid, or a group, or a groupoid.

Let M be an algebra over $\mathbb{C}$, and assume there exists a well-defined action α of X acting on M, i.e., $\alpha(x)$ is a well-defined function on M, for all $x \in X$, satisfying that:

$$\alpha(x_1 \cdot x_2) = \alpha(x_1) \circ \alpha(x_2) \quad \text{on } M,$$

for all $x_1, x_2 \in X$, where $x_1 \cdot x_2$ means the multiplication of x_1 and x_2 under operation $(\cdot)$ on X, and $(\circ)$ means the usual functional composition.

For convenience, we denote $\alpha(x)$ simply by α_x.

Definition 2.7. Then the triple (M, X, α) is called the *dynamical system induced by* X on M via α.

For such a dynamical system (M, X, α), one can define a *crossed product algebra*

$$\mathbb{M}_X = M \times_\alpha X,$$

by the algebra generated by M and $\alpha(X)$, satisfying the α-*relation*:

$$(m_1\alpha_{x_1})\,(m_2\alpha_{x_2}) = (m_1\alpha_{x_1}(m_2))\,\alpha_{x_1x_2} \quad \text{in } \mathbb{M}_X$$

for all $m_j\alpha_{x_j} \in \mathbb{M}_X$, for $j = 1, 2$. Every element T of $\mathbb{M}_X$ has its expression,

$$T = \sum_{x \in X} m_x\alpha_x \quad \text{with} \quad m_x \in M.$$

Remark that, if $\mathbb{M}_X$ is pure-algebraic, then $\sum$ is a finite sum, meanwhile, if $\mathbb{M}_X$ is topological, then $\sum$ is a finite or infinite (equivalently, limit of finite) sum (under topology).

If M is a $*$-algebra, then one may have an additional condition for α-*relation*;

$$(m\alpha_x)^* = \alpha_x(m^*)\alpha_x^* \quad \text{in } \mathbb{M}_X$$

for all $m\alpha_x \in \mathbb{M}_X$.

In this paper, we are interested in cases where given algebras M are von Neumann algebras. In such cases, we call the corresponding topological dynamical systems, W^*-*dynamical systems*, and the corresponding crossed product algebra, the *crossed product W^*-algebras*.

3. p-ADIC W^*-DYNAMICAL SYSTEMS

Let us establish W^*-dynamical systems on a fixed von Neumann algebra M. Throughout this section, we fix a prime p, and a von Neumann algebra M in an operator algebra $B(H)$ on a Hilbert space H.

3.1. p-PRIME VON NEUMANN ALGEBRAS $L^{\infty}(\mathbb{Q}_p)$

As a measure space, the p-adic number field $\mathbb{Q}_p$ has its corresponding L^2-*Hilbert space* H_p, defined by

$$H_p \overset{def}{=} L^2\left(\mathbb{Q}_p, \rho_p\right). \tag{3.1}$$

We call H_p, the *p-prime Hilbert space*, i.e., all elements of H_p are the square ρ_p-integrable functions on $\mathbb{Q}_p$. Remark that all elements of H_p are the functions approximated by simple functions,

$$\sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S$$

generated by characteristic functions χ_X,

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{Q}_p$, with $t_X \in \mathbb{C}$. So, one can understand each element f of H_p as

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \quad \text{with} \quad t_S \in \mathbb{C},$$

where $\sum$ is a finite or an infinite (equivalently, limit of finite) sum.

By definition, the inner product $\langle \cdot, \cdot \rangle_p$ on H_p is defined by

$$\langle f_1, f_2 \rangle_p \overset{def}{=} \int_{\mathbb{Q}_p} f_1 \overline{f_2}\, d\rho_p$$

for all $f_1, f_2 \in H_p$, having the corresponding norm $\|\cdot\|_p$ on H_p,

$$\|f\|_p \overset{def}{=} \sqrt{\langle f, f \rangle_p} = \sqrt{\int_{\mathbb{Q}_p} |f|^2\, d\rho_p}$$

for all $f \in H_p$. Thus, if $f = \sum\limits_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S$ in H_p, then

$$\int_{\mathbb{Q}_p} f d\rho_p = \sum_{X \in \sigma(\mathbb{Q}_p)} t_X \rho_p(X).$$

Let us fix a function $g \in L^{\infty}\left(\mathbb{Q}_p, \rho_p\right)$, which is an essential-norm bounded function. Similar to H_p-case, one can understand g as the approximation of simple functions. Then $gf \in H_p$, too, for all $f \in H_p$.

Definition 3.1. The von Neumann subalgebras $\mathfrak{M}_p = L^{\infty}\left(\mathbb{Q}_p, \rho_p\right)$ of $B(H_p)$ are called the p-prime von Neumann algebras, for all primes p.

By locally compactness, and Hausdorff property of $\mathbb{Q}_p$, for any $x \in \mathbb{Q}_p$, there exist $a \in \mathbb{Q}_p$, and $n \in \mathbb{Z}$, such that $x \in a + p^n U_p$ (see Section 2.1). Therefore, we obtain the following lemma.

Lemma 3.2. *Let $X \in \sigma(\mathbb{Q}_p)$. Then there exists $N \in \mathbb{N} \cup \{\infty\}$ such that:*

(i) *there are corresponding $a_1, \ldots, a_N \in \mathbb{Q}_p$, and $n_1, \ldots, n_N \in \mathbb{Z}$,*
(ii) *X is covered by the unions of $a_k + p^{n_k} U_p$, for $k = 1, \ldots, N$, i.e.,*

$$X \subseteq \bigcup_{k=1}^{N} \left(a_k + p^{n_k} U_p\right), \tag{3.2}$$

where U_p is the unit circle of $\mathbb{Q}_p$, which is the boundary of the unit disk $\mathbb{Z}_p$.

Proof. The proof of (3.2) is done by (2.5) and [18]. □

In (3.2), we show that every measurable subset X of $\mathbb{Q}_p$ is covered by a union of transformed boundaries $a + p^k U_p$ of $a, +p^k \mathbb{Z}_p$ $(a \in \mathbb{Q}_p, k \in \mathbb{Z})$. It shows that the measure $\rho_p(X)$ is less than or equal to the sum of $\rho_p(p^k U_p)$, for some $k \in \mathbb{Z}$.

Lemma 3.3. *Let X be a measurable subset of the unit circle $U_p \in \mathbb{Q}_p$, for primes p. Then there exists*

$$0 \leq r_X \leq 1 \quad \text{in } \mathbb{R}, \tag{3.3}$$

such that

$$\rho_p(X) = r_X \left(1 - \frac{1}{p}\right).$$

Proof. The existence of the quantities of (3.3) is guaranteed by (3.2) and (2.9), (2.10). □

By (3.3), we can obtain the following theorem.

Theorem 3.4. *Let χ_S be a characteristic function for $S \in \sigma(\mathbb{Q}_p)$. Then there exist $N \in \mathbb{N} \cup \{\infty\}$, and $k_1, \ldots, k_N \in \mathbb{Z}, r_1, \ldots, r_N \in \mathbb{R}$, such that*

$$\int_{\mathbb{Q}_p} \chi_S \, d\rho_p = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right). \tag{3.4}$$

Proof. Let S be a measurable subset of $\mathbb{Q}_p$. Then, by (3.2) and (3.3), there exist $N \in \mathbb{N} \cup \{\infty\}$, and $k_1, \ldots, k_N \in \mathbb{Z}$, and $r_1, \ldots, r_N \in [0, 1]$ in $\mathbb{R}$, such that

$$S \subseteq \bigcup_{j=1}^{N} \left(a_j + p^{k_j} U_p\right) \quad \text{for} \quad a_j \in \mathbb{Q}_p.$$

Thus, there exist measurable subsets $S_1, \ldots, S_N$ of S, such that

$$S_j \subseteq a_j + p^{k_j} U_p \quad \text{for} \quad j = 1, \ldots, N,$$

satisfying

$$S = \bigsqcup_{j=1}^{N} S_j \quad \text{and} \quad \rho_p(S_j) \le \rho_p\left(p^{k_j} U_p\right) = \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_{j+1}}}\right).$$

So, there exists $r_j \in [0,1]$ in $\mathbb{R}$, such that

$$\rho_p(S_j) = r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right),$$

for all $j = 1, \ldots, N$, and hence, one has

$$\int_{\mathbb{Q}_p} \chi_S d\rho_p = \rho_p(S) = \sum_{j=1}^{N} \rho_p(S_j) = \sum_{j=1}^{N} r_j \rho_p\left(a_j + p^{k_j} U_p\right)$$
$$= \sum_{j=1}^{N} r_j \rho_p\left(p^{k_j} U_p\right) = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right). \qquad \square$$

The above formula (3.4) characterizes the identically distributedness under integration in $\mathfrak{M}_p$.

Corollary 3.5. *Let $g = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S$ be an element of the p-prime von Neumann algebra $\mathfrak{M}_p$. Then there exist $r_j \in [0,1]$ in $\mathbb{R}$, $k_j \in \mathbb{Z}$, and $t_j \in \mathbb{C}$, and*

$$h = \sum_{j=-\infty}^{\infty} (t_j r_j p^{k_j}) \chi_{U_p} \tag{3.5}$$

such that g and h are identically distributed under integration $\int_{\mathbb{Q}_p} \bullet \, d\rho_p$.

The above theorem and corollary show that the analytic data of $g \in \mathfrak{M}_p$ is characterized by the analytic data of certain types of "good" functions of $\mathfrak{M}_p$, under identically-distributedness.

3.2. p-ADIC SEMIGROUP W^*-DYNAMICAL SYSTEMS

Now, let M be a fixed von Neumann algebra in $B(H)$, and $\mathbb{Q}_p$, a fixed p-adic number field, and let $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ be the p-prime von Neumann algebra in the sense of Section 3.1.

Let $\mathcal{H}_p$ be the tensor product Hilbert space $H_p \otimes H$ of the p-prime Hilbert space H_p and the Hilbert space H where M acts, where $\otimes$ means the topological tensor product of Hilbert spaces.

Define an action α of the σ-algebra $\sigma(\mathbb{Q}_p)$ of the p-adic number field $\mathbb{Q}_p$ acting on the von Neumann algebra M in $B(\mathcal{H}_p)$ by

$$\alpha(S)(m) \overset{def}{=} \chi_S m \chi_S^* = \chi_S m \chi_S \tag{3.6}$$

for all $S \in \sigma(\mathbb{Q}_p)$, and $m \in M$, in $B(\mathcal{H}_p)$, by understanding

$$\chi_S = \chi_S \otimes 1_M, \text{ and } m = 1_{\mathfrak{M}_p} \otimes m \text{ in } B(\mathcal{H}_p)$$

where $1_{\mathbb{Q}_p}$ is the function $\chi_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, and 1_M is the identity element of M, i.e., one can understand $\alpha(S)(m)$ as compressions of m in $B(\mathcal{H}_p)$ with respect to projections χ_S. Then α is an action of $\sigma(\mathbb{Q}_p)$ acting on M in the following sense:

$$\begin{aligned}\alpha(S_1 \cap S_2)(m) &= \chi_{S_1 \cap S_2} m \chi_{S_1 \cap S_2} = \chi_{S_1}\chi_{S_2} m \chi_{S_1}\chi_{S_2} \\ &= \chi_{S_1}\chi_{S_2} m \chi_{S_2}\chi_{S_1} = \chi_{S_1}\left(\alpha(S_2)(m)\right)\chi_{S_1} \\ &= \alpha(S_1)\left(\alpha(S_2)(m)\right) = (\alpha(S_1) \circ \alpha(S_2))(m)\end{aligned}$$

for all $m \in M$, and $S_1, S_2 \in \sigma(\mathbb{Q}_p)$, i.e.,

$$\alpha(S_1 \cap S_2) = \alpha(S_1) \circ \alpha(S_2) \quad \text{for all} \quad S_1, S_2 \in \sigma(\mathbb{Q}_p). \tag{3.7}$$

Observe now that the algebraic structure $(\sigma(\mathbb{Q}_p), \cap)$ forms a semigroup. Indeed, the intersection $\cap$ is well-defined on $\sigma(\mathbb{Q}_p)$, and it is associative;

$$S_1 \cap (S_2 \cap S_3) = (S_1 \cap S_2) \cap S_3,$$

for $S_j \in \sigma(\mathbb{Q}_p)$, for all $j = 1, 2, 3$. Moreover, this semigroup $\sigma(\mathbb{Q}_p)$ contains $\mathbb{Q}_p$, acting as the semigroup-identity satisfying that

$$S \cap \mathbb{Q}_p = S = \mathbb{Q}_p \cap S,$$

for all $S \in \sigma(\mathbb{Q}_p)$, and hence, this semigroup $\sigma(\mathbb{Q}_p)$ forms a monoid with its identity $\mathbb{Q}_p$.

Lemma 3.6. *The triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ forms a monoid W^*-dynamical system.*

Proof. The action α of (3.6) is indeed a well-defined monoid action of $\sigma(\mathbb{Q}_p) = (\sigma(\mathbb{Q}_p), \cap)$ acting on M in $B(\mathcal{H}_p)$ by (3.7). □

For convenience, we denote $\alpha(S)$ simply by α_S, for all $S \in \sigma(\mathbb{Q}_p)$.

The action α of (3.6) is extended to a linear morphism, also denoted by α, from $\mathfrak{M}_p$ into $B(\mathcal{H}_p)$, by

$$\begin{aligned}\alpha(f)(m) &= \alpha\left(\sum_{S \in Supp(f)} t_S \chi_S\right)(m) \\ &\overset{def}{=} \sum_{S \in Supp(f)} t_S \alpha_S(m) = \sum_{S \in Supp(f)} t_S \chi_S m \chi_S\end{aligned} \tag{3.8}$$

for all $f \in \mathfrak{M}_p$.

Proposition 3.7. *Let $\mathfrak{M}_p$ be the p-prime von Neumann algebra, and let M be a von Neumann subalgebra of $B(H)$. Then there exists an action α of $\mathfrak{M}_p$ acting on M in $B(\mathcal{H}_p)$.*

Proof. It is proven by construction (3.8). □

Definition 3.8. Let $\sigma(\mathbb{Q}_p)$ be the σ-algebra of the p-adic number field $\mathbb{Q}_p$, understood as a monoid $(\sigma(\mathbb{Q}_p), \cap)$, and let α be the action of $\sigma(\mathbb{Q}_p)$ on a given von Neumann algebra M in the sense of (3.6). Then the mathematical triple $(M, \sigma(\mathbb{Q}_p), \alpha)$ is called the p-adic (monoidal) W^*-dynamical system. For convenience, we denote it simply by $\mathcal{Q}(M,p)$. For a given p-adic W^*-dynamical system $\mathcal{Q}(M,p)$, define the crossed product algebra

$$\mathcal{M}_p \overset{def}{=} M \times_\alpha \sigma(\mathbb{Q}_p) \tag{3.9}$$

by the von Neumann subalgebra of $B(\mathcal{H}_p)$ generated by M and $\alpha(\sigma(\mathbb{Q}_p))$, satisfying (3.8). The von Neumann subalgebra $\mathcal{M}_p$ of $B(\mathcal{H}_p)$ is called the p-adic dynamical W^*-algebra induced by the p-adic W^*-dynamical system $\mathcal{Q}(M,p)$.

Note that all elements of the p-adic dynamical W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ induced by the p-adic W^*-dynamical system $\mathcal{Q}(M,p)$ have their expressions

$$\sum_{S \in \sigma\mathbb{Q}_p)} m_S \chi_S \quad \text{with} \quad m_S \in M,$$

where the sum $\sum$ means a finite or an infinite sum (under topology).

Define the support $Supp(T)$ of a fixed element $T = \sum_{S \in \sigma(\mathbb{Q}_p)} m_S \chi_S$ of $\mathcal{M}_p$ by

$$Supp(T) \overset{def}{=} \{S \in \alpha(\mathbb{Q}_p) : m_S \neq 0_M\}. \tag{3.10}$$

Now, let $m_1\chi_{S_1}, m_2\chi_{S_2} \in \mathcal{M}_p$, with $m_1, m_2 \in M$, and $S_1, S_2 \in \sigma(\mathbb{Q}_p)$. Then

$$\begin{aligned}(m_1\chi_{S_1})(m_2\chi_{S_2}) &= m_1\chi_{S_1} m_2 \chi_{S_1}\chi_{S_2} = m_1\chi_{S_1} m_2 \chi_{S_1}^2 \chi_{S_2} \\ &= m_1\chi_{S_1} m_2 \chi_{S_1}\chi_{S_1}\chi_{S_2},\end{aligned}$$

since $\chi_S = 1_M \otimes \chi_S$ (in $B(\mathcal{H}_p)$) are projections ($\chi_S^2 = \chi_S = \chi_S^*$) for all $S \in \sigma(\mathbb{Q}_p)$,

$$\begin{aligned}&= m_1 \alpha_{S_1}(m_2)\chi_{S_1}\chi_{S_2} \\ &= m_1 \alpha_{S_1}(m_2)\chi_{S_1 \cap S_2}.\end{aligned}$$

For convenience, if there is no confusion, we denote $\alpha_S(m)$ by m^S for all $S \in \sigma(\mathbb{Q}_p)$ and $m \in M$.

With help of the above notation, we have

$$(m_1\chi_{S_1})(m_2\chi_{S_2}) = m_1 m_2^{S_1} \chi_{S_1 \cap S_2}$$

for $m_k \chi_{S_k} \in \mathcal{M}_p$, for $k = 1, 2$. More generally, one has that

$$\begin{aligned}\prod_{j=1}^{N} (m_j \chi_{S_j}) &= m_1 m_2^{S_1} m_3^{S_1 \cap S_2} \ldots m_N^{S_1 \cap \ldots \cap S_{N-1}} \chi_{S_1 \cap \ldots \cap S_N} \\ &= \left(m_1 \prod_{j=2}^{N-1} m_j^{S_1 \cap \ldots \cap S_j} \right) \left(\chi_{\overset{N}{\underset{j=1}{\cap}} S_j} \right)\end{aligned} \tag{3.11}$$

for all $N \in \mathbb{N}$. Also, we obtain that

$$(m\chi_S)^* = \chi_S m^* \chi_S \chi_S = (m^*)^S \chi_S \tag{3.12}$$

for all $m\chi_S \in \mathcal{M}_p$, since $\chi_S = \chi_S \otimes 1_M$ is a projection on $\mathcal{H}_p$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$. So, let

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k} \in \mathcal{M}_p, \quad k = 1, 2.$$

Then

$$\begin{aligned} T_1 T_2 &= \sum_{(S_1, S_2) \in Supp(T_1) \times Supp(T_2)} m_{S_1} \chi_{S_1} m_{S_2} \chi_{S_2} \\ &= \sum_{(S_1, S_2) \in Supp(T_1) \times Supp(T_2)} m_{S_1} m_{S_2}^{S_1} \chi_{S_1 \cap S_2}, \end{aligned} \tag{3.13}$$

by (3.11). Also, if $T = \sum_{S \in Supp(T)} m_S \chi_S$ in $\mathcal{M}_p$, then

$$T^* = \sum_{S \in Supp(T)} (m_S^*)^S \chi_S, \tag{3.14}$$

by (3.12).

By (3.13) and (3.14), one can have that if

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k} \in \mathcal{M}_p, \quad k = 1, \ldots, n,$$

for $n \in \mathbb{N}$, then

$$\begin{aligned} T_1^{r_1} T_2^{r_2} \ldots T_n^{r_n} &= \prod_{j=1}^{n} \left(\sum_{S_j \in Supp(T_j)} [m_{S_j}^{r_j}]^{S_j} \chi_{S_j} \right) \\ &= \sum_{(S_1, \ldots, S_n) \in \prod_{j=1}^{n} Supp(T_j)} \left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \chi_{S_j} \right) \right) \\ &= \sum_{(S_1, \ldots, S_n) \in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)} \right) \left(\chi_{\bigcap_{j=1}^{n} S_j} \right) \right), \end{aligned} \tag{3.15}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, where

$$[m_{S_j}^{r_j}]^{S_j} \overset{def}{=} \begin{cases} m_{S_j} & \text{if } r_j = 1, \\ (m_{S_j}^*)^{S_j} & \text{if } r_j = * \end{cases} \tag{3.16}$$

for $j = 1, \ldots, n$.

Lemma 3.9. *Let $T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k}$ be elements of the p-adic semigroup W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ in $B(\mathcal{H}_p)$, for $k = 1, \ldots, n$, for $n \in \mathbb{N}$. Then*

$$\prod_{j=1}^{n} T_j^{r_j} = \sum_{(S_1,\ldots,S_n) \in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n} \left([m_{S_j}^{r_j}]^{S_j} \right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)} \right) \left(\chi_{\bigcap_{j=1}^{n} S_j} \right) \right) \tag{3.17}$$

for all $r_1, \ldots, r_n \in \{1, *\}$, where $[m_{S_j}^{r_j}]^{S_j}$ are in the sense of (3.16).

Proof. The proof of (3.17) is by (3.15). □

3.3. STRUCTURE THEOREM OF $M \times_\alpha \sigma(\mathbb{Q}_p)$

Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic dynamical W^*-algebra induced by a p-adic W^*-dynamical system $\mathcal{Q}(M,p) = (M, \sigma(\mathbb{Q}_p), \alpha)$. In this section, we consider a structure theorem for this crossed product von Neumann algebra $\mathcal{M}_p$.

First, define the usual tensor product W^*-subalgebra

$$\mathcal{M}_0 = M \otimes_{\mathbb{C}} \mathfrak{M}_p \quad \text{of } B(\mathcal{H}_p),$$

where $\mathfrak{M}_p = L^\infty(\mathbb{Q}_p, \rho_p)$ is the p-prime von Neumann algebra in the sense of Section 3.1, and where $\otimes_{\mathbb{C}}$ means the topological tensor product of topological operator algebras over $\mathbb{C}$. By definition, clearly, $\mathcal{M}_p$ is a W^*-subalgebra of $\mathcal{M}_0$ in $B(\mathcal{H}_p)$, i.e.,

$$\mathcal{M}_p \overset{W^*\text{-subalgebra}}{\subseteq} \mathcal{M}_0.$$

Now, define the "*conditional*" *tensor product W^*-algebra*

$$\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p,$$

by a W^*-subalgebra of $\mathcal{M}_0$ dictated by the following *α-relation* (3.18) and (3.19);

$$(m_1 \otimes \chi_{S_1})(m_2 \otimes \chi_{S_2}) = (m_1 m_2^{S_1}) \otimes \chi_{S_1}\chi_{S_2}, \tag{3.18}$$

and

$$(m \otimes \chi_S)^* = (m^*)^S \otimes \chi_S^*, \tag{3.19}$$

for all $m_1, m_2, m \in M$, and $S_1, S_2, S \in \sigma(\mathbb{Q}_p)$.

Theorem 3.10. *Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic dynamical W^*-algebra induced by the p-adic W^*-dynamical system $\mathcal{Q}(M,p)$, and let $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$ be the conditional tensor product W^*-algebra of M and the p-prime von Neumann algebra $\mathfrak{M}_p$ satisfying the α-relations* (3.18) *and* (3.19). *Then these von Neumann algebras $\mathcal{M}_p$ and $\mathcal{M}_0^p$ are $*$-isomorphic in $B(\mathcal{H}_p)$, i.e.,*

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \overset{*\text{-iso}}{=} M \otimes_\alpha \mathfrak{M}_p = \mathcal{M}_0^p. \tag{3.20}$$

Proof. Define a morphism $\Phi : \mathcal{M}_p \to \mathcal{M}_0^p$ by

$$\Phi\left(\sum_{S\in\sigma(\mathbb{Q}_p)} m_S\chi_S\right) \overset{def}{=} \sum_{S\in\sigma(\mathbb{Q}_p)} (m_S \otimes \chi_S) \tag{3.21}$$

for all $\sum_{S\in\sigma(\mathbb{Q}_p)} m_S\chi_S \in \mathcal{M}_p$.

By the very definition (3.22), Φ is a well-defined linear transformation, furthermore, it preserves the generators, and hence, it is bounded (or continuous) and injective.

Moreover, this linear morphism Φ satisfies that

$$\begin{aligned}\Phi\left((m_1\chi_S)(m_2\chi_{S_2})\right) &= \Phi\left(m_1m_2^{S_1}\chi_{S_1\cap S_2}\right)\\ &= m_1m_2^{S_1}\otimes\chi_{S_1\cap S_2} = m_1m_2^{S_1}\otimes\chi_{S_1}\chi_{S_2}\\ &= (m_1\otimes\chi_{S_1})(m_2\otimes\chi_{S_2}),\end{aligned}$$

by (3.18),

$$= \Phi\left(m_1\chi_{S_1}\right)\Phi(m_2\chi_{S_2})$$

for all $m_1, m_2 \in M$, and $S_1, S_2 \in \sigma(\mathbb{Q}_p)$. And it also satisfies that

$$\begin{aligned}\Phi((m\chi_S)^*) &= \Phi((m^*)^S\chi_S) = (m^*)^S\otimes\chi_S\\ &= (m^*)^S\otimes\chi_S^* = (m\otimes\chi_S)^* = (\Phi(m\chi_S))^*\end{aligned}$$

for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Therefore, for any $T_1, T_2, T \in \mathcal{M}_p$, we have

$$\Phi(T_1T_2) = \Phi(T_1)\Phi(T_2)$$

and

$$\Phi(T^*) = (\Phi(T))^*.$$

So, the linear morphism Φ is a $*$-monomorphism. It shows that $\mathcal{M}_p$ is a W^*-subalgebra of $\mathcal{M}_0^p$.

Now, consider the linear transformation $\Phi' : \mathcal{M}_0^p \to \mathcal{M}_p$ by the morphism satisfying

$$\Phi'\left(m\otimes\chi_S\right) = m\chi_S \quad \text{for all} \quad m\in M \text{ and } S\in\sigma(\mathbb{Q}_p). \tag{3.22}$$

It is bounded and injective by the very definition (3.22), and it satisfies:

$$\begin{aligned}\Phi'\left((m_1\otimes\chi_{S_1})(m_2\otimes\chi_{S_2})\right) &= \Phi'\left(m_1m_2^{S_1}\otimes\chi_{S_1\cap S_2}\right)\\ &= m_1m_2^{S_1}\chi_{S_1\cap S_2} = (m_1\chi_{S_1})(m_2\chi_{S_2})\\ &= \Phi'(m_1\otimes\chi_{S_1})\Phi'(m_2\otimes\chi_{S_2}),\end{aligned}$$

and

$$\Phi'\left((m \otimes \chi_S)^*\right) = \Phi((m^*)^S \otimes \chi_S^*) = (m^*)^S \chi_S$$
$$= (m\chi_S)^* = \Phi'(m \otimes \chi_S)^*$$

for all $m_1, m_2, m \in M$, and $S_1, S_2, S \in \sigma(\mathbb{Q}_p)$. Thus, for all $T_1', T_2', T' \in \mathcal{M}_0^p$, we have

$$\Phi'(T_1'T_2') = \Phi'(T_1')\Phi'(T_2')$$

and

$$\Phi'((T')^*) = (\Phi'(T'))^*.$$

Therefore, this injective linear morphism Φ' is a $*$-monomorphism. It shows that $\mathcal{M}_0^p$ is a W^*-subalgebra of $\mathcal{M}_p$.

As a consequence, these two W^*-subalgebras $\mathcal{M}_p$ and $\mathcal{M}_0^p$ are $*$-isomorphic in $B(\mathcal{H}_p)$. $\square$

The above characterization (3.20) shows that our p-adic dynamical W^*-algebra $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ is $*$-isomorphic to the conditional tensor product W^*-algebra $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$.

4. FREE PROBABILITY ON p-ADIC DYNAMICAL W^*-ALGEBRAS

Throughout this section, let us fix a prime p, and a p-adic W^*-dynamical system $\mathcal{Q}(M,p) = (M, \sigma(\mathbb{Q}_p), \alpha)$. In this section, we are interested in free probability on the p-adic dynamical W^*-algebra

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$$

induced by $\mathcal{Q}(M,p)$.

By Section 3.3, the von Neumann subalgebra $\mathcal{M}_p$ is $*$-isomorphic to the conditional tensor product W^*-algebra $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$. So, throughout this section, we use $\mathcal{M}_p$ and $\mathcal{M}_0^p$ alternatively.

We will establish free probability on $\mathcal{M}_p$ by putting certain additional conditions: first, we assume that a fixed von Neumann algebra M is equipped with a well-defined linear functional ψ on it, i.e., the pair (M, ψ) is a W^*-probability space in the sense of Voiculescu (see Section 4.1 below, or [17] and [19]). Moreover, assume that the linear functional ψ is *unital* on M, in the sense that:

$$\psi(1_M) = 1,$$

for the identity element 1_M of M.

By understanding $\mathcal{M}_p$ as $\mathcal{M}_0^p$, we obtain a well-defined conditional expectation

$$E_p : \mathcal{M}_0^p \overset{*\text{-iso}}{=} \mathcal{M}_p \to M_p, \tag{4.1}$$

where

$$M_p \overset{def}{=} M \otimes_\alpha \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p), S \subseteq U_p\}\right],$$

where U_p is the unit circle of the p-adic number field $\mathbb{Q}_p$, which is the boundary of the unit disk $\mathbb{Z}_p$ of $\mathbb{Q}_p$, satisfying that:

$$E_p(m\chi_S) = E_p(m \otimes \chi_S) \overset{def}{=} m\chi_{S \cap U_p}$$

for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$.

Define now a morphism

$$F_p : M_p \to M_p \tag{4.2}$$

by a linear transformation satisfying that:

$$F_p\left(m\chi_S\right) = m\left(r_S \chi_{U_p}\right)$$

for all $m\chi_S \in M_p$, where $r_S \in [0,1]$ satisfies that:

$$\int_{\mathbb{Q}_p} \chi_S d\rho_p = r_S \int_{\mathbb{Q}_p} \chi_{U_p} d\rho_p = r_S \left(1 - \frac{1}{p}\right). \tag{4.3}$$

Such a quantity r_S exists for $S \cap U_p$, by (3.3) and (3.4). And then define a linear functional

$$\gamma : M_p \to \mathbb{C}$$

by a linear functional on $\mathcal{M}_p$, satisfying that: for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$,

$$\gamma \overset{def}{=} \left(\psi \otimes \int_{\mathbb{Q}_p} \bullet \, d\rho_p\right) \circ F_p, \tag{4.4}$$

i.e., a linear functional γ of (4.4) satisfies

$$\gamma\left(m \otimes \chi_S\right) \overset{def}{=} \psi(m) \int_{\mathbb{Q}_p} \left(r_S \chi_{U_p}\right) d\rho_p = r_S \psi(m)\Big(1 - \frac{1}{p}\Big),$$

where $r_S \in [0,1]$ satisfies (4.3).

Remark that, in general, the tensor product $\varphi_1 \otimes \varphi_2$ of two linear functionals φ_1 and φ_2 are "not" a linear functional, however, our linear functional γ of (4.2) is active under the linear morphism F_p. And hence, it becomes a well-defined linear functional (see Section 4.2 below).

And then define a linear functional

$$\gamma_p : \mathcal{M}_p \overset{*\text{-iso}}{=} \mathcal{M}_p^0 \to \mathbb{C}$$

by

$$\gamma_p = \gamma \circ E_p, \tag{4.5}$$

where γ and E_p are in the sense of (4.4) and (4.1), respectively, i.e., for all $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$,

$$\gamma_p\left(m\chi_S\right) = \gamma\left(E_p(m\chi_S)\right) = \gamma\left(m\chi_{S\cap U_p}\right) \\ = \psi(m)\int\limits_{\mathbb{Q}_p}\left(r_S\chi_{U_p}\right)d\rho_p = r_S\psi(m)\left(1-\frac{1}{p}\right)$$

for some $r_S \in [0,1]$, satisfying (4.3).

Then the pair $(\mathcal{M}_p, \gamma_p)$ is a W^*-probability space in the sense of Section 4.1 (below). We consider the free distributional data of certain elements of $(\mathcal{M}_p, \gamma_p)$.

4.1. FREE PROBABILITY

For more about free probability theory, see [17] and [19]. In this section, we briefly introduce *Speicher*'s combinatorial free probability (e.g., [17]), which is the combinatorial characterization of the original Voiculescu's analytic free probability (e.g., [19]). An important application for free probability theory can be found in [16]: the isomorphism theorem for free group factors.

Let $B \subset A$ be von Neumann algebras with $1_B = 1_A$ and assume that there exists a *conditional expectation* $E_B : A \to B$ satisfying that:

(i) $E_B(b) = b$ for all $b \in B$,
(ii) $E_B(bab') = bE_B(a)b'$ for all $b, b' \in B$ and $a \in A$,
(iii) E_B is bounded (or continuous),
(iv) $E_B(a^*) = E_B(a)^*$ for all $a \in A$.

Then the pair (A, E_B) is called a *B-valued (amalgamated) W^*-probability space (with amalgamation over B)*.

For any fixed B-valued random variables $a_1, \ldots, a_s$ in (A, E_B), we can have the *B-valued free distributional data* of them:

a) $(i_1, \ldots, i_n)$-*th B-valued joint $*$-moments*:

$$E_B\left(b_1 a_{i_1}^{r_1} b_2 a_{i_2}^{r_2} \ldots b_n a_{i_n}^{r_n}\right),$$

b) $(j_1, \ldots, j_m)$-*th B-valued joint $*$-cumulants*:

$$k_m^B\left(b_1' a_{j_1}^{t_1}, b_2' a_{j_2}^{t_2}, \ldots, b_m' a_{j_m}^{t_m}\right),$$

which provide the equivalent B-valued free distributional data of $a_1, \ldots, a_s$ for all $(i_1, \ldots, i_n) \in \{1, \ldots, s\}^n, (j_1, \ldots, j_m) \in \{1, \ldots, s\}^m$, for all $n, m \in \mathbb{N}$, where $b_1, \ldots, b_n, b_1', \ldots, b_m' \in B$ are arbitrary and $r_1, \ldots, r_n, t_1, \ldots, t_m \in \{1, *\}$. By the *Möbius inversion*, indeed, they provide the same, or equivalent, B-valued free distributional data of $a_1, \ldots, a_s$, i.e., they satisfy

$$E_B\left(b_1 a_{i_1}^{r_1} \ldots b_n a_{i_n}^{r_n}\right) = \sum_{\pi \in NC(n)} k_\pi^B\left(b_1 a_{i_1}^{r_1}, \ldots, b_n a_{i_n}^{r_n}\right)$$

and

$$k_m^B\left(b_1' a_{j_1}^{r_1}, \ldots, b_m' a_{j_m}^{t_m}\right) = \sum_{\theta \in NC(m)} E_{B:\theta}\left(b_1' a_{j_1}^{t_1}, \ldots, b_m' a_{j_m}^{t_m}\right) \mu(\theta, 1_m),$$

where $NC(k)$ is the *lattice of all noncrossing partitions over* $\{1, \ldots, k\}$ for $k \in \mathbb{N}$, and $k_\pi^B(\ldots)$ and $E_{B:\theta}(\ldots)$ are the *partition-depending cumulant* and the *partition-depending moment*, and where μ is the *Möbius functional* in the *incidence algebra* I_2.

Recall that, for $k \in \mathbb{N}$, the *partial ordering on* $NC(k)$ is defined as follows:

$$\pi \leq \theta \overset{def}{\Longleftrightarrow} \text{for each block } V \text{ in } \pi \text{ there exists a block } B \text{ in } \theta \text{ such that } V \subseteq B.$$

Under such a partial ordering $\leq$, the set $NC(k)$ is a lattice with its *maximal element* $1_k = \{(1, \ldots, k)\}$ and its *minimal element* $0_k = \{(1), (2), \ldots, (k)\}$. The notation $(\ldots)$ inside partitions $\{\ldots\}$ means the *blocks of the partitions.* For example, 1_k is the one-block partition and 0_k is the k-block partition, for $k \in \mathbb{N}$. Also, recall that the *incidence algebra* I_2 is the collection of all functionals

$$\xi : \bigcup_{k=1}^{\infty} (NC(k) \times NC(k)) \to \mathbb{C},$$

satisfying $\xi(\pi, \theta) = 0$, whenever $\pi > \theta$, with its usual function addition $(+)$ and its *convolution* $(*)$ defined by

$$\xi_1 * \xi_2(\pi, \theta) \overset{def}{=} \sum_{\pi \leq \sigma \leq \theta} \xi_1(\pi, \sigma)\xi_2(\sigma, \theta)$$

for all $\xi_1, \xi_2 \in I_2$. Then this algebra I_2 has the *zeta functional* ζ, defined by

$$\zeta(\pi, \theta) \overset{def}{=} \begin{cases} 1 & \text{if } \pi \leq \theta, \\ 0 & \text{otherwise.} \end{cases}$$

The *Möbius functional* μ is the convolution-inverse of ζ in I_2. So, it satisfies

$$\sum_{\pi \in NC(k)} \mu(\pi, 1_k) = 0 \quad \text{and} \quad \mu(0_k, 1_k) = (-1)^{k-1} c_{k-1} \tag{4.6}$$

for all $k \in \mathbb{N}$, where

$$c_m \overset{def}{=} \frac{1}{m+1}\binom{2m}{m}$$

is the *m-th Catalan number*, for all $m \in \mathbb{N}$.

The *amalgamated freeness* is characterized by the amalgamated $*$-cumulants. Let (A, E_B) be given as above. Two W^*-subalgebras A_1 and A_2 of A, having their common W^*-subalgebra B in A, are *free over B in* (A, E_B), if and only if all their "mixed" $*$-cumulants vanish. Two subsets X_1 and X_2 of A are *free over B in* (A, E_B), if $vN(X_1, B)$ and $vN(X_2, B)$ are free over B in (A, E_B), where $vN(S_1, S_2)$ means the von Neumann algebra generated by S_1 and S_2. In particular, two B-valued random variable x_1 and x_2 are *free over B in* (A, E_B), if $\{x_1\}$ and $\{x_2\}$ are free over B in (A, E_B).

Suppose two W^*-subalgebras A_1 and A_2 of A, containing their common W^*-subalgebra B, are free over B in (A, E_B). Then we can construct a W^*-subalgebra $vN(A_1, A_2) = \overline{B[A_1 \cup A_2]}^w$ of A generated by A_1 and A_2. Such W^*-subalgebra of A is denoted by $A_1 *_B A_2$. If there exists a family $\{A_i : i \in I\}$ of W^*-subalgebras of A, containing their common W^*-subalgebra B, satisfying $A = {*_B}_{i\in I} A_i$, then we call A, the *B-valued free product algebra of* $\{A_i : i \in I\}$.

Assume now that the W^*-subalgebra B is $*$-isomorphic to $\mathbb{C} = \mathbb{C} \cdot 1_A$. Then the conditional expectation E_B becomes a linear functional on A. By φ, denote E_B. Then, for $a_1, \ldots, a_n \in (A, \varphi)$,

$$k_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \ldots, a_n)\mu(\pi, 1_n)$$

by the Möbius inversion

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi_V(a_1, \ldots, a_n) \right) \mu(\pi, 1_n),$$

since the images of φ are in $\mathbb{C}$.

For example, if $\pi = \{(1,3), (2), (4,5)\}$ in $NC(5)$, then

$$\varphi_\pi(a_1, \ldots, a_5) = \varphi(a_1\varphi(a_2)a_3)\varphi(a_4a_5) = \varphi(a_1a_3)\varphi(a_2)\varphi(a_4a_5).$$

Remember here that, if φ is an arbitrary conditional expectation E_B, and if $B \neq \mathbb{C}\cdot 1_A$, then the above second equality does not hold in general. So, we have

$$k_n(a_1, \ldots, a_n) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi_V(a_1, \ldots, a_n)\mu(0_{|V|}, 1_{|V|}) \right) \tag{4.7}$$

by the multiplicativity of μ.

4.2. FREE STRUCTURE OF $(\mathcal{M}_p, \gamma_p)$

Let $\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p)$ be the p-adic dynamical W^*-algebra in $B(\mathcal{H}_p)$, understood also as its $*$-isomorphic von Neumann algebra, $\mathcal{M}_0^p = M \otimes_\alpha \mathfrak{M}_p$, the conditional tensor product W^*-algebra of M and the p-prime von Neumann algebra $\mathfrak{M}_p$. Let $\gamma_p = \gamma \circ E_p$ be the linear functional in the sense of (4.5) on $\mathcal{M}_0^p = \mathcal{M}_p$, where γ is in the sense of (4.4) and E_p is in the sense of (4.1), with (4.3), i.e., γ_p is a linear functional on $\mathcal{M}_p$, satisfying that

$$\gamma_p(m\chi_S) = \gamma\left(E_p(m\chi_S)\right) = \gamma\left(m(r_S\chi_{U_p})\right) = r_S\psi(m)\left(1 - \frac{1}{p}\right),$$

for some $r_S \in [0,1]$, satisfying (4.3), for all $m \in M$ and $S \in \sigma(\mathbb{Q}_p)$.

First, we will check indeed E_p and γ are well-defined conditional expectation and linear functional, respectively.

Let us check E_p is indeed a well-defined conditional expectation from

$$\mathcal{M}_p \overset{*\text{-iso}}{=} M \otimes_\alpha \mathfrak{M}_p = \mathcal{M}_p^0$$

onto

$$M \overset{*\text{-iso}}{=} M \otimes_\alpha \left(\mathbb{C}\left[\{\chi_{U_p}\}\right]\right) \overset{denote}{=} M_p.$$

Observe that the following properties hold.

(i) For all $m(t\chi_{U_p}) \in M_p$, with $m \in M$, $t \in \mathbb{C}$, one has that

$$E_p\left(m(t\chi_{U_p})\right) = E_p\left((tm)\chi_{U_p}\right) = tm\chi_{U_p} = m\left(t\chi_{U_p}\right),$$

by (4.2) and hence, for all $x \in M_p$, we have

$$E_p(x) = x.$$

(ii) For all $m_1\chi_{U_p}, m_2\chi_{U_p} \in M_p$, and for $m\chi_Y \in \mathcal{M}_p$, we have

$$\begin{aligned} E_p\left(\left(m_1\chi_{U_p}\right)\left(m\chi_Y\right)\left(m_2\chi_{U_p}\right)\right) &= E_p\left(m_1 m^{U_p} m^{U_p\cap Y}\chi_{U_p\cap Y\cap U_p}\right) \\ &= E_p\left(\left(m_1 m^{U_p} m_2^{U_p\cap Y}\right)\chi_{Y\cap U_p}\right) \\ &= m_1 m^{U_p} m_2^{U_p\cap Y}\left(r_Y\chi_{U_p}\right) \end{aligned}$$

for some $r_Y \in [0,1]$ in $\mathbb{R}$, satisfying (4.3),

$$= \left(m_1\chi_{U_p}\right)\left(r_Y m\chi_{U_p}\right)\left(m_2\chi_{U_p}\right)$$

by α-relations

$$= \left(m_1\chi_{U_p}\right)\left(E_p\left(m\chi_Y\right)\right)\left(m_2\chi_{U_p}\right),$$

and hence, under the linearity, one has that, if $x_1, x_2 \in M_p$, and $y \in \mathcal{M}_p$, then

$$E_p\left(x_1 y x_2\right) = x_1 E_p(y) x_2.$$

(iii) By definition, E_p is bounded (or continuous).

So, by (i), (ii) and (iii), the morphism E_p of (4.1) is indeed a well-defined conditional expectation from $\mathcal{M}_p$ onto $M_p \overset{*\text{-iso}}{=} M$.

Now, let us consider the linear functional $\gamma : M_p \to \mathbb{C}$ of (4.4) is indeed a linear functional on M_p. Let $t_j m_j\chi_{U_p} \in M_p$, with $m_j \in M$, $t_j \in \mathbb{C}$ for $j = 1, 2$. Then

$$\begin{aligned} \left(t_1 m_1\chi_{U_p} + t_2 m_2\chi_{U_p}\right) &= \gamma\left((t_1 m_1 + t_2 m_2)\chi_{U_p}\right) \\ &= \psi(t_1 m_1 + t_2 m_2)\int_{\mathbb{Q}_p}\chi_{U_p}\,d\rho_p \\ &= t_1\psi(m_1)\left(1 - \frac{1}{p}\right) + t_2\psi(m_2)\left(1 - \frac{1}{p}\right) \\ &= t_1\gamma(m_1\chi_{U_p}) + t_2\gamma(m_2\chi_{U_p}). \end{aligned}$$

Thus it is indeed a linear functional on M_p, furthermore, by definition, it is bounded. Therefore, the morphism $\gamma_p = \gamma \circ E_p : \mathcal{M}_p \to \mathbb{C}$ of (4.5) is a well-defined bounded linear functional on $\mathcal{M}_p \overset{*\text{-iso}}{=} \mathcal{M}_p^0$.

Definition 4.1. The pair $(\mathcal{M}_p, \gamma_p)$ is called the p-adic dynamical W^*-probability space. Remark that our p-adic dynamical W^*-probability spaces are defined by fixing the unit circle U_p of $\mathbb{Q}_p$.

The following lemma is obtained by the straightforward computations.

Lemma 4.2. *Let $m\chi_S$ be a free random variable in the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Then*

$$\gamma_p\left((m\chi_S)^n\right) = r_S \psi\left(m(m^S)^{n-1}\right)\left(1 - \frac{1}{p}\right) \tag{4.8}$$

for all $n \in \mathbb{N}$, where $r_S \in [0,1]$ satisfies (4.3).

Proof. By (3.16), if $m\chi_S \in \mathcal{M}_p$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$, then

$$(m\chi_S)^n = mm^S m^{S\cap S} \dots m^{S\cap S\cap\dots\cap S}\chi_{S\cap\dots\cap S} = mm^S m^S \dots m^S \chi_S = m(m^S)^{n-1}\chi_S$$

for all $n \in \mathbb{N}$. Therefore, one can have that

$$\gamma_p\left((m\chi_S)^n\right) = \gamma_p\left(m(m^S)^{n-1}\chi_S\right) = \psi\left(m(m^S)^{n-1}\right)\left(\int_{\mathbb{Q}} r_S \chi_{U_p} d\rho_p\right),$$

where $r_S \in [0,1]$ satisfies (4.3),

$$= r_S\psi\left(m(m^S)^{n-1}\right)\left(\rho_p(\chi_{U_p})\right) = r_S\psi\left(m(m^S)^{n-1}\right)\left(1 - \frac{1}{p}\right)$$

for all $n \in \mathbb{N}$. $\square$

More general to (4.8), we obtain the following result.

Lemma 4.3. *Let $m_1\chi_{S_1}, \dots, m_n\chi_{S_n}$ be free random variables in the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$, with $m_k \in M, S_k \in \sigma(\mathbb{Q}_p)$ for $k = 1, \dots, n$ and $n \in \mathbb{N}$. Then there exists $r_0 \in [0,1]$, such that*

$$\gamma_p\left(\prod_{j=1}^{n} m_j\chi_{S_j}\right) = r_0\left(\psi\left(m_1 \prod_{j=2}^{N} m_j^{\bigcap_{i=1}^{j-1} S_i}\right)\right)\left(1 - \frac{1}{p}\right). \tag{4.9}$$

Proof. By (3.11), if $m_k\chi_{S_k} \in (\mathcal{M}_p, \gamma_p)$ are given as above, for $k = 1, \dots, n$, then

$$\prod_{j=1}^{n}(m_j\chi_{S_j}) = m_1 m_2^{S_1} m_3^{S_1\cap S_2} \dots m_N^{S_1\cap\dots\cap S_{n-1}}\chi_{S_1\cap\dots\cap S_n}$$

$$= \left(m_1 \prod_{j=2}^{n} m_j^{\bigcap_{i=1}^{j-1} S_i}\right)\left(\chi_{\bigcap_{j=1}^{n} S_j}\right),$$

in $\mathcal{M}_p$ for all $n \in \mathbb{N}$. Thus,

$$\gamma_p\left(\prod_{j=1}^{n}(m_j\chi_{S_j})\right) = \gamma\left(\left(m_1\prod_{j=2}^{n} m_j^{\bigcap_{i=1}^{j-1} S_i}\right)(r_0\chi_{U_p})\right)$$

where $r_0 \in [0,1]$ satisfies (4.3) for $\bigcap_{j=1}^{n} S_j \in \sigma(\mathbb{Q}_p)$, i.e.,

$$\chi^0_{(\bigcap_{j=1}^{n} S_j)\cap U_p} = r_0\chi_{U_p},$$

under E_p, where χ_Y^0 are in the sense of (4.2), thus,

$$= r_0\left(\psi\left(m_1\prod_{j=2}^{n} m_j^{\bigcap_{i=1}^{j-1} S_i}\right)\right)\left(\rho_p\left(\chi_{U_p}\right)\right) = r_0\left(\psi\left(m_1\prod_{j=2}^{n} m_j^{\bigcap_{i=1}^{j-1} S_i}\right)\right)\left(1-\frac{1}{p}\right). \square$$

By (4.8) and (4.9), we obtain the following free distributional data of free random variables of the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$.

Theorem 4.4. *Let $(\mathcal{M}_p, \gamma_p)$ be the p-adic dynamical W^*-probability space, and let*

$$T_k = \sum_{S_k \in Supp(T_k)} m_{S_k}\chi_{S_k}, \quad for \quad k = 1, \ldots, n,$$

be free random variables in $(\mathcal{M}_p, \gamma_p)$ for $n \in \mathbb{N}$. Then

$$\gamma_p\left(\prod_{j=1}^{n} T_j^{r_j}\right) = \sum_{(S_1,\ldots,S_n)\in \prod_{j=1}^{n} Supp(T_j)} r_{(S_1,\ldots,S_n)}\left(\psi\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\left(1-\frac{1}{p}\right), \tag{4.10}$$

where $[m_{S_j}^{r_j}]^{S_j}$ are in the sense of (4.18), and $r_1, \ldots, r_n \in \{1, *\}$, and where $r_{(S_1,\ldots,S_n)} \in [0,1]$ satisfy (4.3) for all $(S_1, \ldots, S_n)$.

Proof. By (3.15), we have that

$$\prod_{j=1}^{n} T_j^{r_j} = \sum_{(S_1,\ldots,S_n)\in \prod_{j=1}^{n} Supp(T_j)}\left(\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\left(\chi_{\bigcap_{j=1}^{n} S_j}\right)\right),$$

where $[m_{S_j}^{r_j}]^{S_j}$ are in the sense of (3.16), i.e.,

$$[m_{S_j}^{r_j}]^{S_j} = \begin{cases} m_{S_j} & \text{if } r_j = 1, \\ (m_{S_j}^*)^{S_j} & \text{if } r_j = * \end{cases}$$

for all $j = 1, \ldots, n$. So,

$$\gamma_p\left(\prod_{j=1}^{n} T_j^{r_j}\right) = \gamma_p\left(T_1^{r_1} T_2^{r_2} \ldots T_n^{r_n}\right)$$

$$= \gamma_p\left(\sum_{(S_1,\ldots,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\left(\chi_{\bigcap_{j=1}^{n} S_j}\right)\right)\right)$$

$$= \sum_{(S_1,\ldots,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(\gamma_p\left(\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\left(\chi_{\bigcap_{j=1}^{n} S_j}\right)\right)\right)$$

$$= \sum_{(S_1,\ldots,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(\left(r_{(S_1,\ldots,S_n)}\psi\left(\prod_{j=1}^{n}\left([m_{S_j}^{r_j}]^{S_j}\right)^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\left(\rho_p\left(U_p\right)\right)\right)$$

by (4.9), where $r_{(S_1,\ldots,S_n)} \in [0,1]$ satisfy (4.3). □

Thanks to (4.10), we obtain the following corollary.

Corollary 4.5. *Let* $T = \sum_{S\in Supp(T)} m_S \chi_S$ *be a free random variable in* $(\mathcal{M}_p, \gamma_p)$. *Then*

$$\gamma_p(T^n) = \sum_{(S_1,\ldots,S_n)\in Supp(T)^n} \left(r_{(S_1,\ldots,S_n)}\left(\psi\left(\prod_{j=1}^{n}(m_{S_j})^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\left(1-\frac{1}{p}\right)\right), \tag{4.11}$$

$$\gamma_p((T^*)^n) = \sum_{(S_1,\ldots,S_n)\in Supp(T)^n} \left(r_{(S_1,\ldots,S_n)}\left(\psi\left(\prod_{j=1}^{n}((m_{S_j}^*)^{S_j})^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\left(1-\frac{1}{p}\right)\right), \tag{4.12}$$

$$\gamma_p\left(\prod_{j=1}^{n} T^{r_j}\right) = \sum_{(S_1,\ldots,S_n)\in Supp(T)^n} \left(r_{(S_1,\ldots,S_n)}\left(\psi\left(\prod_{j=1}^{n}([m_{S_j}^{r_j}]^{S_j})^{\left(\bigcap_{i=1}^{j-1} S_i\right)}\right)\right)\left(1-\frac{1}{p}\right)\right), \tag{4.13}$$

where Y^n *means the Cartesian product* $Y \times \ldots \times Y$ *of* n*-copies of an arbitrary set* Y *for all* $n \in \mathbb{N}$ *and* $r_1, \ldots, r_n \in \{1, *\}$.

Let us now consider certain specific cases.

Definition 4.6. Let M be a von Neumann subalgebra of $B(H)$, and assume Γ is an algebraic object (a semigroup, or a monoid, or a group, or a groupoid, etc.), acting on M via an action β. Also, suppose that there exists a well-defined linear functional ψ on M, inducing a W^*-probability space (M, ψ) in $B(H)$. We say that ψ is β-invariant, if $\psi\left(\beta(g)(m)\right) = \psi(m)$ for all $m \in M$ and $g \in \Gamma$.

Assume now that our linear functional ψ on M is α-invariant for the monoidal action α of $\sigma(\mathbb{Q}_p)$, i.e., assume that $\psi\left(m^S\right) = \psi(m)$ for all $m \in M$ and $S \in \sigma(\mathbb{Q}_p)$.

Corollary 4.7. *Let $T_k = \sum_{S_k \in Supp(T_k)} m_{S_k} \chi_{S_k}$ be free random variables in the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$, for $k = 1, \ldots, n$, for $n \in \mathbb{N}$. Assume that the linear functional ψ on M is α-invariant. Then*

$$\gamma_p\left(\prod_{j=1}^{n} T_j^{r_j}\right) = \sum_{(S_1,\ldots,S_n)\in \prod_{j=1}^{n} Supp(T_j)} \left(r_{(S_1,\ldots,S_n)} \left(\psi\left(\prod_{j=1}^{n}\left(m_{S_j}^{r_j}\right)\right)\right)\left(1-\frac{1}{p}\right)\right), \tag{4.14}$$

and hence, if $T = \sum_{S \in Supp(T_k)} m_S \chi_S \in (\mathcal{M}_p, \gamma_p)$, then

$$\gamma_p\left(T^n\right) = \sum_{(S_1,\ldots,S_n)\in Supp(T)^n} \left(r_{(S_1,\ldots,S_n)} \left(\psi\left(\prod_{j=1}^{n}\left(m_{S_j}\right)\right)\right)\left(1-\frac{1}{p}\right)\right), \tag{4.15}$$

$$\gamma_p\left((T^*)^n\right) = \sum_{(S_1,\ldots,S_n)\in Supp(T)^n} \left(r_{(S_1,\ldots,S_n)} \left(\psi\left(\prod_{j=1}^{n}\left(m_{S_j}^{*}\right)\right)\right)\left(1-\frac{1}{p}\right)\right), \tag{4.16}$$

$$\gamma_p\left(\prod_{j=1}^{n} T^{r_j}\right) = \sum_{(S_1,\ldots,S_n)\in Supp(T)^n} \left(r_{(S_1,\ldots,S_n)} \left(\psi\left(\prod_{j=1}^{n}\left(m_{S_j}^{r_j}\right)\right)\right)\left(1-\frac{1}{p}\right)\right) \tag{4.17}$$

for all $n \in \mathbb{N}$ and $r_1, \ldots, r_n \in \{1, *\}$, where $r_{(S_1,\ldots,S_n)}$ satisfy (4.3).

Proof. Clearly, the formula (4.14) is by (4.10), and the formulae (4.15), (4.16) and (4.17) are proven by (4.11), (4.12) and (4.13), respectively, under the α-invariance of ψ. □

Now, let us go back to the general case without α-invariance of ψ. Let $(\mathcal{M}_p, \gamma_p)$ be the p-adic dynamical W^*-probability space, and let $m_1\chi_{S_1}, \ldots, m_n\chi_{S_n}$ be free random variables in it, for $n \in \mathbb{N}$, where $m_1, \ldots, m_n \in M$, and $S_1, \ldots, S_n \in \sigma(\mathbb{Q}_p)$. Then, we have

$$\begin{aligned}\gamma_p\left(\prod_{j=1}^{n}\left(m_j\chi_{S_j}\right)^{r_j}\right) &= \gamma_p\left(\prod_{j=1}^{n}[m_j^{r_j}]^{S_j}\chi_{\bigcap_{j=1}^{n} S_j}\right) \\ &\overset{\text{by (4.10)}}{=} r_0\left(\psi\left(\prod_{j=1}^{n}\left([m_j^{r_j}]^{S_j}\right)^{\bigcap_{j=1}^{n-1} S_j}\right)\right)\left(1-\frac{1}{p}\right),\end{aligned} \tag{4.18}$$

where $[m_j^{r_j}]^{S_j}$ are in the sense of (3.16), and $r_0 \in [0,1]$ satisfies (4.3). So, one can obtain that

$$
\begin{aligned}
& k_n\left((m_1\chi_{S_1})^{r_1}, \ldots, (m_n\chi_{S_n})^{r_n}\right) \\
& = \sum_{\pi\in NC(n)} (\gamma_p)_\pi \left([m_1^{r_1}]^{S_1}\chi_{S_1}, \ldots, [m_n^{r_n}]^{S_n}\chi_{S_n}\right)\mu(\pi, 1_n) \\
& = \sum_{\pi\in NC(n)} \left(\prod_{V\in\pi} (\gamma_p)_V \left([m_1^{r_1}]^{S_1}\chi_{S_1}, \ldots, [m_n^{r_n}]^{S_n}\chi_{S_n}\right)\mu\left(0_{|V|}, 1_{|V|}\right)\right) \\
& \text{(by the Möbius inversion (see Section 4.1))} \qquad (4.19) \\
& = \sum_{\pi\in NC(n)} \left(\prod_{V=(i_1,\ldots,i_k)\in\pi} \gamma_p\left([m_{i_1}^{r_{i_1}}]^{S_{i_1}}\chi_{S_{i_1}} \cdots [m_{i_k}^{r_{i_k}}]^{S_{i_k}}\chi_{S_{i_k}}\right)\mu\left(0_k, 1_k\right)\right) \\
& \overset{\text{by (4.18)}}{=} \sum_{\pi\in NC(n)} \left(\prod_{V=(i_1,\ldots,i_k)\in\pi} \left(r_V\left(\psi\Big(\prod_{t=1}^{k}\Big([m_{i_t}^{r_{i_t}}]^{S_{i_t}}\Big)^{\cap_{t=1}^{k-1} S_{i_t}}\Big)\right)\left(1-\frac{1}{p}\right)\right)\mu(0_k, 1_k)\right),
\end{aligned}
$$

where $r_V \in [0,1]$ satisfy (4.3).

By (4.19), we obtain the following inner free structure of the p-adic dynamical W^*-algebra $\mathcal{M}_p$, with respect to γ_p.

Theorem 4.8. *Let $m_1\chi_S$, and $m_2\chi_S$ be free random variables in the p-adic dynamical W^*-probability space $(\mathcal{M}_p, \gamma_p)$, with $m_1, m_2 \in M$, and $S \in \sigma(\mathbb{Q}_p) \setminus \{\varnothing\}$. Also, assume that S is not a measure-zero element in $\sigma(\mathbb{Q}_p)$. Then $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in the W^*-probability space (M, ψ), if and only if $m_1\chi_S$ and $m_2\chi_S$ are free in $(\mathcal{M}_p, \gamma_p)$.*

Proof. Assume that S is not of measure-zero.

$(\Rightarrow)$ Assume that $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in (M, ψ), i.e., the W^*-subalgebras M_1 and M_2 generated by them, respectively, are free in (M, ψ). Then, by definition, all mixed free $*$-cumulants of them (with respect to the linear functional ψ) vanish (see Section 4.1), i.e.,

$$k_n^{\psi}\left(u_{i_1}^{r_1}, \ldots, u_{i_n}^{r_n}\right) = 0 \quad \text{in } \mathbb{C}$$

for all $n \in \mathbb{N} \setminus \{1\}$, where $(u_{i_1}, \ldots, u_{i_n}) \in \{m_1, m_2, m_1^S, m_2^S\}$ is "mixed", and $(i_1, \ldots, i_n) \in \{1,2\}^n$, and $(r_1, \ldots, r_n) \in \{1, *\}^n$. Here, k_n^{ψ} means the free cumulant with respect to ψ.

Consider now the mixed free $*$-cumulants of $m_1\chi_S$ and $m_2\chi_S$ in $(\mathcal{M}_p, \gamma_p)$ for a fixed $S \in \sigma(\mathbb{Q}_p)$. By (4.19),

$$
\begin{aligned}
& k_n\left((m_{i_1}\chi_S)^{r_1}, \ldots, (m_{i_n}\chi_S)^{r_n}\right) \\
& = \sum_{\pi\in NC(n)} \left(\prod_{V=(j_1,\ldots,j_n)\in\pi} \left(r_V\Big(\psi\big(\prod_{t=1}^{k}\big([m_{j_t}^{r_{j_t}}]^{S_{i_t}}\big)^{\cap_{t=1}^{k-1} S_{j_t}}\big)\Big)\left(1-\frac{1}{p}\right)\right)\mu(0_k, 1_k)\right),
\end{aligned}
$$

where $r_V \in [0,1]$ satisfy (4.3), and where all S_{j_t} are identical to S, and hence,

$$= \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,\ldots,j_n)\in\pi} \left(r_V \left(\psi \left(\prod_{t=1}^{k} \left([m_{j_t}^{r_{j_t}}]^S \right)^S \right) \right) \left(1 - \frac{1}{p}\right) \right) \mu\left(0_k, 1_k\right) \right)$$

$$= r_S \left(1 - \frac{1}{p}\right) \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,\ldots,j_n)\in\pi} \left(\psi \left(\prod_{t=1}^{k} \left([m_{j_t}^{r_{j_t}}]^S \right)^S \right) \right) \mu\left(0_k, 1_k\right) \right),$$

since all quantities r_V are identical in $[0,1]$, say r_S, then

$$= r_S \left(1 - \frac{1}{p}\right) \left(k_n^{\psi}(u_{i_1}^{r_1}, \ldots, u_{i_n}^{r_n}) \right) = r_S \left(1 - \frac{1}{p}\right) \cdot 0 = 0$$

for all $n \in \mathbb{N} \setminus \{1\}$, where $(u_{i_1}, \ldots, u_{i_n}) \in \{m_1, m_1^S, m_2, m_2^S\}$ is a mixed n-tuple. It shows that two free random variables $m_1\chi_S$ and $m_2\chi_S$ are free in $(\mathcal{M}_p, \gamma_p)$, whenever m_1 and m_2 are free in (M, ψ).

($\Leftarrow$) Assume now that two free random variables $m_1\chi_S$ and $m_2\chi_S$ are free in $(\mathcal{M}_p, \gamma_p)$, i.e., all their mixed free $*$-cumulants vanish, i.e.,

$$\begin{aligned} &k_n \left((m_{i_1}\chi_S)^{r_1}, \ldots, (m_{i_n}\chi_S)^{r_n} \right) \\ &= r_S \left(1 - \frac{1}{p}\right) \\ &\cdot \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1,\ldots,j_n)\in\pi} \left(\psi \left(\prod_{t=1}^{k} \left([m_{j_t}^{r_{j_t}}]^S \right)^S \right) \right) \mu\left(0_k, 1_k\right) \right) = 0 \end{aligned} \tag{4.20}$$

for all $n \in \mathbb{N}$, where $(i_1, \ldots, i_n) \in \{1,2\}^n$, and $(r_1, \ldots, r_n) \in \{1, *\}^n$, and where r_S satisfies (4.3). Here, we need to notice that each block V induces $r_V \in [0,1]$ satisfying (4.3), but they are identical to r_S, because S is uniquely fixed now.

The formula (4.20) is identical to

$$r_S \left(1 - \frac{1}{p}\right) \left(k_n^{\psi}(u_{i_1}^{r_1}, \ldots, u_{i_n}^{r_n}) \right)$$

for the mixed n-tuple $(u_{i_1}, \ldots, u_{i_n})$ of $\{m_1, m_1^S\} \cup \{m_2, m_2^S\}$. Since S is assumed not to be of measure-zero, to satisfy

$$r_S \left(1 - \frac{1}{p}\right) \left(k_n^{\psi}(u_{i_1}^{r_1}, \ldots, u_{i_n}^{r_n}) \right) = 0,$$

as in (4.20), one must have

$$k_n^{\psi}(u_{i_1}^{r_1}, \ldots, u_{i_n}^{r_n}) = 0$$

for all mixed n-tuple $(u_{i_1}, \ldots, u_{i_n})$. Equivalently, $\{m_1, m_1^S\}$ and $\{m_2, m_2^S\}$ are free in (M, ψ).

It shows that if two free random variables $m_1\chi_S$ and $m_2\chi_S$ are free in $(\mathcal{M}_p, \gamma_p)$, then m_1 and m_2 are free in (M, ψ), whenever S is not of measure-zero. $\square$

The above theorem shows that, the freeness of (M, ψ) acts like a kind of free-filterizations for the inner freeness of $(\mathcal{M}_p, \gamma_p)$.

Corollary 4.9. *Let M_1 and M_2 be W^*-subalgebras of M in $B(H)$. Assume that $S \in \sigma(\mathbb{Q}_p)$ and S is not of measure-zero in $\sigma(\mathbb{Q}_p)$. Then*

$$\begin{aligned}&\{M_1, \alpha_S(M_1)\} \text{ and } \{M_2, \alpha_S(M_2)\} \text{ are free in } (M, \psi) \text{ if and only if} \\ &M_1 \otimes_\alpha \mathbb{C}\left[\{\chi_S\}\right] \text{ and } M_2 \otimes_\alpha \mathbb{C}\left[\{\chi_S\}\right] \text{ are free in } (\mathcal{M}_p, \gamma_p).\end{aligned} \tag{4.21}$$

It is not difficult to check that if $S \cap U_p = \varnothing$, then the families

$$\{m\chi_S : m \in M\} \quad \text{and} \quad \{m\chi_Y : m \in M, Y \subseteq U_p \text{ in } \sigma(\mathbb{Q}_p)\}$$

are free in $(\mathcal{M}_p, \gamma_p)$. Indeed, let $m_1\chi_S$ and $m_2\chi_{U_p} \in \mathcal{M}_p$, with $m_1, m_2 \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Assume that $S \cap U_p$ is empty. Since $S \cap U_p = \varnothing$, all mixed cumulants of $m_1\chi_S$ and $m_2\chi_{U_p}$ have $r_V = 0$, for some $V \in \pi$ in (4.19), for all $\pi \in NC(n)$. Therefore, one obtains the following inner freeness condition of $(\mathcal{M}_p, \gamma_p)$.

Proposition 4.10. *Let $S \in \sigma(\mathbb{Q}_p)$ such that $S \cap U_p = \varnothing$. Then the subsets*

$$\{m\chi_S : m \in M\} \quad \text{and} \quad \{m\chi_Y : m \in M, Y \subseteq U_p \text{ in } \sigma(\mathbb{Q}_p)\}$$

are free in $(\mathcal{M}_p, \gamma_p)$.

Proof. The proof is done by the discussion of the very above paragraph. □

Motivated by the above proposition, we obtain the following general result.

Theorem 4.11. *Let $S_1, S_2 \in \sigma(\mathbb{Q}_p)$ be such that $S_1 \neq S_2$.*

$$\begin{aligned}&\text{If } S_1 \cap S_2 = \varnothing, \text{ then the subsets } \{m\chi_{S_1} : m \in M\} \text{ and } \{a\chi_{S_2} : a \in M\} \\ &\text{are free in } (\mathcal{M}_p, \gamma_p).\end{aligned} \tag{4.22}$$

Proof. The proof is a little modification of the proof of the above proposition. Indeed, we can check that

$$S_1 \cap S_2 = \varnothing \implies (S_1 \cap U_p) \cap (S_1 \cap U_p) = \varnothing.$$

So, we can apply the above proposition. □

4.3. p-ADIC DYNAMICAL W^*-PROBABILITY SPACES

In this section, we extend the results we obtained in Section 4.2. Remark that, in Section 4.2, we fix an element χ_{U_p} of $\mathfrak{M}_p$ to construct

$$M_p = M \otimes_\alpha \mathbb{C}\left[\{\chi_{U_p}\}\right] \overset{*\text{-iso}}{=} M,$$

i.e., we needed to define a suitable conditional expectation E_p of (4.1) from $\mathcal{M}_p$ onto $M = M_p$, satisfying

$$E_p\left(m\chi_S\right) = m\left(r\chi_{U_p}\right),$$

for $r \in [0,1]$ satisfying (4.3). In such a case, the linear functional $\gamma_p = \gamma \circ E_p$ of (4.4) on $\mathcal{M}_p$ is well-determined by the well-defined linear functional γ of (4.2) on M_p.

One may do the same process by fixing $p^k U_p$ instead of fixing U_p, for $k \in \mathbb{Z}$. Recall that $p^k U_p$ are the boundaries $p^k \mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p$ of $p^k \mathbb{Z}_p$, for all $k \in \mathbb{Z}$, i.e., for a fixed $k \in \mathbb{Z}$, define

$$M_{p:k} \overset{def}{=} M \otimes_\alpha \mathbb{C}\left[\{\chi_{p^k U_p}\}\right] \overset{*\text{-iso}}{=} M, \tag{4.23}$$

(i.e., M_p of Section 4.2 is identical to $M_{p:0}$ under (4.23)), and construct a conditional expectation

$$E_{p:k} : \mathcal{M}_p = \mathcal{M}_p^0 \to M_{p:k}$$

by a linear morphism satisfying that

$$E_{p:k}\left(m\chi_S\right) = m\chi^0_{S \cap p^k U_p}, \tag{4.24}$$

with

$$\chi^0_{S \cap p^k U_p} = r\chi_{p^k U_p},$$

where $r \in [0,1]$ satisfying

$$\int_{\mathbb{Q}_p} \chi^0_{S\cap p^k U_p} d\rho_p = r \int_{\mathbb{Q}_p} \chi_{p^k U_p} d\rho_p = r\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right). \tag{4.25}$$

The quantities r of (4.24), satisfying (4.25), have to be chosen in the interval $[0,1]$ of $\mathbb{R}$, since

$$\rho_p\left(S \cap p^k U_p\right) \leq \rho_p\left(p^k U_p\right),$$

in general, for fixed $k \in \mathbb{Z}$.

Then, similar to Section 4.2, $E_{p:k}$ is a well-defined conditional expectation from $\mathcal{M}_p$ onto $M_{p:k} = M$.

And then, for the fixed $k \in \mathbb{Z}$, define a linear functional

$$\gamma_k : M_{p:k} \to \mathbb{C}$$

by a linear morphism satisfying

$$\gamma_k\left(m\chi_{p^k U_p}\right) \overset{def}{=} \psi(m) \int_{\mathbb{Q}_p} \left(\chi_{p^k U_p}\right) d\rho_p = \psi(m)\left(\tfrac{1}{p^k} - \tfrac{1}{p^{k+1}}\right). \tag{4.26}$$

Then one has a well-defined linear functional

$$\gamma_{p:k} : \mathcal{M}_p \to \mathbb{C}$$

defined by

$$\gamma_{p:k} \overset{def}{=} \gamma_k \circ E_{p:k} \quad \text{for all} \quad k \in \mathbb{Z}. \tag{4.27}$$

Note that our linear functional γ_p in the sense of (4.5) is identified with $\gamma_{p:0}$ of (4.27).

Observation 4.12. *Let us replace $M_p = M_{p:0}$ of Section 4.2 to $M_{p:k}$, for $k \in \mathbb{Z}$. Then the formulae* (4.8) *through* (4.20) *can be re-obtained by replacing factors* $(1-\frac{1}{p})$ *to* $(\frac{1}{p^k} - \frac{1}{p^{k+1}})$. *So, the freeness conditions* (4.21) *and* (4.22) *are same under* $(\mathcal{M}_p, \gamma_{p:k})$*-settings. For instance, if $m_j \chi_{S_j} \in (\mathcal{M}_p, \gamma_{p:k})$, for $j = 1, \ldots, n$, for $n \in \mathbb{N}$, then*

$$\gamma_{p:k}\left(\prod_{j=1}^{n} m_j \chi_{S_j}\right) = r_0 \left(\psi\left(m_1 \prod_{j=2}^{N} m_j^{\bigcap_{i=1}^{j-1} S_i}\right)\right)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)$$

for some $r_0 \in [0,1]$ satisfying (4.25) (*see* (4.9)).

The first main result of this section, Observation 4.12, shows that one can have systems of W^*-probability spaces

$$\{(\mathcal{M}_p, \gamma_{p:k})\}_{k \in \mathbb{Z}},$$

sharing similar free probability with $(\mathcal{M}_p, \gamma_p = \gamma_{p:0})$ of Section 4.2.

Moreover, one can consider the following. By taking pairwise distinct $k_1, \ldots, k_n$ in $\mathbb{Z}$, for $n \in \mathbb{N}$, define a ρ_p-measurable subset

$$U_{k_1,\ldots,k_n} \overset{def}{=} \bigcup_{i=1}^{n} (p^{k_i} U_p) = \bigsqcup_{i=1}^{n} (p^{k_i} U_p)$$

in $\sigma(\mathbb{Q}_p)$. Define now a subalgebra $M_{p:k_1,\ldots,k_n}$ of the p-adic dynamical W^*-algebra $\mathcal{M}_p$ by

$$M_{p:k_1,\ldots,k_n} = M \otimes \mathbb{C}\left[\{\chi_Y : Y = p^{k_j} U_p \in U_{k_1,\ldots,k_n}\}\right] \overset{*\text{-iso}}{=} M^{\oplus n}.$$

Define a conditional expectation

$$E_{p:k_1,\ldots,k_n} : \mathcal{M}_p = \mathcal{M}_p^0 \to M_{p:k_1,\ldots,k_n} = M^{\oplus n}$$

by a linear morphism satisfying that

$$E_{p:k_1,\ldots,k_n}(m\chi_S) = m\left(\sum_{j=1}^{n} r_j \chi_{p^j U_p}\right) \tag{4.28}$$

for all $m \in M$, $S \in \sigma(\mathbb{Q}_p)$, where $r_j \in [0,1]$ satisfy (3.3) and (3.4) (as in (4.3) and (4.25)) for $j = 1, \ldots, n$, where

$$\int_{\mathbb{Q}_p} \chi_{S \cap U_{k_1,\ldots,k_n}} d\rho_p = \int_{\mathbb{Q}_p} \left(\sum_{j=1}^{n} r_j \chi_{p^j U_p}\right) d\rho_p, \tag{4.29}$$

as in (3.5).

By (4.24) with (4.25), and (4.28) with (4.29), one can re-define the conditional expectation $E_{p:k_1,\ldots,k_n}$ by the morphism from $\mathcal{M}_p$ onto $M^{\oplus n} = M_{p:k_1,\ldots,k_n}$ by

$$E_{p:k_1,\ldots,k_n} \overset{def}{=} \bigoplus_{j=1}^{n} E_{p:k_j}, \tag{4.30}$$

where $E_{p:k_j}$ are in the sense of (4.24), for all $j = 1, \ldots, n$.

Similarly, define now a linear functional

$$\gamma_{k_1,\ldots,k_n} : M_{p:k_1,\ldots,k_n} = M^{\oplus n} \to \mathbb{C}$$

by

$$\gamma_{k_1,\ldots,k_n} \overset{def}{=} \sum_{j=1}^{n} \gamma_{k_j}, \tag{4.31}$$

where γ_{k_j} are in the sense of (4.26). By the bounded linearity of γ_{k_j}, the morphism $\gamma_{k_1,\ldots,k_n}$ is again bounded linear on $M_{p:k_1,\ldots,k_n}$.

Thus, by (4.30) and (4.31), we can define a linear functional $\gamma_{p:k_1,\ldots,k_n}$ on $\mathcal{M}_p = \mathcal{M}_p^0$ by

$$\gamma_{p:k_1,\ldots,k_n} \overset{def}{=} \gamma_{k_1,\ldots,k_n} \circ E_{p:k_1,\ldots,k_n} : \mathcal{M}_p \to \mathbb{C}. \tag{4.32}$$

So, we obtain a well-defined W^*-probability space $(\mathcal{M}_p, \gamma_{p:k_1,\ldots,k_n:})$. More generally, we have a system of W^*-probability spaces,

$$\bigcup_{n=1}^{\infty} \{(\mathcal{M}_p, \gamma_{p:k_1,\ldots,k_n}) : (k_1, \ldots, k_n) \in \mathbb{Z}^n\}, \tag{4.33}$$

generalizing the system $\{(\mathcal{M}_p, \gamma_{p:k})\}_{k\in\mathbb{Z}}$.

Remark that, since

$$E_{p:k_1,\ldots,k_n} = \bigoplus_{j=1}^{n} E_{p:k_j} \quad \text{(the direct sum of morphisms)},$$

we have

$$\begin{aligned}\gamma_{p:k_1,\ldots,k_n} &\overset{def}{=} \gamma_{k_1,\ldots,k_n} \circ E_{p:k_1,\ldots,k_n} \\ &= (\gamma_{k_1,\ldots,k_n}) \circ \left(\bigoplus_{j=1}^{n} E_{p:k_j}\right) = \sum_{j=1}^{n} \gamma_{k_j} \circ E_{p:k_j} = \sum_{j=1}^{n} \gamma_{p:k_j},\end{aligned}$$

i.e., we obtain

$$\gamma_{p:k_1,\ldots,k_n} = \sum_{j=1}^{n} \gamma_{p:k_j} \tag{4.34}$$

for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $n \in \mathbb{N}$.

Observation 4.13. *By the construction* (4.34) *of* $(\mathcal{M}_p,\ \gamma_{p:k_1,\ldots,k_n})$ *the formulae* (4.8) *through* (4.20) *of Section 4.2 can be extendable to similar corresponding results under the* $(\mathcal{M}_p, \gamma_{p:k_1,\ldots,k_n})$*-settings for all* $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ *and all* $n \in \mathbb{N}$. *For example, if* $m_j\chi_{S_j} \in (\mathcal{M}_p,\ \gamma_{p:k_1,\ldots,k_n})$ *for* $j = 1,\ldots,l$ *and* $l \in \mathbb{N}$, *then the formula* (4.9) *can be extendable to the following*

$$\gamma_{p:k_1,\ldots,k_n}\left(\prod_{j=1}^{l} m_j\chi_{S_j}\right) = \sum_{j=1}^{n}\gamma_{p:k_j}\left(\prod_{j=1}^{l} m_j\chi_{S_j}\right)$$
$$= \sum_{j=1}^{n} r_j\left(\psi\left(m_1\prod_{j=2}^{N} m_j^{\bigcap_{i=1}^{j-1}S_i}\right)\right)\left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right),$$

where $r_j \in [0,1]$ *satisfy* (4.25). *Furthermore, the freeness conditions* (4.21) *and* (4.22) *can be extendable to the similar results under* $(\mathcal{M}_p, \gamma_{p:k_1,\ldots,k_n})$*-settings, by* [19].

The above second main result of this section, Observation 4.13, shows that we can naturally extend our $(\mathcal{M}_p, \gamma_{p:k})$ -settings (extended from the $(\mathcal{M}_p, \gamma_p = \gamma_{p:0})$-setting) to $(\mathcal{M}_p, \gamma_{p:k_1,\ldots,k_n})$ -settings. By (3.2), whenever one takes an element S of $\sigma(\mathbb{Q}_p)$, the corresponding element χ_S has its identically-distributed element $\sum_{j=1}^{N} r_j\chi_{p^{k_j}U_p}$, by (3.3) and (3.4). Thus, one may obtain full free-distributional data for $m\chi_S \in \mathcal{M}_p$ as a free random variable of $(\mathcal{M}_p, \gamma_{p:k_1,\ldots,k_n})$ for $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ and all $n \in \mathbb{N}$.

4.4. FREE DISTRIBUTIONAL DATA OF CERTAIN OPERATORS IN $(\mathcal{M}_p, \gamma_{p:k})$

In this section, we concentrate on certain elements of a p-adic dynamical W^*-algebra $\mathcal{M}_p = \times_\alpha\sigma(\mathbb{Q}_p)$, and study their free distributional data by understanding them as free random variables in $(\mathcal{M}, \gamma_{p:k})$, for some $k \in \mathbb{Z}$.

For convenience, for all $k \in \mathbb{Z}$, we denote p^kU_p simply by $U_{p:k}$.

We obtain the following proposition.

Proposition 4.14. *Let* $T = m\chi_{U_{p:k}} \in (\mathcal{M}_p, \gamma_{p:k})$ *be a free random variable with* $m \in M, k \in \mathbb{Z}$. *Then*

$$\gamma_p(T^n) = \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\left(\psi\left(m\left(m^{U_{p:k}}\right)^{n-1}\right)\right), \tag{4.35}$$

$$\gamma_p(T^{*\ n}) = \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\left(\psi\left((m^*)^{U_{p:k}}\left((m^*)^{U_{p:k}}\right)^{n-1}\right)\right), \tag{4.36}$$

and

$$\gamma_p(T^{r_1}\ldots T^{r_n}) = \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\left(\psi\left(\prod_{j=1}^{N}[m^{r_j}]^{U_{p:k}}\right)\right) \tag{4.37}$$

for all $n \in \mathbb{N}$, *where* $(r_1,\ldots,r_n) \in \{1,*\}^n$ *in* (4.40), *and* $[m^{r_j}]^{U_{p:k}}$ *are in the sense of* (3.16).

Proof. By (4.15), if $T = m\chi_{U_{p:k}}$ in $(\mathcal{M}_p, \gamma_p)$, then

$$\gamma_p(T^n) = \left(\psi\left(m(m^{U_{p:k}})^{n-1}\right)\right)(\rho_p(U_{p:k})) = \left(\psi\left(m(m^{U_{p:k}})^{n-1}\right)\right)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right),$$

by Observation 4.12. Thus, we obtain (4.35).

Similarly, by (4.16), one can have that

$$\begin{aligned}\gamma_p((T^*)^n) &= \left(\psi\left((m^*)^{U_{p:k}}\left((m^*)^{U_{p:k}}\right)^{U_{p:k}})^{n-1}\right)\right)(\rho_p(U_{p:k})) \\ &= \left(\psi\left((m^*)^{U_{p:k}}\left((m^*)^{U_{p:k}}\right)^{U_{p:k}})^{n-1}\right)\right)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right),\end{aligned}$$

by Observation 4.12. So, we can get (4.36).

Also, by (3.14), we have that

$$\begin{aligned}\gamma_p(T^{r_1}\dots T^{r_n}) &= \left(\psi\left(\prod_{j=1}^{n}[m^{r_j}]^{U_{p:k}}\right)\right)(\rho_p(U_{p:k})) \\ &= \left(\psi\left(\prod_{j=1}^{n}[m^{r_j}]^{U_{p:k}}\right)\right)\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right),\end{aligned}$$

by Observation 4.12, where $[m^{r_j}]^{U_{p:k}}$ are in the sense of (4.18). Therefore, one can get (4.37). □

Recall that if χ_S be an element of the p-prime von Neumann algebra $\mathfrak{M}_p$, for $S \in \sigma(\mathbb{Q}_p)$, then there exist $N \in \mathbb{N} \cup \{\infty\}, k_j \in \mathbb{Z}$, and $0 \leq r_j \leq 1$ in $\mathbb{R}$, for $j = 1, \dots, N$, such that

$$\rho_p(S) = \int_{\mathbb{Q}_p} \chi_S d\rho_p = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right). \tag{4.38}$$

By (4.35), (4.36), (4.37) and (4.38), we obtain the following theorem.

Theorem 4.15. *Let $T = m\chi_S \in \mathcal{M}_p$, with $m \in M$, and $S \in \sigma(\mathbb{Q}_p)$. Assume further that there exist $N \in \mathbb{N}$, $k_j \in \mathbb{Z}$, and $0 \leq w_j \leq 1$ in $\mathbb{R}$ for $j = 1, \dots, N$, satisfying (4.38). Then, by understanding T as a free random variable of $(\mathcal{M}_p, \gamma_{p:k_1,\dots,k_N})$, we have*

$$\gamma_{p:k_1,\dots,k_N}(T^{r_1}\dots T^{r_n}) = \left(\psi\left(\prod_{t=1}^{n}[m^{r_t}]^S\right)\right)\left(\sum_{j=1}^{N} w_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right)\right) \tag{4.39}$$

for $(r_1, \dots, r_n) \in \{1, *\}^n$ and all $n \in \mathbb{N}$.

Proof. Since one has

$$\rho_p(\chi_S) = \sum_{j=1}^{N} r_j \rho_p(\chi_{U_{p:k_j}}) = \sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}} \right), \tag{4.40}$$

we have that

$$\begin{aligned}\gamma_{p:k_1,\ldots,k_N}(T^{r_1} \ldots T^{r_n}) &= \left(\psi \left(\prod_{j=1}^{n} [m^{r_j}]^S \right) \right) (\rho_p(S)) \\ &= \left(\psi \left(\prod_{j=1}^{n} [m^{r_j}]^S \right) \right) \left(\sum_{j=1}^{N} r_j \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}} \right) \right),\end{aligned}$$

by (4.36) and (4.40). □

5. AMALGAMATED FREE PROBABILITY ON $\mathcal{M}_p$ OVER $\mathbb{C}^{\oplus n}$

In this final section, we study amalgamated free probability on the p-adic dynamical W^*-algebra

$$\mathcal{M}_p = M \times_\alpha \sigma(\mathbb{Q}_p) \overset{*\text{-iso}}{=} M \otimes_\alpha \mathfrak{M}_p = \mathcal{M}_p^0$$

with amalgamation over $M^{\oplus n}$ (for some $n \in \mathbb{N}$), in terms of a certain conditional expectation $E_{(k_1,\ldots,k_n)}$,

$$E_{(k_1,\ldots,k_n)} : \mathcal{M}_p \to \mathbb{C}^{\oplus n}$$

for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and all $n \in \mathbb{N}$, defined by

$$E_{(k_1,\ldots,k_n)} \overset{def}{=} \bigoplus_{j=1}^{n} \gamma_{p:k_j} \tag{5.1}$$

where $\gamma_{p:k}$ are linear functionals in the sense of (4.26) for all $k \in \mathbb{Z}$.

Remark 5.1. Remark the difference between the conditional expectation $E_{(k_1,\ldots,k_n)}$ of (5.1), and $E_{p:k_1,\ldots,k_n}$ of (4.30). Indeed, the conditional expectations $E_{p:k_1,\ldots,k_n}$ of (4.30) are from $\mathcal{M}_p$ onto

$$M^{\oplus n} \overset{*\text{-iso}}{=} \bigoplus_{j=1}^{n} \left(M \otimes_\alpha \mathbb{C}\left[\{\chi_{p^{k_j} U_p}\} \right] \right),$$

not onto $\mathbb{C}^{\oplus n}$. We are considering different free-probabilistic structures here compared with those in Section 4.4.

Since each summand $\gamma_{p:k_j}$ of $E_{(k_1,\dots,k_n)}$ is a well-defined linear functionals on $\mathcal{M}_p$, the morphism $E_{(k_1,\dots,k_n)}$ becomes a well-defined conditional expectation from $\mathcal{M}_p$ onto $\mathbb{C}^{\oplus n}$ (e.g., see [10]), i.e., for any $(t_1,\dots,t_n) \in \mathbb{C}^{\oplus n}$,

$$\begin{aligned} & E_{(k_1,\dots,k_n)}\left((t_1,\dots,t_n)\right) = \left(\bigoplus_{j=1}^{n} \gamma_{p:k_j}\right)\left((t_1,\dots,t_n)\right) \\ & = \left(\gamma_{p:k_1}(t_1),\dots,\gamma_{p:k_n}(t_n)\right) = \left(t_1,\dots,t_n\right), \end{aligned} \tag{5.2}$$

and hence, for any $v \in \mathbb{C}^{\oplus n}$, $E_{(k_1,\dots,k_n)}(v) = v$. Moreover for any $(t_{j1},\dots,t_{jn}) \in \mathbb{C}^{\bigoplus n}$, for $k = 1, 2$ and all $m\chi_S \in \mathcal{M}_p$,

$$\begin{aligned} & E_{(k_1,\dots,k_n)}\left((t_{11},\dots,t_{1n})\,(m\chi_S)\,(t_{21},\dots,t_{2n})\right) \\ & = \left(\bigoplus_{j=1}^{n} \gamma_{p:k_j}\right)\left(\bigoplus_{j=1}^{n} (t_{1j}(m\chi_S)t_{2j})\right) \\ & = \bigoplus_{j=1}^{n} \left(\gamma_{p:k_j}\left(t_{1j}(m\chi_S)t_{2j}\right)\right) \\ & = \bigoplus_{j=1}^{n} t_{1j} r_j \psi(m) \left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right) t_{2j} \\ & = \bigoplus_{j=1}^{n} t_{1j}\left(\gamma_{p:k_j}(m\chi_S)\right) t_{2j} \\ & = \left((t_{11},\dots,t_{1n})\right)\left(\left(\bigoplus_{j=1}^{n} \gamma_{p:k_j}\right)(m\chi_S)\right)\left((t_{21},\dots,t_{2n})\right) \\ & = \left(t_{11},\dots,t_{1n}\right)\left(E_{(k_1,\dots,k_n)}(m\chi_S)\right)\left((t_{21},\dots,t_{2n})\right), \end{aligned} \tag{5.3}$$

where $r_j \in [0,1]$ satisfies (4.25). Thus, for any $x \in \mathcal{M}_p$, and $v_1, v_2 \in \mathbb{C}^{\bigoplus n}$, one has that

$$E_{(k_1,\dots,k_n)}\left(v_1 x v_2\right) = v_1\left(E_{(k_1,\dots,k_n)}(x)\right) v_2.$$

Also, one can get that

$$(m\chi_S)^* = (m^*)^S \chi_S, \text{ for any } m\chi_S \in \mathcal{M}_p, \tag{5.4}$$

so

$$E_{(k_1,\ldots,k_n)}\left((m\chi_S)^*\right) = \left(\bigoplus_{j=1}^{n}\gamma_{p:k_j}\right)\left((m^*)^S\chi_S\right)$$
$$= \bigoplus_{j=1}^{n}\gamma_{p:k_j}\left((m^*)^S\chi_S\right) = \bigoplus_{j=1}^{n}\left(\gamma_{p:k_j}\left(m\chi_S\right)\right)^*$$
$$= \left(\bigoplus_{j=1}^{n}\gamma_{p:k_j}(m\chi_S)\right)^* = \left(E_{(k_1,\ldots,k_n)}(m\chi_S)\right)^*,$$

and hence, for any $x \in \mathcal{M}_p$, we obtain

$$E_{(k_1,\ldots,k_n)}(x^*) = \left(E_{(k_1,\ldots,k_n)}(x)\right)^*.$$

Proposition 5.2. *The pairs $(\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$ are well-defined $\mathbb{C}^{\oplus n}$-valued W^*-probability space in the sense of Section 4.1 for all $(k_1,\ldots,k_n) \in \mathbb{Z}^n$ and all $n \in \mathbb{N}$.*

Proof. By (5.2), (5.3) and (5.4), the morphisms $E_{(k_1,\ldots,k_n)}$ of (5.1) are well-defined algebraic conditional expectations, moreover, by the boundedness of linear functionals $\{\gamma_{p:k}\}_{k\in\mathbb{Z}}$ on $\mathcal{M}_p$, they become bounded (or continuous) conditional expectations, for all $(k_1,\ldots,k_n) \in \mathbb{Z}^n$, for all $n \in \mathbb{N}$. Therefore, the pairs $(\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$ form $\mathbb{C}^{\oplus n}$-valued W^*-probability spaces in the sense of Section 4.1. □

Now, fix $n \in \mathbb{N}$, and $(k_1,\ldots,k_n) \in \mathbb{Z}^n$. Let $m\chi_S \in \mathcal{M}_p$, as a $\mathbb{C}^{\oplus n}$-valued free random variable in $(\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$, with $m \in (M,\psi)$ and $S \in \sigma(\mathbb{Q}_p)$. Then one can get that

$$\begin{aligned} E_{(k_1,\ldots,k_n)}\left(m\chi_S\right) &= \left(\bigoplus_{j=1}^{n}\gamma_{p:k_j}\right)\left(m\chi_S\right) \\ &= \bigoplus_{j=1}^{n}\left(\gamma_{p:k_j}\left(m\chi_S\right)\right) = \bigoplus_{j=1}^{n}\left(r_j\psi(m)\left(\frac{1}{p^{k_j}} - \frac{1}{p^{k_j+1}}\right)\right) \end{aligned} \tag{5.5}$$

where $r_j \in [0,1]$ satisfy (4.25).

If we denote the quantities $\frac{1}{p^k} - \frac{1}{p^{k+1}}$ by

$$\theta_{p:k}, \text{ for all primes } p \text{ and } k \in \mathbb{Z}, \tag{5.6}$$

then the formula (5.5) can be re-written by

$$E_{(k_1,\ldots,k_n)}\left(m\chi_S\right) = \psi(m)\left(r_1\theta_{p:k_1},\ldots,r_n\theta_{p:k_n}\right), \tag{5.7}$$

as a form of vectors in $\mathbb{C}^{\oplus n}$, with $r_j \in [0,1]$ satisfying (4.25), where $\theta_{p:k_j}$ are in the sense of (5.6).

Now, let $m_j\chi_j \in (\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$ and $j = 1, \ldots, s$ for $s \in \mathbb{N}$, with $m_j \in (M, \psi)$, and $S_j \in \sigma(\mathbb{Q}_p)$. Then, for any $(i_1, \ldots, i_l) \in \{1, \ldots, s\}^l$, one has

$$\begin{aligned} &E_{(k_1,\ldots,k_n)}\left(m_{i_1}\chi_{S_{i_1}}\ldots m_{i_l}\chi_{S_{i_l}}\right) \\ &= E_{(k_1,\ldots,k_n)}\left(\left(m_{i_1}m_{i_2}^{S_{i_1}}m_{i_3}^{S_{i_1}\cap S_{i_2}}\ldots m_{i_l}^{S_{i_1}\cap\ldots\cap S_{i_{l-1}}}\right)\chi_{S_{i_1}\cap\ldots\cap S_{i_l}}\right) \\ &= \psi\left(m_{i_1}m_{i_2}^{S_{i_1}}\ldots m_{i_l}^{S_{i_1}\cap\ldots\cap S_{i_{l-1}}}\right)(r_1\theta_{p:k_1},\ldots,r_n\theta_{p:k_n}), \end{aligned} \tag{5.8}$$

where $r_1, \ldots, r_n$ are in $[0, 1]$ satisfying

$$\rho_p\left(\bigcap_{i=1}^{l} S_{i_i}\right) = \sum_{j=1}^{n} r_j\theta_{p:k_j}, \tag{5.9}$$

by (5.5) and (5.7). Recall that if the intersection of $\bigcap_{i=1}^{l} S_{i_i}$ and $p^{k_j}U_p$ are empty for some $j \in \{1, \ldots, n\}$, then $r_{k_j} = 0$ in (5.8).

The following lemma is nothing but the re-written format of (5.8).

Lemma 5.3. *Let $m_j\chi_{S_j} \in (\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$ with $m_j \in (M, \psi)$ and $S_j \in \sigma(\mathbb{Q}_p)$ for $j = 1, \ldots, s$ and $s \in \mathbb{N}$. Then, for any $(j_1, \ldots, j_l) \in \{1, \ldots, s\}^l$ and $l \in \mathbb{N}$, one has*

$$\begin{aligned} &E_{(k_1,\ldots,k_n)}\left(\prod_{i=1}^{l} m_{j_i}\chi_{S_{j_i}}\right) \\ &= \psi\left(m_{j_1}\left(\prod_{i=1}^{l} m_{j_{i+1}}^{\bigcap_{u=1}^{i} S_{j_u}}\right)\right)\left(r_1^{(j_1,\ldots,j_l)}\theta_{p:k_1},\ldots,r_n^{(j_1,\ldots,j_l)}\theta_{p:k_n}\right), \end{aligned} \tag{5.10}$$

where $r_1^{(j_1,\ldots,j_l)}, \ldots, r_n^{(j_1,\ldots,j_l)}$ satisfy (5.9).

Under the same settings with the very above lemma, denote elements

$$m_{j_1}\left(\prod_{i=1}^{l} m_{j_{i+1}}^{\bigcap_{u=1}^{i} S_{j_u}}\right)$$

of (M, ψ) by $m_{(j_1,\ldots,j_l)}$. Then, the formula (5.10) can be re-written by

$$E_{(k_1,\ldots,k_n)}\left(\prod_{i=1}^{l} m_{j_i}\chi_{S_{j_i}}\right) = \psi\left(m_{(j_1,\ldots,j_l)}\right)\left(r_1^{(j_1,\ldots,j_l)}\theta_{p:k_1},\ldots,r_n^{(j_1,\ldots,j_l)}\theta_{p:k_n}\right) \tag{5.11}$$

for all $(j_1, \ldots, j_l) \in \{1, \ldots, s\}^l$ and $l \in \mathbb{N}$.

By (5.10) and (5.11), we can obtain the following $\mathbb{C}^{\oplus n}$-valued freeness condition on $\mathcal{M}_p$.

Theorem 5.4. *Let $m_j \chi_{S_j} \in \mathcal{M}_p$ with $m_j \in (M, \psi)$ and $S_j \in \sigma(\mathbb{Q}_p)$ for $j = 1, 2$. Then, $m_1 \chi_{S_1}$ and $m_2 \chi_{S_2}$ are free in $(\mathcal{M}_p, \gamma_{p:k_j})$ for all $j = 1, \ldots, n$, if and only if they are $\mathbb{C}^{\oplus n}$-valued free in $(\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$.*

Proof. ($\Rightarrow$) Assume that $m_1 \chi_{S_1}$ and $m_2 \chi_{S_2}$ are free in $(\mathcal{M}_p, \gamma_{p:k_j})$, i.e., all mixed free cumulants of them vanish for $\gamma_{p:k_j}$ for all $j = 1, \ldots, n$. By (5.11) (also, generally by [10]), we obtain that, for $T_i = m_i \chi_{S_i}$ $(i = 1, 2)$,

$$k_l^{(k_1,\ldots,k_n)}(T_{i_1}, \ldots, T_{i_l}) = \left(k_l^{(k_1)}(T_{i_1}, \ldots, T_{i_l}), \ldots, k_l^{(k_n)}(T_{i_1}, \ldots, T_{i_l})\right),$$

in $\mathbb{C}^{\oplus n}$ for all $(i_1, \ldots, i_l) \in \{1, 2\}^l$, and all $l \in \mathbb{N}$, where $k_l^{(k_j)}(\ldots)$ mean the free cumulants in terms of $\gamma_{p:k_j}$, for all $j = 1, \ldots, n$, and $k_l^{(k_1,\ldots,k_n)}(\ldots)$ means the $\mathbb{C}^{\oplus n}$-valued (amalgamated) free cumulant in terms of the conditional expectation $E_{(k_1,\ldots,k_n)}$ in the sense of Section 4.1 (see also [17]).

Therefore, if $(i_1, \ldots, i_l) \in \{1, 2\}^l$ are "mixed" for all $l \in \mathbb{N} \setminus \{1\}$, then

$$k_l^{(k_1,\ldots,k_n)}(T_{i_1}, \ldots, T_{i_l}) = (0, 0, \ldots, 0) \quad \text{in } \mathbb{C}^{\oplus n},$$

i.e., whenever $(i_1, \ldots, i_l)$ are mixed, the $\mathbb{C}^{\oplus n}$-valued mixed free cumulants of $m_1 \chi_{S_1}$ and $m_2 \chi_{S_2}$ vanish in $\mathbb{C}^{\oplus n}$. Equivalently, they are free in $(\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$.

($\Leftarrow$) Suppose T_1 and T_2 are $\mathbb{C}^{\oplus n}$-valued free in $(\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$, where $T_j = m_j \chi_{S_j}$ for $j = 1, 2$. Assume that there exists at least one $k_{j_0} \in \{k_1, \ldots, k_n\}$ in $\mathbb{Z}$, such that T_1 and T_2 are not free in $(\mathcal{M}_p, \gamma_{p:k_{j_0}})$. Then, there exists at least one mixed l-tuple $(i_1, \ldots, i_l) \in \{1, 2\}^l$ for some $l \in \mathbb{N} \setminus \{1\}$ such that

$$k_l^{(k_{j_0})}(T_{i_1}, \ldots, T_{i_l}) \neq 0 \quad \text{in } \mathbb{C}.$$

Let us fix such a mixed l-tuple $(i_1, \ldots, i_l)$ of $\{1, 2\}^l$. Then we have that

$$\begin{aligned} &k_l^{(k_1,\ldots,k_n)}(T_{i_1}, \ldots, T_{i_l}) \\ &= \left(k_l^{(k_1)}(T_{i_1}, \ldots, T_{i_l}), \ldots, k_l^{(k_{j_0})}(T_{i_1}, \ldots, T_{i_l}), \ldots, k_l^{(k_n)}(T_{i_1}, \ldots, T_{i_l})\right), \end{aligned}$$

which is a nonzero vector in $\mathbb{C}^{\oplus n}$. It contradicts our assumption that T_1 and T_2 are $\mathbb{C}^{\oplus n}$-valued free in $(\mathcal{M}_p, E_{(k_1,\ldots,k_n)})$.

Therefore, by ($\Rightarrow$) and ($\Leftarrow$), the relation (5.10) holds. □

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