q-ANALOGUE OF SUMMABILITY OF FORMAL SOLUTIONS OF SOME LINEAR q-DIFFERENCE-DIFFERENTIAL EQUATIONS

Hidetoshi Tahara and Hiroshi Yamazawa

Abstract. Let $q > 1$. The paper considers a linear $q$-difference-differential equation: it is a $q$-difference equation in the time variable $t$, and a partial differential equation in the space variable $z$. Under suitable conditions and by using $q$-Borel and $q$-Laplace transforms (introduced by J.-P. Ramis and C. Zhang), the authors show that if it has a formal power series solution $\hat{X}(t, z)$ one can construct an actual holomorphic solution which admits $\hat{X}(t, z)$ as a $q$-Gevrey asymptotic expansion of order 1.

Keywords: $q$-difference-differential equations, summability, formal power series solutions, $q$-Gevrey asymptotic expansions, $q$-Laplace transform.

Mathematics Subject Classification: 35C10, 35C20, 39A13.

1. INTRODUCTION

Let $m \geq 1$ be an integer, and let $(t, z) = (t, z_1, \ldots, z_d) \in \mathbb{C}_t \times \mathbb{C}_z^d$ be complex variables. For $r > 0$ we write $D_r = \{ t \in \mathbb{C} ; |t| \leq r \}$ and $D_r^* = \{ t \in \mathbb{C} ; 0 < |t| \leq r \}$. For $R > 0$ we write $D_R = \{ z \in \mathbb{C}_z^d ; |z| \leq R \}$ with $|z| = \max_{1 \leq i \leq d} |z_i|$. We denote by $O_R$ the set of all holomorphic functions in a neighbourhood of $D_R$, and by $O_R[[t]]$ the set of all formal power series in $t$ with coefficients in $O_R$.

For a holomorphic function $f(t, z)$ in a neighbourhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d$, we define the order of the zeros of the function $f(t, z)$ at $t = 0$ (we denote this by $\text{ord}_t(f)$) by

$$\text{ord}_t(f) = \min \{ k \in \mathbb{N} ; (\partial_t^k f)(0, z) \neq 0 \text{ near } z = 0 \},$$

where $\mathbb{N} = \{ 0, 1, 2, \ldots \}$.

Let us consider the linear partial differential equation

$$\sum_{j+|\alpha| \leq m} a_{j, \alpha}(t, z)(t \partial_t)^j \partial_z^\alpha X = F(t, z) \quad (1.1)$$

Where $a_{j, \alpha}(t, z)$ are coefficients depending on $t$ and $z$. The authors show that under suitable conditions and by using $q$-Borel and $q$-Laplace transforms, one can construct an actual holomorphic solution which admits the formal power series solution as a $q$-Gevrey asymptotic expansion of order 1.
with the unknown function $X = X(t, z)$, where $a_{j, \alpha}(t, z)$ ($j + |\alpha| \leq m$) and $F(t, z)$ are holomorphic functions in a neighbourhood of $(0, 0) \in \mathbb{C}_t \times \mathbb{C}^d_z$. The Newton polygon $N(1.1)$ of (1.1) is defined by

$$N(1.1) = \text{the convex hull of } \bigcup_{j + |\alpha| \leq m} C(j + |\alpha|, \text{ord}_t(a_{j, \alpha}))$$

in $\mathbb{R}^2$, where $C(a, b) = \{(x, y) \in \mathbb{R}^2; x \leq a, y \geq b\}$. See Miyake [8] and Ouchi [10] (though Ouchi used the word “the characteristic polygon” instead of “the Newton polygon”). Let us consider the following two cases:

Case 1. $N(1.1) = \{(x, y) \in \mathbb{R}^2; x \leq m, y \geq 0\}$.

Case 2. There is an integer $0 \leq m_0 < m$ such that

$$N(1.1) = \{(x, y) \in \mathbb{R}^2; x \leq m, y \geq \max\{0, x - m_0\}\}.$$

In Case 1, about the convergence of formal solutions of (1.1), by Baouendi-Goulaouic [1] we have the following result.

**Theorem 1.1.** Suppose the condition in Case 1,

$$a_{m, 0}(0, 0) \neq 0, \quad \text{and} \quad \text{ord}_t(a_{j, \alpha}) \geq 1 \text{ if } |\alpha| > 0.$$

Then, if (1.1) has a formal solution $\hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in O_R[[t]]$, it is convergent in a neighbourhood of the origin $(0, 0) \in \mathbb{C}_t \times \mathbb{C}^d_z$.

In Case 2, even if (1.1) has a formal solution, it is not convergent in general, but we can give a meaning to this formal solution by using the notion of Borel summability. By [10], we have the following theorem.

**Theorem 1.2.** Suppose the condition in Case 2,

$$a_{m_0, 0}(0, 0) \neq 0, \quad \frac{a_{m, 0}(t, 0)}{t^{m - m_0}} \bigg|_{t=0} \neq 0,$$

and

$$\text{ord}_t(a_{j, \alpha}) \geq \max\{1, j + |\alpha| - m_0 + 1\} \text{ if } |\alpha| > 0.$$

Then, if (1.1) has a formal solution $\hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in O_R[[t]]$, it is Borel summable in $t$ (uniformly in $z$ near $z = 0$) in a suitable direction.

Let $q > 1$. For a function $f(t, z)$ we define the $q$-difference operator $D_q$ by

$$(D_q f)(t, z) = \frac{f(qt, z) - f(t, z)}{qt - t}.$$  

In this paper, we will try to $q$-discretize equation (1.1) with respect to the time variable $t$ in the form

$$\sum_{j + |\alpha| \leq m} a_{j, \alpha}(t, z)(tD_q)^j \partial_z^\alpha X = F(t, z), \quad (1.2)$$

and we will consider the following problem.
Problem 1.3. Let \( \hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]] \) be a formal solution of (1.2). Then:

(1) \((q\text{-analogue of Theorem 1.1})\) Under what condition can we get the convergence of the formal solution \( \hat{X}(t, z) \)?

(2) \((q\text{-analogue of Theorem 1.2})\) Under what condition can we get a true solution \( W(t, z) \) of (1.2) which admits \( \hat{X}(t, z) \) as a \( q \)-Gevrey asymptotic expansion of order 1 (in the sense of Definition 1.4 given below)?

For \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( \epsilon > 0 \), we set

\[
\mathcal{Z}_\lambda = \{-\lambda q^m \in \mathbb{C} ; m \in \mathbb{Z}\},
\]

\[
\mathcal{Z}_{\lambda, \epsilon} = \bigcup_{m \in \mathbb{Z}} \{ t \in \mathbb{C} \setminus \{0\} ; |1 + \lambda q^m / t| \leq \epsilon \}.
\]

It is easy to see that if \( \epsilon > 0 \) is sufficiently small the set \( \mathcal{Z}_{\lambda, \epsilon} \) is a disjoint union of closed disks. The following definition is due to Ramis-Zhang [11].

**Definition 1.4.** Let \( \hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]] \) and let \( W(t, z) \) be a holomorphic function on \( (D^*_r \setminus \mathcal{Z}_\lambda) \times D_R \) for some \( r > 0 \). We say that \( W(t, z) \) admits \( \hat{X}(t, z) \) as a \( q \)-Gevrey asymptotic expansion of order 1, if there are \( M > 0 \) and \( H > 0 \) such that

\[
\left| W(t, z) - \sum_{n=0}^{N-1} X_n(z)t^n \right| \leq \frac{MH^N}{\epsilon} q^{N(N-1)/2} |t|^N
\]

holds on \( (D^*_r \setminus \mathcal{Z}_{\lambda, \epsilon}) \times D_R \) for any \( N = 0, 1, 2, \ldots \) and any sufficiently small \( \epsilon > 0 \).

To solve Problem 1.3 we will use the framework of \( q \)-Laplace and \( q \)-Borel transforms via the Jacobi theta function, developed by Ramis-Zhang [11] and Zhang [15]. In the case of \( q \)-difference equations (corresponding to ordinary differential equations), \( q \)-analogues of summability of formal solutions have been studied quite well by Zhang [14], Marotte-Zhang [7] and Ramis-Sauloy-Zhang [12]. In the case of \( q \)-difference-differential equations, we have some references, Malek [5, 6], Lastra-Malek [3] and Lastra-Malek-Sanz [4], but their equations are different from ours.

2. MAIN RESULTS

Throughout this paper, we let \( q > 1 \) be a real number, \( m \geq 1 \) be an integer, and \( \sigma > 0 \) be a real number. As a generalization of (1.2), we will treat the following equation

\[
\sum_{j+\sigma|\alpha| \leq m} a_{j, \alpha}(t, z) (tD_q)^j \partial_z^\alpha X = F(t, z)
\]

with the unknown function \( X = X(t, z) \), where \( a_{j, \alpha}(t, z) \) \((j + \sigma|\alpha| \leq m)\) and \( F(t, z) \) are holomorphic functions in a neighbourhood of \((0, 0) \in \mathbb{C}_t \times \mathbb{C}_z^d\).
In this case, we will use the $t$-Newton polygon (see the paper by Tahara-Yamazawa [13]): the $t$-Newton polygon $N_t(2.1)$ of equation (2.1) is defined by

$$N_t(2.1) = \text{the convex hull of } \bigcup_{j+\sigma|\alpha| \leq m} C(j, \text{ord}_t(a_{j,\alpha})).$$

in $\mathbb{R}^2$. Let us consider the following two cases:

**Case 1.** $N_t(2.1) = \{(x, y) \in \mathbb{R}^2; x \leq m, y \geq 0\}$.

**Case 2.** There is an integer $0 \leq m_0 < m$ such that

$$N_t(2.1) = \{(x, y) \in \mathbb{R}^2; x \leq m, y \geq \max\{0, x - m_0\}\}.$$

In Case 1, we can give a $q$-analogue of Theorem 1.1 in the following form:

**Theorem 2.1.** Suppose the condition in Case 1,

$$a_{m,0}(0,0) \neq 0, \quad \text{and} \quad \text{ord}_t(a_{j,\alpha}) \geq 1 \text{ if } |\alpha| > 0.$$

Then, if (2.1) has a formal solution $\tilde{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in O_R[[t]]$, it is convergent in a neighbourhood of the origin $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z$.

**Example 2.2.** Let us consider

$$(tD_q)^m X = A(z)t + B(z)t^p (tD_q)^j \partial_z^\alpha X,$$

where $A(z)$ and $B(z)$ are holomorphic functions in a neighbourhood of $z = 0$. In the case when $|\alpha| = 0$, if $j \leq m$ and $p \geq 1$ we can apply Theorem 2.1 to this equation. In the case when $|\alpha| > 0$, if $j \leq m - 1$ and $p \geq 1$ we can apply Theorem 2.1 to this equation. We note that for any $|\alpha| > 0$ by setting $\sigma = 1/|\alpha| > 0$ we have $j + \sigma|\alpha| \leq m$.

In Case 2, by assumption we have the expression

$$a_{j,0}(t, z) = t^{j-m_0}b_{j,0}(t, z) \quad \text{for } m_0 < j \leq m$$

for some holomorphic functions $b_{j,0}(t, z)$ ($m_0 < j \leq m$) in a neighbourhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_z$. We set

$$P(\xi, z) = \sum_{m_0 < j \leq m} \frac{b_{j,0}(0, z)}{(q-1)^{j/(j-1)/2}} \xi^{j-m_0} + \frac{a_{m_0,0}(0, z)}{(q-1)^{m_0} q^{m_0 (m_0 - 1)/2}}.$$

If the conditions $a_{m_0,0}(0,0) \neq 0$ and $b_{m,0}(0,0) \neq 0$ are satisfied, we see that $P(\xi, 0)$ is a polynomial of degree $m - m_0$ and it has $m - m_0$ non-zero roots $\tau_1, \ldots, \tau_{m-m_0}$. Then, the set $S$ of singular directions is defined by

$$S = \bigcup_{i=1}^{m-m_0} \{t\tau_i; t > 0\}.$$

As to a $q$-analogue of Theorem 1.2, we have the following result.
Theorem 2.3.

(1) Suppose the condition in Case 2,

\[ a_{m_0,0}(0,0) \neq 0, \quad \text{and} \quad \text{ord}_t(a_{j,\alpha}) \geq \max\{1, j - m_0 + 1\} \text{ if } |\alpha| > 0. \]

Then, if (2.1) has a formal solution \( \tilde{X}(t, z) = \sum_{n\geq0} X_n(z)t^n \in \mathcal{O}_R[[t]], \) we can find \( A > 0, h > 0 \) and \( 0 < R_1 < R \) such that \( |X_n(z)| \leq Ah^n q^{(n-1)/2} \) on \( D_{R_1} \) for any \( n = 0, 1, 2, \ldots. \)

(2) In addition, if the conditions

\[ \frac{a_{m_0,0}(t,0)}{t^{m-m_0}} \bigg|_{t=0} \neq 0 \quad (\text{this is equivalent to } b_{m,0}(0,0) \neq 0), \]

\[ \text{ord}_t(a_{j,\alpha}) \geq j - m_0 + 2, \quad \text{if } |\alpha| > 0 \text{ and } m_0 \leq j < m \]

are satisfied, for any \( \lambda \in \mathbb{C} \setminus (\{0\} \cup S) \) there are \( r > 0, R_1 > 0 \) and a holomorphic solution \( W(t, z) \) of (2.1) on \( (D_r^* \setminus \mathcal{L}_k) \times D_{R_1} \) such that \( W(t, z) \) admits \( \tilde{X}(t, z) \) as a q-Gevrey asymptotic expansion of order 1.

Example 2.4. Let \( 0 \leq m_0 < m \) and let us consider

\[ (tD_q)^{m_0}X = A(z)t + t^{m-m_0}(tD_q)^mX + B(z)t^n(tD_q)^j\partial_z^\alpha X, \]

where \( A(z) \) and \( B(z) \) are holomorphic functions in a neighbourhood of \( z = 0. \) In the case \( |\alpha| = 0, \) if \( j \leq m \) and \( p \geq \max\{1, j - m_0 + 1\} \) we can apply Theorem 2.3 to this equation. In the case \( |\alpha| > 0, \) if \( j \leq m - 1 \) and \( p \geq \max\{1, j - m_0 + 2\} \) we can apply Theorem 2.3 to this equation. In both cases, \( S \) is given by

\[ S = \{ z = te^{\sqrt{-1}\theta} \in \mathbb{C}; \ t > 0, \ \theta = 2\pi k/(m - m_0), 0 \leq k \leq m - m_0 - 1 \}. \]

The rest of this paper is organised as follows. In Section 3 we give a proof of Theorem 2.1, in Section 4 we show part (1) of Theorem 2.3, and in Sections 5 and 6 we prove part (2) of Theorem 2.3.

By the definition of \( D_q, \) we have

\[ (tD_q f)(t, z) = \frac{f(qt, z) - f(t, z)}{q - 1}. \]

If we define the operator \( \sigma_q \) by \( \sigma_q(f)(t, z) = f(qt, z), \) we can rewrite equation (2.1) to the following linear equation

\[ \sum_{j + \sigma|\alpha| \leq m} a_{j,\alpha}(t, z)(q - 1)^{-j}(\sigma_q - 1)^j\partial_z^\alpha X = F(t, z) \]

which is written in the form

\[ \sum_{j + \sigma|\alpha| \leq m} a_{j,\alpha}^*(t, z)(\sigma_q^j)\partial_z^\alpha X = F(t, z) \quad (2.2) \]
with
\[ a^*_{j,\alpha}(t, z) = \sum_{j \leq k \leq m - \sigma|\alpha|} a_{k,\alpha}(t, z)(q - 1)^{-k} \binom{k}{j} (-1)^{k-j}, \quad j + \sigma|\alpha| \leq m. \]

Therefore, in the proof of Theorems 2.1 and 2.3 in Sections 3–6 we will treat equation (2.2) instead of the original equation (2.1). In the discussion, we will use the norm \( \|\varphi\|_s = \max_{|z| \leq s} |\varphi(z)| \) and the following lemma.

**Lemma 2.5.** If a holomorphic function \( \varphi(z) \) on \( D_R \) satisfies
\[ \|\varphi\|_s \leq \frac{A}{(R - s)^a} \quad \text{for any} \quad 0 < s < R, \]
for some \( A > 0 \) and \( a \geq 0 \), we have the estimates
\[ \|\partial_z^i \varphi\|_s \leq \frac{(a + 1)eA}{(R - s)^{a+1}} \quad \text{for any} \quad 0 < s < R \text{ and } i = 1, \ldots, d. \]

For the proof, see [9] or Lemma 5.1.3 in [2].

### 3. PROOF OF THEOREM 2.1

Let us consider the equation
\[ \sum_{j + \sigma|\alpha| \leq m} a_{j,\alpha}(t, z)(\sigma_q)^j \partial_z^\alpha X = F(t, z), \quad (3.1) \]
where \( a_{j,\alpha}(t, z) \) \((j + \sigma|\alpha| \leq m)\) and \( F(t, z) \) are holomorphic functions in a neighbourhood of \((0, 0) \in \mathbb{C}_t \times \mathbb{C}^d_z\). To prove Theorem 2.1 it is enough to show the following proposition.

**Proposition 3.1.** Suppose the conditions
\[ a_{m,0}(0, 0) \neq 0, \quad \text{and} \quad \text{ord}_t(a_{j,\alpha}) \geq 1 \text{ if } |\alpha| > 0. \quad (3.2) \]

Then, if (3.1) has a formal solution \( \hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]] \), it is convergent in a neighbourhood of the origin \((0, 0) \in \mathbb{C}_t \times \mathbb{C}^d_z\).

**Proof.** By the assumption, we can expand \( a_{j,\alpha}(t, z) \) \((j + \sigma|\alpha| \leq m)\) and \( F(t, z) \) into the forms:
\[ a_{j,0}(t, z) = \sum_{k \geq 0} c_{j,0,k}(z)t^k \quad (0 \leq j \leq m), \]
\[ a_{j,\alpha}(t, z) = \sum_{k \geq 1} c_{j,\alpha,k}(z)t^k \quad (|\alpha| > 0), \]
\[ F(t, z) = \sum_{k \geq 0} F_k(z)t^k. \]
We may suppose that $R > 0$ is sufficiently small. Therefore, we may suppose $0 < R < 1$, that $c_{j,\alpha,k}(z)$ and $F_k(z)$ are all holomorphic functions on $D_R$, and that there are $B > 0$ and $h > 0$ satisfying $|c_{j,\alpha,k}(z)| \leq Bh^k$ ($j + \sigma|\alpha| \leq m$ and $k \geq 1$) and $|F_k(z)| \leq Bh^k$ ($k \geq 0$) on $D_R$. Since $a_m,0(0,0) \neq 0$ is supposed, we may also assume that $a_{m,0}(0,z) \neq 0$ on $D_R$. We set

$$C(\lambda, z) = \sum_{j \leq m} a_{j,0}(0,z)\lambda^j.$$  

It is clear that there are constant $c_0 > 0$ and a positive integer $\mu$ such that

$$|C(q^n, z)| \geq c_0(q^n)^m \quad \text{on } D_R \text{ for any } n \geq \mu. \quad (3.3)$$

Since $a_{j,0}(0,z) = c_{j,0,0}(z)$ ($0 \leq j \leq m$) holds, our equation (3.1) is written in the form

$$C(\sigma q, z)X = F(t,z) - \sum_{j + \sigma|\alpha| \leq m} \sum_{k \geq 1} c_{j,\alpha,k}(z)t^k(\sigma q)^j \partial_x^{\alpha} X. \quad (3.4)$$

Let

$$\hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]]$$

be a formal solution of (3.1). By substituting this into (3.4) and by comparing the coefficients of $t^n$ in both sides of the equation, we have the following recurrent formulas:

$$C(q^n, z)X_n = F_n(z)$$

and for $n \geq 1$

$$C(q^n, z)X_n = F_n(z) - \sum_{j + \sigma|\alpha| \leq m} \sum_{k=1}^n c_{j,\alpha,k}(z)(q^j)^{n-k} \partial_x^{\alpha} X_{n-k}. \quad (3.5)$$

We set $L = m/\sigma$; if $j + \sigma|\alpha| \leq m$ we have $|\alpha| \leq L$. To prove Proposition 3.1 it is enough to show the following lemma.

**Lemma 3.2.** There are $A > 0$ and $H > 0$ such that the estimate

$$\|\partial_x^{\alpha} X_n\|_s \leq \frac{AH^n}{(R-s)^{Ln}} \quad \text{for any } 0 < s < R \text{ and } |\alpha| \leq L \quad (3.6)$$

holds for any $n = 0, 1, 2, \ldots$.

**Proof of Lemma 3.2.** Let $\mu$ be as in (3.3). Since $\partial_x^{\alpha} X_n(z)$ ($n = 0, 1, \ldots, \mu$ and $|\alpha| \leq L$) are holomorphic functions on $D_R$, by taking $A > 0$ and $H > 0$ sufficiently large we have the condition (3.6) for $n = 0, 1, \ldots, \mu$. 

Let $n > \mu$, and suppose that (3.6) with $n$ replaced by $p$ is already proved for all $p < n$. Then, by (3.3), (3.5) and the induction hypothesis, we have
\[
\|X_n\|_s \leq \frac{1}{c_0(q^n)^m} \left[ Bh^n + \sum_{j+\sigma|\alpha| \leq m} \sum_{1 \leq k \leq n} Bh^k(q^j)^{n-k} \times \frac{AH^{n-k}}{(R-s)^{L(n-k)}} \right] 
\leq \frac{AH^n}{(R-s)^{L(n-1)}c_0(q^n)^m} \left[ \frac{B}{A} \left( \frac{h}{H} \right)^n + \sum_{j+\sigma|\alpha| \leq m} \sum_{1 \leq k \leq n} B \left( \frac{h}{H} \right)^k (q^j)^{n-k} \right],
\]
and so, by Lemma 2.5, we have
\[
\|\partial_z^\alpha X_n\|_s \leq \frac{AH^n e^{\alpha|L(n-1)+1| \ldots (L(n-1)+|\alpha|)}}{(R-s)^{L(n-1)+|\alpha|c_0(q^n)^m}} \times [B/A \left( \frac{h}{H} \right)^n + \sum_{j+\sigma|\alpha| \leq m} \sum_{1 \leq k \leq n} B \left( \frac{h}{H} \right)^k (q^j)^{n-k}].
\] (3.7)

for any $0 < s < R$. Here, we note that $n/q^n \to 0$ (as $n \to \infty$), and so $n/q^n \leq c_1$ ($n = 1, 2, \ldots$) hold for some $c_1 > 1$. Since
\[
(L(n-1)+1) \ldots (L(n-1)+|\alpha|) \leq (Ln)^{|\alpha|} \leq L^{|\alpha|}(c_1 q^n)^{|\alpha|}
\]
holds, by applying this to (3.7) and by using $(q^n)^{j+\sigma|\alpha|} \leq (q^n)^m$ we have
\[
\|\partial_z^\alpha X_n\|_s \leq \frac{AH^n}{(R-s)^n} \times \frac{(eLc_1)^L}{c_0} \left[ B/A \left( \frac{h}{H} \right)^n + \sum_{j+\sigma|\alpha| \leq m} \sum_{1 \leq k \leq n} B \left( \frac{h}{H} \right)^k \right].
\]
Thus, if $A \geq B$ and $H$ is sufficiently large with $H > h$, we have
\[
\frac{(eLc_1)^L}{c_0} \left[ \left( \frac{h}{H} \right)^n + \sum_{j+\sigma|\alpha| \leq m} \sum_{1 \leq k \leq n} B \left( \frac{h}{H} \right)^k \right] 
\leq \frac{(eLc_1)^L}{c_0} \left[ \left( \frac{h}{H} \right)^n + \sum_{j+\sigma|\alpha| \leq m} B \times \frac{h}{1-h} \right] \leq 1.
\]
This proves that if we take $A > 0$ and $H > 0$ sufficiently large we have the estimate (3.6). This proves Lemma 3.2. □

Thus, we have proved Proposition 3.1. □

**Example 3.3.** Let $A > 0$, $B > 0$, $m \in \mathbb{N}$, $j \in \mathbb{N}$, $p \in \mathbb{N}^*$ ($= \{1, 2, \ldots\}$), $\alpha \in \mathbb{N}^*$, and let us consider
\[
(\sigma_q)^m X = \frac{A}{1-z} t + B t^p (\sigma_q)^j \partial_z^\alpha X.
\]
This equation has a unique formal power series solution and it is given by
\[
\hat{X}(t, z) = \sum_{n \geq 0} AB^n q^n (q^{n+1})^j \ldots (q^{(n-1)p+1})^j \ldots (q^{np+1})^j \frac{(na)!}{q^n (q^{n+1})^m \ldots (q^{np+1})^m} (1-z)^{n\alpha+1} t^{np+1}.
\]
It is easy to see that $\hat{X}(t, z)$ is convergent if and only if $j \leq m - 1$ holds: in this case, by setting $\sigma = 1/\alpha$ we have $j + \sigma \alpha \leq m$. □
4. PROOF OF (1) OF THEOREM 2.3

Let us consider the same equation (3.1) under the assumption that there is an integer $m_0$ with $0 \leq m_0 < m$ such that

$$\begin{cases}
\text{ord}_t(a_{j,\alpha}) \geq \max\{0, j - m_0\}, & \text{if } |\alpha| = 0, \\
\text{ord}_t(a_{j,\alpha}) \geq \max\{1, j - m_0 + 1\}, & \text{if } |\alpha| > 0
\end{cases} \tag{4.1}$$

and that $a_{m_0,0}(0,z) \neq 0$ on $D_R$ for some $R > 0$. We set

$$C(\lambda,z) = \sum_{j=0}^{m_0} a_{j,0}(0,z)\lambda^j$$

which is a polynomial of degree $m_0$ in $\lambda$ with holomorphic coefficients. Since the condition $a_{m_0,0}(0,z) \neq 0$ is assumed, we have a constant $c_0 > 0$ and a positive integer $\mu$ such that

$$|C(q^n,z)| \geq c_0(q^n)^{m_0} \quad \text{on } D_R \text{ for any } n \geq \mu. \tag{4.2}$$

For simplicity, we set $\Lambda = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^d ; j + \sigma|\alpha| \leq m\}$ and let $L = m/\sigma$. We have $(j,0) \in \Lambda$ for any $j = 0,1,\ldots,m$, and if $(j,\alpha) \in \Lambda$ we have $|\alpha| \leq L$. By condition (4.1), we see that:

- if $j \leq m_0$ and $|\alpha| = 0$, we have $a_{j,0}(t,z) = a_{j,0}(0,z) + tb_{j,0}(t,z)$,
- if $m_0 < j \leq m$ and $|\alpha| = 0$, we have $a_{j,0}(t,z) = t^{j-m_0}b_{j,0}(t,z)$,
- if $|\alpha| > 0$, we have $a_{j,\alpha}(t,z) = t^{\max\{1,j-m_0+1\}}b_{j,\alpha}(t,z)$

for some holomorphic functions $b_{j,\alpha}(t,z)$ in a neighbourhood of $(0,0) \in \mathbb{C} \times \mathbb{C}^d$. By setting

$$\begin{align*}
p_{j,0} &= 1, & \text{if } j \leq m_0 \text{ and } |\alpha| = 0, \\
p_{j,0} &= j - m_0, & \text{if } m_0 < j \leq m \text{ and } |\alpha| = 0, \\
p_{j,\alpha} &= \max\{1,j-m_0+1\}, & \text{if } |\alpha| > 0
\end{align*} \tag{4.3}$$

we see that our equation (3.1) is written in the form

$$C(\sigma_q,z)X + \sum_{(j,\alpha) \in \Lambda} t^{p_{j,\alpha}}b_{j,\alpha}(t,z)(\sigma_q)^j\partial_z^\alpha X = F(t,z). \tag{4.4}$$

Since $|\alpha|/L \leq 1$ holds for any $(j,\alpha) \in \Lambda$, by the definition of $p_{j,\alpha}$ ($(j,\alpha) \in \Lambda$) we have

$$1 \geq \frac{j + |\alpha|/L - m_0}{p_{j,\alpha}}, \quad (j,\alpha) \in \Lambda. \tag{4.5}$$

To prove (1) of Theorem 2.3 it is enough to show the following result.

**Proposition 4.1.** Suppose the conditions (4.2), (4.3) and (4.5) hold. Then, if

$$\hat{X}(t,z) = \sum_{n \geq 0} X_n(z)t^n \in \mathcal{O}_R[[t]].$$
is a formal solution of (4.4), there are $A > 0$, $H > 0$ and $R_1 > 0$ such that
\[ |X_n(z)| \leq AH^n q^{n(n-1)/2} \text{ on } D_{R_1}, \quad n = 0, 1, 2, \ldots. \tag{4.6} \]

**Proof.** By assumption, we can expand $b_{j,\alpha}(t, z)$ ($(j, \alpha) \in \Lambda$) and $F(t, z)$ into the forms:
\[
 b_{j,\alpha}(t, z) = \sum_{k \geq 0} b_{j,\alpha,k}(z)t^k \quad ((j, \alpha) \in \Lambda),
\]
\[
 F(t, z) = \sum_{k \geq 0} F_k(z)t^k.
\]

We may suppose that $R > 0$ is sufficiently small. Therefore, we may suppose $0 < R < 1$, that $b_{j,\alpha,k}(z)$ and $F_k(z)$ are all holomorphic functions on $D_R$, and that there are $B > 0$ and $h > 0$ such that $|b_{j,\alpha,k}(z)| \leq Bh^k ((j, \alpha) \in \Lambda)$ and $|F_k(z)| \leq Bh^k (k \geq 0)$ hold on $D_R$.

Let
\[
 \hat{X}(t, z) = \sum_{n=0}^{\infty} X_n(z)t^n \in \mathcal{O}_R[[t]]
\]
be a formal solution of (4.4). By a calculation we have the following recurrent formulas:
\[
 C(q^n, z)X_0 = F_0(z)
\]
and for $n \geq 1$
\[
 C(q^n, z)X_n = F_n(z) - \sum_{(j,\alpha) \in \Lambda} \sum_{0 \leq k \leq n-p_j,\alpha} b_{j,\alpha,k}(z)(q^j)^{n-k-p_j,\alpha} \partial_z^\alpha X_{n-k-p_j,\alpha}. \tag{4.7}
\]

To prove Proposition 4.1 it is enough to show the following lemma.

**Lemma 4.2.** There are $A > 0$ and $H > 0$ such that the estimate
\[
 \|\partial_z^\alpha X_n\|_s \leq \frac{AH^n q^{n(n-1)/2}}{(R-s)^{Ln}} \quad \text{for any } 0 < s < R \text{ and } |\alpha| \leq L \tag{4.8}
\]
holds for any $n = 0, 1, 2, \ldots$.\n
**Proof of Lemma 4.2.** Let $\mu$ be as in (4.2). Since $\partial_z^\alpha X_n(z)$ $(n = 0, 1, \ldots, \mu$ and $|\alpha| \leq L$) are holomorphic functions on $D_R$, by taking $A > 0$ and $H > 0$ sufficiently large we have condition (4.8) for $n = 0, 1, \ldots, \mu$.

Let $n > \mu$, and suppose that (4.8) with $n$ replaced by $p$ is already proved for all $p < n$. Since (4.2) is known, $X_n$ can be expressed in the form
\[
 X_n = X_{n,F} + \sum_{(j,\alpha) \in \Lambda} X_{n,j,\alpha}
\]
where $X_{n,F}$ and $X_{n,j,\alpha}$ ($(j, \alpha) \in \Lambda$) are defined by $C(q^n, z)X_{n,F} = F_n(z)$ and
\[
 C(q^n, z)X_{n,j,\alpha} = -\sum_{0 \leq k \leq n-p_j,\alpha} b_{j,\alpha,k}(z)(q^j)^{n-k-p_j,\alpha} \partial_z^\alpha X_{n-k-p_j,\alpha}. \tag{4.9}
\]
Then, if $H \geq h$ we have
\[
\|X_{n,F}\|_s \leq \frac{Bh^n}{c_0(q^n)m_0} \leq \frac{AH^n}{c_0} \times B \left( \frac{h}{H} \right)^\mu,
\]
(4.10)
and by (4.2), (4.9) and the induction hypothesis we have
\[
\|X_{n,j,\alpha}\|_s \leq \frac{1}{c_0(q^n)m_0} \sum_{0 \leq k \leq n-p_{j,\alpha}} Bh^k q^{(n-k-p_{j,\alpha})j} \times \frac{AH^{n-k-p_{j,\alpha}}q^{(n-k-p_{j,\alpha})(n-k-p_{j,\alpha}-1)/2}}{(R-s)L(n-k-p_{j,\alpha})}.
\]
(4.11)
We recall that by (4.5) we have $p_{j,\alpha} - j + m_0 \geq |\alpha|/L$ and so
\[
-nm_0 + (n-k-p_{j,\alpha}j + (n-k-p_{j,\alpha})(n-k-p_{j,\alpha}-1)/2 \\
= n(n-1)/2 - (k+p_{j,\alpha} - j + m_0)(n-k-p_{j,\alpha}) \\
- (k+p_{j,\alpha})(k+p_{j,\alpha}-1)/2 - m_0(k+p_{j,\alpha}) \\
\leq n(n-1)/2 - (|\alpha|/L)(n-k-p_{j,\alpha}) \\
\leq n(n-1)/2 - (|\alpha|/L)(n-k-p_{j,\alpha}).
\]
By applying this to (4.11), we have
\[
\|X_{n,j,\alpha}\|_s \leq \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)L(n-k-p_{j,\alpha})} \times \frac{1}{q(|\alpha|/L)(n-k-p_{j,\alpha})} \sum_{0 \leq k \leq n-p_{j,\alpha}} B \left( \frac{h}{H} \right)^k \frac{1}{H^{p_{j,\alpha}}},
\]
and if $H \geq 2h$ holds, we have
\[
\|X_{n,j,\alpha}\|_s \leq \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)L(n-k-p_{j,\alpha})} \times \frac{1}{q(|\alpha|/L)(n-k-p_{j,\alpha})} \frac{2B}{H^{p_{j,\alpha}}}
\]
(4.12)
for any $0 < s < R$.

Now, let us apply Lemma 2.5 to these estimates (4.10) and (4.12). Namely, for any $|\alpha| \leq L$, we have
\[
\|\partial_x^\alpha X_{n,F}\|_s \leq \frac{AH^n |\alpha||\alpha|!}{c_0(R-s)|\alpha|} \times \frac{B}{A} \left( \frac{h}{H} \right)^\mu \leq \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)Ln} \times \frac{e^{L}L!B}{A} \left( \frac{h}{H} \right)^\mu
\]
(4.13)
and
\[
\|\partial_x^\alpha X_{n,j,\alpha}\|_s \leq \frac{AH^n q^{n(n-1)/2}}{c_0(R-s)L(n-k-p_{j,\alpha})+|\alpha|} \frac{2B}{H^{p_{j,\alpha}}} \times \frac{e^{L}|\alpha|!}{q(|\alpha|/L)(n-k-p_{j,\alpha})}.
\]
Since \((n+1)/(q^{1/L})^n \to 0\) (as \(n \to \infty\)) holds, we have the estimate \((n+1) \leq c_1 (q^{1/L})^n\) \((n = 0, 1, 2, \ldots)\) for some \(c_1 > 0\). Then,

\[
\frac{e^{\mid \alpha \mid} (L(n - k - p_{j,\alpha}) + 1) \ldots (L(n - k - p_{j,\alpha}) + |\alpha|)}{q^{(|\alpha|/L)(n-k-p_{j,\alpha})}} \leq \frac{e^{\mid \alpha \mid} (L(n - k - p_{j,\alpha} + 1))^{\mid \alpha \mid}}{q^{(|\alpha|/L)(n-k-p_{j,\alpha})}} \leq (eLc_1)^{\mid \alpha \mid},
\]

and so we have

\[
\| \partial_z^n X_{n,j,\alpha} \| \leq \frac{AH^n q^{n(n-1)/2}}{c_0 (R-s)^{L(n-k-p_{j,\alpha})+\mid \alpha \mid}} \times \frac{2B}{H^{p_{j,\alpha}}} (eLc_1)^{\mid \alpha \mid}
\]

(4.14)

for any \(0 < s < R\).

By (4.13) and (4.14), we have

\[
\| \partial_z^n X_n \| \leq \frac{AH^n q^{n(n-1)/2}}{(R-s)^{Ln}} \times C_1 \quad \text{for any } 0 < s < R
\]

with

\[
C_1 = \frac{e^{L} B}{c_0 A} \left( \frac{h}{H} \right)^{\mu} + \sum_{(j,\alpha) \in \Lambda} \frac{2B}{c_0 H^{p_{j,\alpha}}} (eLc_1)^{\mid \alpha \mid}.
\]

Thus, if \(C_1 \leq 1\) we can obtain the result (4.8). We note that if we take \(A > 0\) and \(H > 0\) sufficiently large, we have the condition \(C_1 \leq 1\). This completes the proof of Lemma 4.2.

Thus, by (4.8) \((n = 0, 1, 2, \ldots)\), we have the condition (4.6). This proves Proposition 4.1.

**Example 4.3.** Let \(A > 0\), \(B > 0\), \(p \in \mathbb{N}^*\) and \(\alpha > 0\). The following equation is a particular case of (4.4) with \(m_0 = 0\) and \(m = 1\):

\[
X = \frac{A}{1-z} t + t \sigma_q X + Bt^p \partial_z^a X.
\]

This equation has a unique formal power series solution and we can apply Proposition 4.1 to this case. In the case \(p = 1\) the formal solution is given by

\[
\hat{X}(t, z) = \frac{A}{1-z} t + \sum_{n \geq 2} \left( (q^1 + B \partial_z^a) \ldots (q^{n-1} + B \partial_z^a) \frac{A}{1-z} \right) t^n.
\]

Since \(q > 1\) holds, we have \((n\alpha)^a \leq cq^n\) \((n = 1, 2, \ldots)\) for some \(c > 0\). We have the following majorant relation:

\[
\hat{X}(t, z) \ll \sum_{n \geq 1} \frac{A(1 + Bc)^{n-1} q^{n(n-1)/2}}{(1-z)^{1+(n-1)\alpha}} t^n.
\]
5. PROOF OF (2) OF THEOREM 2.3

We will consider the same equation

$$C(\sigma_q, z)X + \sum_{(j, \alpha) \in \Lambda} t^{p_{j, \alpha}} b_{j, \alpha}(t, z)(\sigma_q)^j \partial_z^\alpha X = F(t, z) \quad (5.1)$$

as (4.4) under the same conditions as in Section 4. In addition, as is supposed in Theorem 2.3, we assume here that $0 \leq m_0 < m$, $a_{m_0, 0}(0, 0) \neq 0$, $b_{m, 0}(0, 0) \neq 0$, and

$$b_{j, \alpha}(0, z) \equiv 0 \quad \text{for} \ m_0 \leq j < m \ \text{and} \ |\alpha| > 0. \quad (5.2)$$

The last condition is equivalent to the condition that $\text{ord}_t(a_{j, \alpha}) \geq j - m_0 + 2$ if $|\alpha| > 0$ and $m_0 \leq j < m$. We set

$$P(\tau, z) = \sum_{m_0 < j \leq m} \frac{b_{j, 0}(0, z)}{q^{j(j-1)/2}} \tau^{j-m_0} + \frac{a_{m_0, 0}(0, z)}{q^{m_0(m_0-1)/2}} \quad (5.3)$$

which is a polynomial of degree $m - m_0$ with respect to $\tau$. Since $b_{m, 0}(0, 0) \neq 0$ and $a_{m_0, 0}(0, 0) \neq 0$ are supposed, the equation $P(\tau, 0) = 0$ in $\tau$ has $m - m_0$ non-zero roots. We denote them by $\tau_1, \ldots, \tau_{m-m_0}$. We set

$$S = \bigcup_{i=1}^{m-m_0} \{t\tau_i; t > 0\}.$$  

For $\lambda \in \mathbb{C} \setminus \{0\}$ and $\theta > 0$, we write $S_\theta(\lambda) = \{\xi \in \mathbb{C} \setminus \{0\}; |\arg \xi - \arg \lambda| < \theta\}$.

**Lemma 5.1.** For any $\lambda \in \mathbb{C} \setminus \{\{0\} \cup S\}$ we can find $c > 0$, $\theta > 0$, $r > 0$ and $R > 0$ such that $|P(\xi, z)| \geq c(\xi^2 + 1)^{m-m_0}$ holds on $(S_\theta(\lambda) \cup D_r) \times D_R$.

From now, we take any $\lambda \in \mathbb{C} \setminus \{\{0\} \cup S\}$ and fix it. Take also $c > 0$, $\theta > 0$, $r > 0$ and $R > 0$ so that Lemma 5.1 holds, and fix them. We may suppose that $r$ and $R$ are sufficiently small. Set $\Omega = (S_\theta(\lambda) \cup D_r) \times D_R$. Under these settings, we take a sufficiently large $\mu \in \mathbb{N}^*$ so that

$$\beta = \sum_{j<m_0} \frac{\|a_{j, 0}(0)\|_R}{cq^{m_0(m_0-1)/2}(q^{m_0-j})^\mu} < 1. \quad (5.4)$$

This is possible, because $(q^{m_0-j})^\mu \to \infty$ (as $\mu \to \infty$).

5.1. FORMAL $q$-BOREL TRANSFORMS

Let us recall the definition of formal $q$-Borel transforms introduced by Zhang [14]. For a formal series

$$\hat{V}(t, z) = \sum_{n \geq 0} V_n(z) t^n \in \mathcal{O}_R[[t]],$$

the formal $q$-Borel transform $\hat{B}_{q;1}[\hat{V}](\xi, z)$ of $\hat{V}(t, z)$ is defined by
\[
\hat{B}_{q;1}[\hat{V}](\xi, z) = \sum_{n \geq 0} \frac{V_n(z)}{q^{n(n-1)/2}} \xi^n \in \mathcal{O}_R[[\xi]].
\]

The following property is known (see Statement 1.3.3 in [14]).

**Lemma 5.2.** Let $\hat{a}(t, z) = \sum_{k \geq 0} a_k(z)t^k \in \mathcal{O}_R[[t]]$, and let $\hat{V}(t, z) \in \mathcal{O}_R[[t]]$. Set $v(\xi, z) = \hat{B}_{q;1}[V](\xi, z)$. Then, for any $m \in \mathbb{N}$ we have
\[
\hat{B}_{q;1}[\hat{a} \times (\sigma_q)^m \hat{V}](\xi, z) = \sum_{k \geq 0} \frac{a_k(z)}{q^{k(k-1)/2}} \xi^k v(q^{m-k} \xi, z).
\]

**Corollary 5.3.** For any $m \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$, we have

1. $\hat{B}_{q;1}[t^m(\sigma_q)^m \hat{V}](\xi, z) = \frac{\xi^m}{q^{m(m-1)/2}} v(\xi, z)$,
2. $\hat{B}_{q;1}[t^{m+k}(\sigma_q)^m \hat{V}](\xi, z) = \frac{\xi^{m+k}}{q^{(m+k)(m+k-1)/2}} (\sigma_q^{-1})^k v(\xi, z)$,
3. $\hat{B}_{q;1}[t^m(\sigma_q)^{m+k} \hat{V}](\xi, z) = \frac{\xi^m}{q^{m(m-1)/2}} (\sigma_q)^k v(\xi, z)$.

5.2. EQUATION IN THE $q$-BOREL PLANE

Let
\[
\hat{X}(t, z) = \sum_{n \geq 0} X_n(z)t^k \in \mathcal{O}_R[[t]]
\]
be a formal solution of (5.1), and let $\mu$ be as in (5.4). We set
\[
X^*(t, z) = \sum_{n \geq \mu} X_n(z)t^n.
\]

Then, $X^*(t, z)$ is a formal solution of the equation
\[
C(\sigma_q, z)X^* + \sum_{(j, \alpha) \in \Lambda} t^{p_{j, \alpha}} b_{j, \alpha}(t, z)(\sigma_q)^j \partial_z^\alpha X^* = F^*(t, z) \tag{5.5}
\]
for some holomorphic function $F^*(t, z)$ on $D_r \times D_R$ with $\text{ord}_t(F^*) \geq \mu$.

**Lemma 5.4.** By multiplying equation (5.5) by $t^{m_0}$ we have the expression
\[
\sum_{j \leq m_0} t^{m_0} a_{j,0}(0, z)(\sigma_q)^j X^* + \sum_{m_0 < j \leq m} t^j b_{j,0}(0, z)(\sigma_q)^j X^*
\]
\[
+ \sum_{j \leq m_0} t^{m_0+1} b_{j,0}^*(t, z)(\sigma_q)^j X^* + \sum_{m_0 < j \leq m} t^j b_{j,0}^*(t, z)(\sigma_q)^j X^*
\]
\[
+ \sum_{j < m_0, |\alpha| > 0} t^{m_0+1} b_{j,\alpha}^*(t, z)(\sigma_q)^j \partial_z^\alpha X^* \tag{5.6}
\]
\[
+ \sum_{m_0 \leq j < m, |\alpha| > 0} t^{j+2} b_{j,\alpha}(t, z)(\sigma_q)^j \partial_z^\alpha X^* = t^{m_0} F^*(t, z)
\]
for some holomorphic functions $b_{j,\alpha}^*(t, z)$ ($\mathbb{(j, \alpha) \in \Lambda}$) on $D_r \times D_R$. 
**Proof.** By the definition of $p_{j,\alpha}$, we have

$$
\sum_{j \leq m_0} t^{m_0} a_{j,0}(0, z)(\sigma_q)^j X^* + \sum_{j \leq m_0} t^{m_0+1} b_{j,0}(t, z)(\sigma_q)^j X^*
$$

$$
+ \sum_{m_0 < j \leq m} t^j b_{j,0}(t, z)(\sigma_q)^j X^*
$$

$$
+ \sum_{(j,\alpha) \in \Lambda, |\alpha| > 0} t^{\max\{1+m_0,j+1\}} b_{j,\alpha}(t, z)(\sigma_q)^j \partial_x^{\alpha} X^* = t^{m_0} F^*(t, z).
$$

Therefore, by setting

$$
\begin{align*}
&b^*_j(t, z) = (b_{j,0}(t, z) - b_{j,0}(0, z))/t, &\text{if } m_0 < j \leq m, \\
&b^*_{j,\alpha}(t, z) = b_{j,\alpha}(t, z)/t, &\text{if } m_0 \leq j < m \text{ and } |\alpha| > 0, \\
&b^*_{j,\alpha}(t, z) = b_{j,\alpha}(t, z), &\text{in the other case}
\end{align*}
$$

we obtain (5.6). In the case $|\alpha| > 0$ and $m_0 \leq j < m$, we have used condition (5.2). \(\qed\)

Now, let us apply formal $q$-Borel transform to equation (5.6). Under the setting

$$
u(\xi, z) = \hat{B}_{q;1}[X^*](\xi, z),\quad F^*(t, z) = \sum_{n \geq \mu} F^*_n(z)t^n,
$$

$$
t^{m_0+1} b^*_{j,0}(t, z) = \sum_{k \geq m_0+1} c_{j,0,k}(z)t^k \quad (|\alpha| = 0 \text{ and } j \leq m_0),
$$

$$
t^{j+1} b^*_{j,0}(t, z) = \sum_{k \geq j+1} c_{j,0,k}(z)t^k \quad (|\alpha| = 0 \text{ and } m_0 \leq j \leq m),
$$

$$
t^{m_0+1} b^*_{j,\alpha}(t, z) = \sum_{k \geq m_0+1} c_{j,\alpha,k}(z)t^k \quad (|\alpha| > 0 \text{ and } j < m_0),
$$

$$
t^{j+2} b^*_{j,\alpha}(t, z) = \sum_{k \geq j+2} c_{j,\alpha,k}(z)t^k \quad (|\alpha| > 0 \text{ and } m_0 \leq j < m)
$$
we have the equation
\[
\sum_{j \leq m_0} a_{m_0,j}(0, z) \xi^{m_0}(\sigma_{q-1})^{m_0-j} u + \sum_{m_0 < j \leq m} b_{j,0}(0, z) \xi^j u
\]
\[
+ \sum_{j \leq m_0} \sum_{k \geq m_0+1} c_{j,0,k}(z) \frac{\xi^k(\sigma_{q-1})^{j-k} u}{q^j(k-1)/2} + \sum_{m_0 < j \leq m} \sum_{k \geq 1} \frac{c_{j,0,k}(z)}{q^k(k-1)/2} \xi^k(\sigma_{q-1})^{k-j} u
\]
\[
+ \sum_{j < m_0, |\alpha| > 0} \sum_{k \geq m_0+1} \frac{c_{j,\alpha,k}(z)}{q^k(k-1)/2} \xi^k(\sigma_{q-1})^{j-k} \partial_z^\alpha u
\]
\[
+ \sum_{m_0 \leq j < m, |\alpha| > 0} \sum_{k \geq j+2} \frac{c_{j,\alpha,k}(z)}{q^k(k-1)/2} \xi^k(\sigma_{q-1})^{j-k} \partial_z^\alpha u
\]
\[
= \sum_{n \geq \mu} \frac{F^*_n(z)}{q^{n+m_0}(n+m_0-1)/2} \xi^{n+m_0}.
\]

Therefore, by canceling \(\xi^{m_0}\) from both sides of this equation, and then by using \(P(\xi, z)\) in (5.3) and the notations

\[
a_{m_0-i,0}(z) = \frac{a_{m_0-i,0}(0, z)}{q^{m_0(m_0-1)/2}} \quad (i = 1, \ldots, m_0),
\]

\[
c_{j,0,k}(z) = \frac{c_{j,0,k+m_0}(z)}{q^{m_0(m_0-1)/2}q^{m_0k}} \quad (j \leq m_0 \text{ and } k \geq 1),
\]

\[
c_{j,0,k}(z) = \frac{c_{j,0,k+j}(z)}{q^{j(j-1)/2}q^{jk}} \quad (m_0 < j \leq m \text{ and } k \geq 1),
\]

\[
c_{j,\alpha,k}(z) = \frac{c_{j,\alpha,k+m_0}(z)}{q^{m_0(m_0-1)/2}q^{m_0k}} \quad (|\alpha| > 0, j < m_0 \text{ and } k \geq 1),
\]

\[
c_{j,\alpha,k}(z) = \frac{c_{j,\alpha,k+j+1}(z)}{q^{j(j+1)/2}q^{j+1k}} \quad (|\alpha| > 0, m_0 \leq j < m \text{ and } k \geq 1),
\]

\[
f_n(z) = \frac{F^*_n(z)}{q^{m_0(m_0-1)/2}q^{m_0n}}, \quad n \geq \mu,
\]
we can reduce our equation (5.7) into the form

\[
P(\xi, z)u + \sum_{i=1}^{m_0} a_{m_0-i}(z)(\sigma_{q-1})^i u \\
+ \sum_{j \leq m_0} \sum_{k \geq 1} \frac{c_{j,0,k}}{q^{k(k-1)/2}} \xi^k (\sigma_{q-1})^{k+(m_0-j)} u \\
+ \sum_{m_0 < j \leq m} \sum_{k \geq 1} \frac{c_{j,0,k}}{q^{k(k-1)/2}} \xi^k (\sigma_{q-1})^{k+(m_0-j)} u \\
+ \sum_{0 \leq j < m_0} \sum_{|\alpha| > 0 \ k \geq 1} \frac{c_{j,\alpha,k}}{q^{k(k-1)/2}} \xi^k (\sigma_{q-1})^{k+(m_0-j)} \partial^\alpha \xi u \\
+ \sum_{m_0 \leq j < m_0} \sum_{|\alpha| > 0 \ k \geq 1} \frac{c_{j,\alpha,k}}{q^{k(k-1)/2}} \xi^k (\sigma_{q-1})^{k+(j+1-m_0)} \partial^\alpha \xi u
\]

(5.8)

\[
= \sum_{n \geq m} \frac{f_n(z)}{q^{n(n-1)/2}} \xi^n.
\]

The meaning of this equation is as follows:

**Lemma 5.5.**

1. By taking \( r > 0 \) and \( R > 0 \) sufficiently small, we may assume that \( u(\xi, z) = \mathcal{B}_{q;1}[X^+](\xi, z) \) is a holomorphic function on \( D_r \times D_R \).

2. Each sum in (5.8) is a holomorphic function on \( D_r \times D_R \) in the following sense: if \( c_k(z) \in \mathcal{O}_R \) (\( k \geq 1 \)) satisfy the estimates \(|c_k(z)| \leq Ch^k \) on \( D_R \) (\( k \geq 1 \)) for some \( C > 0 \) and \( h > 0 \), the sum

\[
\sum_{k \geq 1} \frac{c_k(z)}{q^{k(k-1)/2}} \xi^k (\sigma_{q-1})^{k+e} u \quad \text{(with } i \in \mathbb{N}, e \in \mathbb{N})
\]

is a holomorphic function on \( D_{r'} \times D_R \) with \( r' = rq^{1+e} \).

**Proof.** By Proposition 4.1, we have the estimates \( \|X_n\|_R \leq Ah^n q^{n(n-1)/2} \) (\( n = 0, 1, 2, \ldots \)) for some \( A > 0 \) and \( H > 0 \). By taking \( 0 < r < 1/H \) we have the result (1). We note that

\[
\sum_{k \geq 1} \frac{|c_k(z)|}{q^{k(k-1)/2}} |\xi|^{k+i} (\sigma_{q-1})^{k+e} u \leq C(|\xi|)W(|\xi|), \quad z \in D_R,
\]

where

\[
C(\xi) = \sum_{k \geq 1} \frac{C h^k}{q^{k(k-1)/2}} \xi^{k+i} \quad \text{and} \quad W(\xi) = \sum_{n \geq m} A H^n \left( \frac{\xi}{q^{1+e}} \right)^n.
\]

Since \( C(\xi) \) is an entire function in \( \xi \) and \( W(\xi) \) is a holomorphic function on \( \{ \xi ; |\xi| < q^{1+e}/H \} \), we have the result (2). \( \square \)
5.3. HOLOMORPHIC EXTENSION OF $u(\xi, z)$

As is seen above, the formal $q$-Borel transform $u(\xi, z) = \hat{B}_{q,1}[X^*](\xi, z)$ is a holomorphic solution of (5.8) on $D_r \times D_R$. The following is the main result on equation (5.8).

**Proposition 5.6.** The local solution $u(\xi, z)$ has a holomorphic extension $u^*(\xi, z)$ to a domain $(S_{\vartheta}(\lambda) \cup D_{r_1}) \times D_R$ for some $r_1 > 0$ that satisfies the following properties:

1. $u^*(\xi, z)$ is also a solution of (5.8).
2. For any $0 < R_1 < R$ there are $A > 0$ and $H > 0$ such that
   \[ |u(\lambda q^m, z)| \leq AH^m q^{m(m+1)/2} \] on $D_{R_1}$ for any $m = 0, 1, 2, \ldots$.

The proof of this result will be given in Section 6. We will admit this result for a while.

5.4. $q$-ANALOGUE OF THE SUMMABILITY OF $\hat{X}(t, z)$

Now, let us return to the situation in Theorem 2.3. Let $u^*(\xi, z)$ be the holomorphic extension of $u(\xi, z)$ to the domain $\Omega_1 = (S_{\vartheta}(\lambda) \cup D_{r_1}) \times D_R$. Let $\vartheta_q(x)$ be the Jacobi theta function defined by

\[ \vartheta_q(x) = \sum_{m \in \mathbb{Z}} \frac{x^m}{q^{m(m-1)/2}} \]

which is a holomorphic function on $\mathbb{C} \setminus \{0\}$. We set

\[ W^*(t, z) = \mathcal{L}_{q,1}^{\lambda}[u^*](t, z) = \sum_{n \in \mathbb{Z}} \frac{u^*(\lambda q^n, z)}{\vartheta_q(\lambda q^n/t)} \]

which is the $q$-Laplace transform of $u^*(\xi, z)$ in the direction $\lambda$ (introduced by Ramis-Zhang [11]). Then, by combining the above Proposition 5.6 with Théorème 1.3.2 in [15] (or Proposition 1 in [4]) we get the following theorem.

**Theorem 5.7.**

1. $W^*(t, z)$ is a holomorphic solution of equation (5.5) on $(D_{r_2} \setminus \{0\} \cup \mathcal{X}_s) \times D_{R_1}$ for some $r_2 > 0$.
2. Moreover, there are $M_1 > 0$ and $H_1 > 0$ such that the following estimate holds
   \[ \left| W^*(t, z) - \sum_{n=\mu}^{N-1} X_n(z)t^n \right| \leq \frac{M_1 H_1}{\epsilon} q^{N(N-1)/2}|t|^N \] for $t \in U_\epsilon$ and $z \in D_{R_1}$

   for any sufficiently small $\epsilon > 0$ and any $N \geq \mu$, where $U_\epsilon = D_{r_2} \setminus \{0\} \cup \mathcal{X}^*_s, \epsilon)$.

By setting

\[ W(t, z) = \sum_{n=0}^{\mu-1} X_n(z)t^n + W^*(t, z) \]

we have a true holomorphic solution of (2.1) which admits $\hat{X}(t, z)$ as a $q$-Gevrey asymptotic expansion of order 1. This proves (2) of Theorem 2.3.
6. PROOF OF PROPOSITION 5.6

Let \( \lambda \in \mathbb{C} \setminus \{0\} \), \( \theta > 0 \), \( r > 0 \), and \( R > 0 \), set \( \Omega = (D_r \cup S_\theta(\lambda)) \times D_R \subset \mathbb{C}_\xi \times \mathbb{C}_z^d \), and set \( N = m - m_0 \). In this section, as a model of (5.8) we will consider the equation

\[
P(\xi, z)u + \sum_{i=1}^{K} a_i(z)(\sigma_{q^{-1}})^i u + \sum_{i=0}^{N} \sum_{(j, \alpha) \in \Lambda^*} \frac{c_{i,j,\alpha,k}(z)}{q^k(k-1)/2} \xi^{k+i}(\sigma_{q^{-1}})^{k+e_{j,\alpha}} \partial_z^\alpha u = \sum_{n \geq \mu} \frac{f_n(z)}{q^n(n-1)/2} \xi^n
\]

(6.1)

on \( \Omega \). We suppose that \( 0 < R < 1 \) and the following conditions \((c_1)-(c_5)\) hold:

\((c_1)\) \( P(\xi, z) = \xi^N + c_1(z)\xi^{N-1} + \ldots + c_N(z) \in \mathcal{O}_R[\xi] \) for some \( N \in \mathbb{N} \). Moreover, \(|P(\xi, z)| \geq c(|\xi| + 1)^N \) holds on \( \Omega \) for some \( c > 0 \).

\((c_2)\) \( K \) and \( \mu \) are positive integers, and \( \Lambda^* \) is a finite subset of \( \mathbb{N} \times \{ \alpha \in \mathbb{N}^d ; |\alpha| \leq L \} \) (where \( L \in \mathbb{N}^* \)).

\((c_3)\) \( e_{j,\alpha} \ ((j, \alpha) \in \Lambda^*) \) are integers satisfying

\[
\begin{cases} 
    e_{j,\alpha} \geq 0, & \text{if } |\alpha| = 0, \\
    e_{j,\alpha} \geq 1, & \text{if } |\alpha| > 0. 
\end{cases}
\]

\((c_4)\) \( a_i(z) \in \mathcal{O}_R \ (i = 1, \ldots, K) \) and satisfy

\[
\beta = \sum_{i=1}^{K} \frac{\|a_i\|_R}{c(q^i)^\mu} < 1 \quad \text{(this corresponds to (5.4))}.
\]

\((c_5)\) \( c_{i,j,\alpha,k}(z) \in \mathcal{O}_R \ (0 \leq i \leq N, (j, \alpha) \in \Lambda^* \) and \( k \geq 1 \)) and \( f_n(z) \in \mathcal{O}_R \ (n \geq \mu) \). Moreover, there are \( B > 0 \) and \( h > 0 \) such that \( \|c_{i,j,\alpha,k}\|_R \leq Bh^k \ (0 \leq i \leq N, (j, \alpha) \in \Lambda^* \) and \( k \geq 1 \) \) and \( \|f_n\|_R \leq Bh^n \ (n \geq \mu) \) hold.

Then, we have the following result which yields Proposition 5.6.

**Proposition 6.1.**

1. **Equation (6.1) has a unique formal solution of the form \( \hat{u}(\xi, z) \in \xi^\mu \times \mathcal{O}_R[[\xi]] \).**
2. **Equation (6.1) has a unique holomorphic solution \( u(\xi, z) \) on \( \Omega \). Moreover, for any \( 0 < R_1 < R \) there are \( A_0 > 0 \) and \( H_0 > 0 \) such that

\[
|u(\lambda q^m, z)| \leq A_0 H_0^{-m} q^{m(m+1)/2} \quad \text{on } D_{R_1} \text{ for any } m = 0, 1, 2, \ldots \quad (6.2)
\]

The part (1) is verified by a simple calculation and the following lemma:

**Lemma 6.2.** For any \( n \geq \mu \) and \( g_n(z) \in \mathcal{O}_R \), the equation

\[
P(0, z)w_n + \sum_{i=1}^{K} a_i(z) \frac{w_n}{(q^i)^n} = g_n(z)
\]

has a unique solution \( w_n(z) \in \mathcal{O}_R \).
Proof. Since $|P(0, z)| \geq c$ holds on $D_R$, by the assumption $(c_4)$ we have

$$\left| P(0, z) + \sum_{i=1}^{K} \frac{a_i(z)}{q^i} \right| \geq |P(0, z)| - \sum_{i=1}^{K} \frac{\|a_i\|_R}{q^i} \geq c(1 - \beta) > 0,$$

and so we have the result. \hfill \Box

The proof of the part (2) will be done in Subsections 6.1–6.3.

6.1. ON EQUATION $\mathcal{L} w = g$

We set

$$\mathcal{L} = P(\xi, z) + \sum_{i=1}^{K} a_i(z)(\sigma q - 1)^i$$

and consider the equation

$$\mathcal{L} w = g(\xi, z) \text{ on } \Omega. \quad (6.3)$$

We denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on $\Omega$.

Lemma 6.3.

(1) Let $g(\xi, z) \in \mathcal{O}(\Omega)$. If $|g(\xi, z)| \leq A|\xi|^b$ holds on $\Omega$ for some $A > 0$ and $b \geq \mu$, equation (6.3) has a unique holomorphic solution $w(\xi, z) \in \mathcal{O}(\Omega)$ satisfying

$$|w(\xi, z)| \leq \frac{A|\xi|^b}{c(1 - \beta)(|\xi| + 1)^N} \text{ on } \Omega. \quad (6.4)$$

(2) Let $g(\xi, z) \in \mathcal{O}(\Omega)$. If it satisfies

$$\|g(\xi)\|_s \leq \frac{A|\xi|^b}{(R - s)^a} \text{ on } D_r \cup S_\theta(\lambda) \text{ for any } 0 < s < R$$

for some $A > 0$, $a \geq 0$ and $b \geq \mu$, equation (6.3) has a unique holomorphic solution $w(\xi, z) \in \mathcal{O}(\Omega)$ satisfying

$$\|w(\xi)\|_s \leq \frac{1}{c(1 - \beta)(R - s)^a(|\xi| + 1)^N} \text{ on } D_r \cup S_\theta(\lambda) \text{ for any } 0 < s < R.$$

Proof. Let us show (1). We construct a solution in the form

$$w(\xi, z) = \sum_{n \geq 0} w_n(\xi, z), \quad (6.5)$$

where $w_n(\xi, z) \ (n = 0, 1, 2, \ldots)$ are solutions of the following recurrent formulas:

$$P(\xi, z)w_0 = g(\xi, z) \quad (6.6)$$

and for $n \geq 1$

$$P(\xi, z)w_n = -\sum_{1 \leq i \leq K} a_i(z)(\sigma q - 1)^i w_{n-1}. \quad (6.7)$$
Since $|P(\xi, z)| \geq c(|\xi| + 1)^N$ on $\Omega$ is supposed, by (6.6) and (6.7) we can uniquely determine $w_n(\xi, z) \in \mathcal{O}(\Omega)$ ($n = 0, 1, 2, \ldots$) inductively on $n$.

By (6.6) and the assumption, we have

$$|w_0(\xi, z)| \leq \frac{A|\xi|^b}{c(|\xi| + 1)^N} \quad \text{on } \Omega.$$ 

Then, we have

$$\left| \sum_{1 \leq i \leq K} a_i(z)(\sigma_q^{-1})^i w_0 \right| \leq \sum_{1 \leq i \leq K} \|a_i\|_R \times |w_0(\xi/q^i, z)|$$
$$\leq \sum_{1 \leq i \leq K} \|a_i\|_R \times \frac{A|\xi/q|^b}{c(|\xi/q| + 1)^N} \leq \sum_{1 \leq i \leq K} \frac{\|a_i\|_R}{c(q^i)^b} \times A|\xi|^b \leq \beta A|\xi|^b.$$ 

Therefore, by (6.7) with $n = 1$, we have the estimate

$$|w_1(\xi, z)| \leq \frac{\beta A|\xi|^b}{c(|\xi| + 1)^N} \quad \text{on } \Omega.$$ 

By repeating the same argument we have the estimates

$$|w_n(\xi, z)| \leq \frac{\beta^n A|\xi|^b}{c(|\xi| + 1)^N} \quad \text{on } \Omega, \quad n = 0, 1, 2, \ldots \tag{6.8}$$

Thus, we can see that the formal solution $w(\xi, z)$ in (6.5) is convergent and it defines a holomorphic solution of (6.3) on $\Omega$. The estimate (6.4) is clear from the estimates (6.8).

As is seen in (1) of Proposition 6.1, it is clear that equation (6.3) has a unique formal solution $\hat{w}(t, z) \in \xi^\mu \times \mathcal{O}_R(\xi)$. This shows the uniqueness of the solution in $\mathcal{O}(\Omega)$.

Thus, part (1) is proved. The result (2) is a consequence of (1). \(\square\)

6.2. ON EQUATION (6.1)

Next, let us solve equation (6.1), that is,

$$\mathcal{L} u + \sum_{i=0}^{N} \sum_{(j, \alpha) \in A^*} \sum_{k \geq 1} \frac{c_{i,j,\alpha,k}(z)}{q^{k(k-1)/2}} \xi^{k+i} (\sigma_q^{-1})^{k+e_j,\alpha} \partial_z u = \sum_{n \geq \mu} \frac{f_n(z)}{q^{n(n-1)/2}} \xi^n$$

on $\Omega$. To do so, we set the formal solution $u(\xi, z)$ in the form

$$u(\xi, z) = \sum_{n \geq \mu} u_n(\xi, z)$$

and we solve the following recurrent formulas:

$$\mathcal{L} u_\mu = \frac{f_\mu(z)}{q^{\mu(\mu-1)/2}} \xi^\mu \quad \tag{6.9}$$
and for  \( n \geq \mu + 1 \)

\[
\mathcal{L} u_n = \frac{f_n(z)}{q^{n(n-1)/2}} \xi^n - \sum_{i=0}^{N} \sum_{(j,\alpha) \in \Lambda^{\ast}} \sum_{1 \leq k \leq n-\mu} c_{i,j,\alpha,k}^{(z)} \frac{q^{k(k-1)/2}}{q^{(k-1)/2}} \xi^{k+i} (\sigma_{q^{-1}})^{k+e_{j,\alpha}} \partial_{z}^{\alpha} u_{n-k}. \tag{6.10}
\]

**Lemma 6.4.** We have a unique solution \( u_n(\xi, z) \in \mathcal{O}(\Omega) (n \geq \mu) \) of the system (6.9) and (6.10) that satisfies the following: there are \( A > 0 \) and \( H > 0 \) such that

\[
\| \partial_{z}^{\alpha} u_n(\xi) \|_s \leq \frac{A H^{n} \alpha^{n}}{q^{n(n-1)/2} (R-s)^{L} n^{\alpha}} |\xi|^{n} \quad \text{on } D_{r} \cup S_{\theta}(\lambda)
\]

for any \( 0 < s < R \) and any \( |\alpha| \leq L \)

holds for any \( n \geq \mu \).

**Proof.** Since \( \| f_{\mu} \|_{R} \leq B h^{\mu} \) is supposed, by applying (1) of Lemma 6.3 to equation (6.9) we have a unique solution \( u_{\mu}(\xi, z) \in \mathcal{O}(\Omega) \) satisfying the estimate

\[
|u_{\mu}(\xi, z)| \leq \frac{1}{c(1-\beta)(|\xi|+1)^{N}} \frac{B h^{\mu} |\xi|^{\mu}}{q^{\mu(\mu-1)/2} (R-s)^{\alpha}} \leq \frac{1}{c(1-\beta)} \times \frac{B h^{\mu} |\xi|^{\mu}}{q^{\mu(\mu-1)/2} (R-s)^{L}} \quad \text{on } \Omega.
\]

By applying Lemma 2.5 to this estimate and by using the condition \( 0 < R < 1 \) we have

\[
\| \partial_{z}^{\alpha} u_{\mu}(\xi) \|_s \leq \frac{1}{c(1-\beta)} \times \frac{B h^{\mu} |\xi|^{\mu}}{q^{\mu(\mu-1)/2} (R-s)^{\alpha}} \leq \frac{L^{L} L^{L}}{c(1-\beta)} \times \frac{B h^{\mu} |\xi|^{\mu}}{q^{\mu(\mu-1)/2} (R-s)^{L}} \quad \text{on } D_{r} \cup S_{\theta}(\lambda)
\]

for any \( 0 < s < R \) and \( |\alpha| \leq L \). Hence, if we take \( A > 0 \) and \( H > 0 \) so that

\[
AH^{\mu} \geq \frac{L^{L} L^{L}}{c(1-\beta)} \times B h^{\mu}, \tag{6.12}
\]

by the condition \( \mu \geq 1 \) we have the estimate (6.11) for \( n = \mu \). Let us show the general case by induction on \( n \).

Let \( n \geq \mu + 1 \), and suppose that we already have \( u_{p}(\xi, z) \in \mathcal{O}(\Omega) (\mu \leq p < n) \) which satisfy estimate (6.11) with \( n \) replaced by \( p \) for all \( \mu \leq p < n \). We set

\[
g_{n}(\xi, z) = \frac{f_{n}(z)}{q^{n(n-1)/2}} \xi^{n} - \sum_{i=0}^{N} \sum_{(j,\alpha) \in \Lambda^{\ast}} \sum_{1 \leq k \leq n-\mu} c_{i,j,\alpha,k}^{(z)} \frac{q^{k(k-1)/2}}{q^{(k-1)/2}} \xi^{k+i} (\sigma_{q^{-1}})^{k+e_{j,\alpha}} \partial_{z}^{\alpha} u_{n-k}.
\]

Then our equation (6.10) is written as \( \mathcal{L} u_n = g_{n}(\xi, z) \). By assumption \( (c_{5}) \) and the induction hypothesis, we can see that \( g_{n}(\xi, z) \in \mathcal{O}(\Omega) \) is known and it satisfies the estimate

\[
\| g_{n}(\xi) \|_s \leq \frac{B h^{n}}{q^{n(n-1)/2}} |\xi|^{n} + \sum_{i=0}^{N} \sum_{(j,\alpha) \in \Lambda^{\ast}} \sum_{1 \leq k \leq n-\mu} \frac{B h^{k}}{q^{k(k-1)/2}} |\xi|^{k+i} \times \frac{A H^{n-k} (n-k)^{\alpha}}{q^{(n-k)(n-k-1)/2} (R-s)^{L(n-k)} (q^{k+e_{j,\alpha}})} \tag{6.13}
\]
on $D_r \cup S_\theta(\lambda)$ for any $0 < s < R$. Since $0 < R < 1$ is supposed and
\[
\frac{n(n-1)}{2} = \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} + k(n-k)
\]
holds, from (6.13) we have
\[
\|g_n(\xi)\|_s \leq \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)L^{(n-1)}} \left[ \frac{B}{A} \frac{h}{H} \right]^n + \sum_{i=0}^N \sum_{(j,0) \in \Lambda^*} B\left(\frac{h}{H}\right) \frac{1}{q^{e_{j,0}(n-k)}} \times |\xi|^i + \sum_{i=0}^N \sum_{(j,\alpha) \in \Lambda^*} (\alpha > 0) \sum_{1 \leq k \leq n-\mu} B\left(\frac{h}{H}\right) \frac{1}{q^{e_{j,\alpha}(n-k)}} \times |\xi|^i \right].
\]
Since $e_{j,0} \geq 0$, we have $1/q^{e_{j,0}(n-k)} \leq 1$. Since $m^L/q^m \to 0$ (as $m \to \infty$), we have $m^L/q^m \leq c_0$ for some $c_0$ (we may assume that $c_0 > 1$ holds). Then for $0 < |\alpha| \leq L$, we have $e_{j,\alpha} \geq 1$ and so
\[
\frac{(n-k)|\alpha|}{q^{e_{j,\alpha}(n-k)}} \leq \frac{(n-k)^L}{q^{n-k}} \leq c_0.
\]
Therefore, if we assume the conditions $A > B$ and $H > h$, we have the estimate
\[
\|g_n(\xi)\|_s \leq \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)L^{(n-1)}} \left[ \frac{h}{H} \right]^n + \sum_{i=0}^N \sum_{(j,\alpha) \in \Lambda^*} c_0 B(h/H) \frac{1}{1-h/H} \times |\xi|^i \right]
\]
for any $0 < s < R$. Thus, by applying Lemma 6.3 to equation $L_{\alpha}u_n = g_n(\xi, z)$ and by using the estimates $|\xi|^i/(|\xi|+1)^N \leq 1$ ($0 \leq i \leq N$) we have
\[
\|u_n(\xi)\|_s \leq \frac{1}{c(1-\beta)} \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)L^{(n-1)}} \left[ \frac{h}{H} \right]^n + \sum_{i=0}^N \sum_{(j,\alpha) \in \Lambda^*} c_0 B(h/H) \frac{1}{1-h/H} \right]
\]
on $D_r \cup S_\theta(\lambda)$ for any $0 < s < R$.

Now, let us apply Lemma 2.5. We get
\[
\|\partial^\alpha_{\xi} u_n(\xi)\|_s \leq \frac{1}{c(1-\beta)} \frac{e^{\alpha|L(n-1)+1| \ldots (L(n-1)+|\alpha|)AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)L^{(n-1)+|\alpha|}} \times \left[ \frac{h}{H} \right]^n + \sum_{i=0}^N \sum_{(j,\alpha) \in \Lambda^*} c_0 B(h/H) \frac{1}{1-h/H} \right]
\]
\[
\leq \frac{1}{c(1-\beta)} \frac{e^{L^LH_{\alpha}}|\alpha|AH^n|\xi|^n}{q^{n(n-1)/2}(R-s)L^n} \times \left[ \frac{h}{H} \right]^n + \sum_{i=0}^N \sum_{(j,\alpha) \in \Lambda^*} c_0 B(h/H) \frac{1}{1-h/H} \right]
\]

on $D_r \cup S_\theta(\lambda)$ for any $0 < s < R$. If
\[
\frac{(eL)^L}{c(1-\beta)} \left( \frac{h}{H} \right)^\mu + \sum_{i=0}^{N} \sum_{(j,\alpha) \in \Lambda^*} c_0 B(h/H) \frac{1}{1 - h/H} \leq 1 \tag{6.14}
\]
holds, we have the result (6.10).

Thus, by taking $A$ and $H$ so that $A > B$, $H > h$, (6.12) and (6.14) are satisfied, we have the result in Lemma 6.4.

\[\square\]

6.3. COMPLETION OF THE PROOF OF PART (2)

By Lemma 6.4, we can easily see that the formal solution
\[
u(\xi, z) = \sum_{n \geq \mu} u_n(\xi, z)
\]
is convergent on $\Omega$ and it defines a holomorphic solution of (6.1). Let us show the estimate (6.2).

Take any $0 < R_1 < R$. By Lemma 6.4, we have
\[
|u_n(\xi, z)| \leq \frac{AH^n|\xi|^n}{q^{n(n-1)/2}(R - R_1)^{Ln}}
\]
on $\Omega_1 = (D_r \cup S_\theta(\lambda)) \times D_{R_1}$ for any $n \geq \mu$. We set $H_2 = H|\lambda|/(R - R_1)^L$; we obtain
\[
|u(\lambda q^m, z)| \leq \sum_{n \geq \mu} |u_n(\lambda q^m, z)| \leq \sum_{n \geq \mu} \frac{AH^n(|\lambda|q^m)^n}{q^{n(n-1)/2}(R - R_1)^{Ln}}
\]
\[
\leq A \sum_{n \geq \mu} \frac{(H|\lambda|/(R - R_1)^L)nq^m}{q^{n(n-1)/2}}
\]
\[
= AH_2^mq^{m(m+1)/2} \sum_{n \geq \mu} \frac{(H_2)^{n-m}}{q^{(n-m)(n-m-1)/2}}
\]
\[
\leq \vartheta_q(H_2)AH_2^mq^{m(m+1)/2}, \quad m = 0, 1, 2, \ldots
\]
where $\vartheta_q(x)$ is the Jacobi theta function. This proves (6.2).

Acknowledgments
The first author is partially supported by the Grant-in-Aid for Scientific Research No. 22540206 of Japan Society for the Promotion of Science.
REFERENCES


Hidetoshi Tahara
h-tahara@hoffman.cc.sophia.ac.jp

Sophia University
Department of Information and Communication Sciences
Kioicho, Chiyoda-ku, Tokyo 102-8554, Japan

Hiroshi Yamazawa
yamazawa@shibaura-it.ac.jp

Shibaura Institute of Technology
College of Engineer and Design
Minuma-ku, Saitama-shi, Saitama 337-8570, Japan

Received: November 11, 2013.
Revised: July 5, 2014.
Accepted: July 27, 2014.