

EXISTENCE, UNIQUENESS AND ESTIMATES OF CLASSICAL SOLUTIONS TO SOME EVOLUTIONARY SYSTEM

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Abstract. The theorem of the local existence, uniqueness and estimates of solutions in Hölder spaces for some nonlinear differential evolutionary system with initial conditions is formulated and proved. This system is composed of one partial hyperbolic second-order equation and an ordinary subsystem with a parameter. In the proof of the theorem we use the Banach fixed-point theorem, the Arzeli-Ascola lemma and the integral form of the differential problem.

Keywords: hyperbolic wave equation, telegraph equation, system of nonlinear equations, existence, uniqueness and estimates of solutions, Hölder space.

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1. INTRODUCTION

Let functions $f : [0, T] \times \mathbb{R}^{2+n} \rightarrow \mathbb{R}$, $g = (g_1, \dots, g_n) : [0, T] \times \mathbb{R}^{2+n} \rightarrow \mathbb{R}^n$, $u_0, u_1 : \mathbb{R} \rightarrow \mathbb{R}$, $v_0 = (v_{01}, \dots, v_{0n}) : \mathbb{R} \rightarrow \mathbb{R}^n$ and a constant $c \geq 0$ be given. Consider a nonlinear second-order partial differential system of $(1 + n)$ equations of the form

$$\begin{cases} u_{tt} - u_{xx} + cu_t = f(t, x, u, v), & (t, x) \in [0, T] \times \mathbb{R}, \\ v_t = g(t, x, u, v), & (t, x) \in [0, T] \times \mathbb{R} \end{cases} \quad (1.1)$$

with the initial conditions

$$\begin{cases} u(0, x) = u_0(x), & x \in \mathbb{R}, \\ v(0, x) = v_0(x), & x \in \mathbb{R}, \\ u_t(0, x) = u_1(x), & x \in \mathbb{R}, \end{cases} \quad (1.2)$$

where $v = (v_1, \dots, v_n)$. The equations are weakly coupled. If $c > 0$, then the first hyperbolic wave equation in (1.1) is called the telegraph equation. The others equations in (1.1) are of first-order with a space parameter x . Our results are true if we consider the first equation in (1.1) only with f not depending on v (see examples in Remark 4.4).

Some information especially about the maximum principles and the existence of time-periodic bounded weak solutions for the wave or telegraph equations is given in [12, 16].

Physical motivation of system (1.1) with $c = 1$ and f, g of a special form together with a construction of the solitary waves solutions and their stability are given in [13]. The existence, uniqueness and continuous dependence on the initial values of global classical solutions to a similar system but with the parabolic leading equation instead of our telegraph or wave equations were studied by J. Evans and N. Shenk [6, 7]. Moreover, J. Evans in [7–10] considered a stability in the suitable sense of stationary and traveling wave classical solutions to such systems. Those systems describe for example the dynamics of a nerve impulse in axons and they cover in particular the Hodgkin-Huxley system. Similar evolutionary systems appear also in [15]. In [11] there is described a connection between fast and slow waves in the FitzHugh-Nagumo system. Realistic view of wave mechanics was first proposed by de Broglie [1]. In his inspiring work Madelung related the linear Schrödinger equation to the hydrodynamic type system [14]. The various aspects of the hydrodynamic [2] and mechanistic [5] formulations of the nonlinear Schrödinger equation are still at the center of interest. These formulations lead to systems composed of the second or the third order partial differential evolution equations and the first order ones.

In this paper we consider local in time bounded together with their first derivatives solutions (u, v) , $u \in C_{loc}^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R}) \cap C^2([0, \tau] \times \mathbb{R}, \mathbb{R})$, $v \in C_{loc}^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R}^n)$ of the initial problem (1.1), (1.2), where $C_{loc}^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R})$ and $C_{loc}^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R}^n)$ ($\alpha \in (0, 1]$) are some Hölder spaces (see Section 2). If $\alpha = 1$, then $C_{loc}^{1+\alpha} = C^{1+\alpha}$ and they are global Lipschitz spaces. The main result of the paper is a theorem on the existence, uniqueness and estimates of such solutions. The estimates can be given a priori and they depend on the estimates of the initial conditions and the right-hand sides of equations. In the proof of the theorem we use the Banach fixed-point theorem, the Arzeli-Ascoli lemma and the integral form of the differential problem. The similar construction of the proof was used by T. Cziłapiński [3, 4] for quasilinear hyperbolic partial differential functional equations of the first order.

The paper is organized in the following way. In Section 2 a notation is introduced and some definitions and assumptions are formulated. The integral system equivalent under some assumptions to the initial differential problem (1.1), (1.2) is given in Section 3. In Section 4 the theorem on the existence, uniqueness and estimates of solutions to (1.1), (1.2) is proved.

2. NOTATION, DEFINITIONS AND ASSUMPTIONS

Let $\|\cdot\|$ be the *maximum norm* in \mathbb{R}^d , i.e.

$$\|y\| = \max_{i=1,\dots,d} |y_i|, \tag{2.1}$$

where $y \in \mathbb{R}^d$. In the space of bounded continuous functions $C_b(\Omega, \mathbb{R}^d)$ we define the *supremum norm*

$$\|z\|_0 = \sup \{ \|z(\omega)\| : \omega \in \Omega \}, \tag{2.2}$$

where $z \in C_b(\Omega, \mathbb{R}^d)$, $\Omega = \mathbb{R}$ or $\Omega \subset \mathbb{R}^2$. Obviously $(C_b(\Omega, \mathbb{R}^d), \|\cdot\|_0)$ is the Banach space. The space of bounded continuous functions together with their first derivatives we denote by $C_b^1(\Omega, \mathbb{R}^d)$ and together with their first and second derivatives by $C_b^2(\Omega, \mathbb{R}^d)$. By the symbol

$$\|z\|_1 = \|z\|_0 + \|z_t\|_0 + \|z_x\|_0 \tag{2.3}$$

we denote the norm in $C_b^1(\Omega, \mathbb{R}^d)$.

For any $z \in C_b(\Omega, \mathbb{R}^d)$ and $\alpha \in (0, 1]$, let

$$[z]_{H,\alpha} = \sup \{ \|z(t, x) - z(\bar{t}, \bar{x})\| [|t - \bar{t}| + |x - \bar{x}|]^{-\alpha} : (t, x), (\bar{t}, \bar{x}) \in \Omega \}. \tag{2.4}$$

Note that $[z]_{H,\alpha} < \infty$ is the smallest Hölder constant for the function z and the exponent α , it is usually called the *Hölder coefficient*. If $\alpha = 1$, then it is the *Lipschitz coefficient*. The *Hölder space* $C^{k+\alpha}(\Omega, \mathbb{R}^d)$, $k = 0, 1$, is the space of functions $z \in C_b^k(\Omega, \mathbb{R}^d)$ with the finite norm

$$\|z\|_{0+\alpha} = \|z\|_0 + [z]_{H,\alpha} \quad \text{if } k = 0, \tag{2.5}$$

$$\|z\|_{1+\alpha} = \|z\|_1 + [z_t]_{H,\alpha} + [z_x]_{H,\alpha} \quad \text{if } k = 1. \tag{2.6}$$

If $\Omega = \mathbb{R}$, then t and z_t in the above definitions do not appear.

Let $\Delta(x, \tau) \subset \mathbb{R} \times [0, \tau]$, $x \in \mathbb{R}$ be any isosceles triangle with the vertices $(x - \tau, 0)$, (x, τ) , $(x + \tau, 0)$. We denote by $C_{loc}^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R}^d)$ the *Hölder space* of functions $z \in C_b^1([0, \tau] \times \mathbb{R}, \mathbb{R}^d)$ such that $z \in C^{1+\alpha}(\Delta(x, \tau), \mathbb{R}^d)$ for any $\Delta(x, \tau)$ with the constants

$$\begin{aligned} & [z_t]_{H,\alpha,\Delta(x,\tau)} \\ &= \sup \{ \|z_t(t, x) - z_t(\bar{t}, \bar{x})\| [|t - \bar{t}| + |x - \bar{x}|]^{-\alpha} : (t, x), (\bar{t}, \bar{x}) \in \Delta(x, \tau) \}, \\ & [z_x]_{H,\alpha,\Delta(x,\tau)} \\ &= \sup \{ \|z_x(t, x) - z_x(\bar{t}, \bar{x})\| [|t - \bar{t}| + |x - \bar{x}|]^{-\alpha} : (t, x), (\bar{t}, \bar{x}) \in \Delta(x, \tau) \} \end{aligned} \tag{2.7}$$

independent of $\Delta(x, \tau)$. We say in this case that z_t, z_x are *uniformly Hölder continuous* in $[0, \tau] \times \mathbb{R}$. It is clear that if $\alpha = 1$, then $C_{loc}^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R}^d) = C^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R}^d)$.

The following assumption on the functions u_0, v_0, u_1 will be needed.

Assumption $\mathcal{H}_1[u_0, v_0, u_1]$. Suppose that $u_0 \in C^{1+\alpha}(\mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}, \mathbb{R})$, $v_0 \in C^{1+\alpha}(\mathbb{R}, \mathbb{R}^n)$, $u_1 \in C^{0+\alpha}(\mathbb{R}, \mathbb{R}) \cap C^1(\mathbb{R}, \mathbb{R})$ and that

$$\|u_0\|_0 \leq \Lambda_1, \quad \|u_1\|_0 \leq \Lambda_1^{(1)}, \quad \|(u_0)_x\|_0 \leq \Lambda_1^{(1)}, \tag{2.8}$$

$$[u_1]_{H,\alpha} \leq \Lambda_1^{(2)}, \quad [(u_0)_x]_{H,\alpha} \leq \Lambda_1^{(2)},$$

$$\|v_0\|_0 \leq \Lambda_2, \quad \|g(0, \cdot, u_0(\cdot), v_0(\cdot))\|_0 \leq \Lambda_2^{(1)}, \quad \|(v_0)_x\|_0 \leq \Lambda_2^{(1)},$$

$$[g(0, \cdot, u_0(\cdot), v_0(\cdot))]_{H,\alpha} \leq \Lambda_2^{(2)}, \quad [(v_0)_x]_{H,\alpha} \leq \Lambda_2^{(2)},$$

where $\Lambda_i, \Lambda_i^{(j)}, i, j = 1, 2$, are some non-negative constants.

Definition 2.1. Suppose that Assumption $\mathcal{H}_1[u_0, v_0, u_1]$ is satisfied and given non-negative constants $Q_i, Q_i^{(j)}, i, j = 1, 2$, such that $Q_i \geq \Lambda_i, Q_i^{(j)} \geq \Lambda_i^{(j)}$ we will denote by $C_{loc}^{1+\alpha}([0, \tau] \times \mathbb{R}, \mathbb{R}^{1+n}), v = (v_1, \dots, v_n)$, with the properties:

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}, \tag{2.9}$$

$$\|u\|_0 \leq Q_1, \quad \|u_t\|_0 \leq Q_1^{(1)}, \quad \|u_x\|_0 \leq Q_1^{(1)}, \tag{2.10}$$

$$[u_t]_{H,\alpha,\Delta(x,\tau)} \leq Q_1^{(2)}, \quad [u_x]_{H,\alpha,\Delta(x,\tau)} \leq Q_1^{(2)},$$

$$\|v\|_0 \leq Q_2, \quad \|v_t\|_0 \leq Q_2^{(1)}, \quad \|v_x\|_0 \leq Q_2^{(1)},$$

$$[v_t]_{H,\alpha,\Delta(x,\tau)} \leq Q_2^{(2)}, \quad [v_x]_{H,\alpha,\Delta(x,\tau)} \leq Q_2^{(2)}.$$

We will use the following assumptions on the functions f, g and their first derivatives.

Assumption $\mathcal{H}_2[f, g]$. The functions f, g and the derivatives $f_x, f_p, f_r = (f_{r_1}, \dots, f_{r_n}), g_x, g_p, g_r = (g_{r_1}, \dots, g_{r_n})$ are continuous. Moreover, there exist non-decreasing functions $M_k, M_k^{(1)}, H_2, H_2^{(1)} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+, k = 1, 2$, such that for all $t, \bar{t} \in [0, T], x, \bar{x} \in \mathbb{R}, (q_1, q_2) \in \mathbb{R}_+^2, |p|, |\bar{p}| \leq q_1, \|r\|, \|\bar{r}\| \leq q_2, i = 1, \dots, n$ we have

$$|f(t, x, p, r)| \leq M_1(q_1, q_2), \tag{2.11}$$

$$|f_x(t, x, p, r)|, |f_p(t, x, p, r)|, |f_{r_i}(t, x, p, r)| \leq M_1^{(1)}(q_1, q_2), \tag{2.12}$$

$$\|g(t, x, p, r)\| \leq M_2(q_1, q_2), \tag{2.13}$$

$$\|g_x(t, x, p, r)\|, \|g_p(t, x, p, r)\|, \|g_{r_i}(t, x, p, r)\| \leq M_2^{(1)}(q_1, q_2), \tag{2.14}$$

$$\|g(t, x, p, r) - g(\bar{t}, \bar{x}, \bar{p}, \bar{r})\| \leq H_2(q_1, q_2)|t - \bar{t}|^\alpha, \tag{2.15}$$

$$\|g_x(t, x, p, r) - g_x(\bar{t}, \bar{x}, \bar{p}, \bar{r})\| \leq H_2^{(1)}(q_1, q_2) [|x - \bar{x}| + |p - \bar{p}| + \|r - \bar{r}\|]^\alpha,$$

$$\|g_p(t, x, p, r) - g_p(\bar{t}, \bar{x}, \bar{p}, \bar{r})\| \leq H_2^{(1)}(q_1, q_2) [|x - \bar{x}| + |p - \bar{p}| + \|r - \bar{r}\|]^\alpha, \tag{2.16}$$

$$\|g_{r_i}(t, x, p, r) - g_{r_i}(\bar{t}, \bar{x}, \bar{p}, \bar{r})\| \leq H_2^{(1)}(q_1, q_2) [|x - \bar{x}| + |p - \bar{p}| + \|r - \bar{r}\|]^\alpha.$$

Put $M_k^* = M_k(Q_1, Q_2)$, $M_k^{(1)*} = M_k^{(1)}(Q_1, Q_2)$, $H_2^* = H_2(Q_1, Q_2)$, $H_2^{(1)*} = H_2^{(1)}(Q_1, Q_2)$, $k = 1, 2$. Some inequalities concerned Q_i , $Q_i^{(j)}$, $i, j = 1, 2$, have to be satisfied.

Assumption $\mathcal{H}_3[Q]$. Suppose that

$$\begin{aligned} Q_1 &> \Lambda_1, \\ Q_1^{(1)} &> \Lambda_1^{(1)}, \\ Q_1^{(2)} &> 2\Lambda_1^{(2)}, & \alpha \in (0, 1), \\ Q_1^{(2)} &> M_1^* + \frac{c^2}{4}Q_1 + \frac{c^2}{4}\Lambda_1 + \frac{3}{2}c\Lambda_1^{(1)} + 2\Lambda_1^{(2)}, & \alpha = 1, \end{aligned} \tag{2.17}$$

$$\begin{aligned} Q_2 &> \Lambda_2, \\ Q_2^{(1)} &\geq M_2^*, \quad Q_2^{(1)} > \Lambda_2^{(1)}, \\ Q_2^{(2)} &> H_2^*, \quad Q_2^{(2)} > \Lambda_2^{(2)}, & \alpha \in (0, 1), \\ Q_2^{(2)} &\geq M_2^{(1)*} \left[1 + 2 \left(Q_1^{(1)} + nQ_2^{(1)} \right) \right] + H_2^*, & \alpha = 1, \\ Q_2^{(2)} &> M_2^{(1)*} \left(1 + Q_1^{(1)} + nQ_2^{(1)} \right) + \Lambda_2^{(2)}, & \alpha = 1. \end{aligned} \tag{2.18}$$

Remark 2.2. The conditions (2.11)–(2.14) in Assumption $\mathcal{H}_2[f, g]$ are equivalent to boundedness of f, g and $f_x, f_p, f_{r_i}, g_x, g_p, g_{r_i}$ on the sets $[0, T] \times \mathbb{R} \times [-q_1, q_1] \times [-q_2, q_2]^n$ for any $(q_1, q_2) \in \mathbb{R}_+^2$, i.e. globally with respect to t, x and locally with respect to p, r . If f, g are globally Lipschitz continuous with respect to x, p, r or f, g do not depend on t, x and are locally Lipschitz continuous with respect to p, r , then (2.12), (2.14) hold. But the global Lipschitz continuity of f, g with respect to t, x and local one with respect to p, r not always imply (2.12), (2.14), e.g. $f(t, x, p, r) = x \sin p$. On the other hand the function $f(t, x, p, r) = p^2 \sin x$ is locally Lipschitz continuous with respect to x, p only and (2.12) is true. The form of (2.11)–(2.14) is useful in applications because we do not have to calculate the norms. It is important for example if we consider differential equations with the polynomial right-hand sides. An analogous situation appears in Assumption $\mathcal{H}_1[u_0, v_0, u_1]$. An analysis of the local or global Hölder continuity of g, g_x, g_p, g_{r_i} in conditions (2.15), (2.16) is similar as the above one concerning the local or global Lipschitz continuity of f, g (see Remark 4.4 and examples (4.23), (4.24)).

Remark 2.3. Note that in Assumption $\mathcal{H}_3[Q]$ a choice of the constants $Q_i, Q_i^{(j)}$, $i, j = 1, 2$, is possible for any given $\Lambda_i, \Lambda_i^{(j)}$, $i, j = 1, 2$. So we may give a priori the estimates of the solution of the differential problem (1.1), (1.2), its first derivatives and the Hölder constants of its first derivatives.

Define the constants

$$\begin{aligned}
 S_{1\tau} &= \frac{1}{2} \left[M_1^* + \frac{c^2}{4} Q_1 \right] \tau^2 + \left[\frac{c}{2} \Lambda_1 + \Lambda_1^{(1)} \right] \tau + \Lambda_1, \\
 S_{1\tau}^{(1.t)} &= \frac{c}{4} \left[M_1^* + \frac{c^2}{4} Q_1 \right] \tau^2 + \left[M_1^* + \frac{c^2}{4} Q_1 + \frac{c^2}{4} \Lambda_1 + \frac{c}{2} \Lambda_1^{(1)} \right] \tau \\
 &\quad + 2^{-1+\alpha} \Lambda_1^{(2)} \tau^\alpha + \Lambda_1^{(1)}, \\
 S_{1\tau}^{(1.x)} &= \left[M_1^* + \frac{c^2}{4} Q_1 + \frac{c}{2} \Lambda_1^{(1)} \right] \tau + 2^{-1+\alpha} \Lambda_1^{(2)} \tau^\alpha + \Lambda_1^{(1)}, \\
 S_{1\tau}^{(2.t)} &= \left[\frac{c^2}{8} M_1^* + \frac{c^4}{32} Q_1 \right] \tau^{3-\alpha} \\
 &\quad + \left[c(1+2^{-\alpha}) M_1^* + (1+2^{1-\alpha}) M_1^{(1)*} (1+Q_1^{(1)} + nQ_2^{(1)}) \right. \\
 &\quad + \frac{c^3}{4} (1+2^{-\alpha}) Q_1 + \frac{c^2}{4} (1+2^{1-\alpha}) Q_1^{(1)} \\
 &\quad + \left. \frac{c^3}{8} \Lambda_1 + \frac{c^2}{4} (1+2^{1-\alpha}) \Lambda_1^{(1)} \right] \tau^{2-\alpha} \\
 &\quad + \frac{c}{2} \Lambda_1^{(2)} \tau + \left[M_1^* + \frac{c^2}{4} Q_1 + \frac{c^2}{4} \Lambda_1 + \frac{3}{2} c \Lambda_1^{(1)} \right] \tau^{1-\alpha} + 2\Lambda_1^{(2)}, \\
 S_{1\tau}^{(2.x)} &= \left[\frac{c}{2} M_1^* + (1+2^{1-\alpha}) M_1^{(1)*} (1+Q_1^{(1)} + nQ_2^{(1)}) + \frac{c^3}{8} Q_1 \right. \\
 &\quad + \left. \frac{c^2}{4} (1+2^{1-\alpha}) Q_1^{(1)} \right] \tau^{2-\alpha} \\
 &\quad + \left[M_1^* + \frac{c^2}{4} Q_1 + \frac{c^2}{4} \Lambda_1 + c \left(1 + \frac{3^{1-\alpha}}{2} \right) \Lambda_1^{(1)} \right] \tau^{1-\alpha} + 2\Lambda_1^{(2)},
 \end{aligned} \tag{2.19}$$

$$\begin{aligned}
 S_{2\tau} &= M_2^* \tau + \Lambda_2, \\
 S_{2\tau}^{(1.t)} &= M_2^*, \\
 S_{2\tau}^{(1.x)} &= M_2^{(1)*} \left[1 + Q_1^{(1)} + nQ_2^{(1)} \right] \tau + \Lambda_2^{(1)}, \\
 S_{2\tau}^{(2.t)} &= M_2^{(1)*} \left[2^{1-\alpha} + (1+2^{1-\alpha}) (Q_1^{(1)} + nQ_2^{(1)}) \right] \tau^{1-\alpha} + H_2^*, \\
 S_{2\tau}^{(2.x)} &= \left[H_2^{(1)*} (1+Q_1^{(1)} + nQ_2^{(1)}) (1+Q_1^{(1)} + Q_2^{(1)})^\alpha \right. \\
 &\quad + M_2^{(1)*} (Q_1^{(2)} + nQ_2^{(2)}) \left. \right] \tau \\
 &\quad + M_2^{(1)*} (1+Q_1^{(1)} + nQ_2^{(1)}) \tau^{1-\alpha} + \Lambda_2^{(2)}.
 \end{aligned} \tag{2.20}$$

Remark 2.4. Note that since

$$\begin{aligned} \lim_{\tau \rightarrow 0^+} S_{1\tau} &= \Lambda_1, & \lim_{\tau \rightarrow 0^+} S_{1\tau}^{(1,t)} &= \lim_{\tau \rightarrow 0^+} S_{1\tau}^{(1,x)} = \Lambda_1^{(1)}, \\ \lim_{\tau \rightarrow 0^+} S_{1\tau}^{(2,t)} &= \lim_{\tau \rightarrow 0^+} S_{1\tau}^{(2,x)} = 2\Lambda_1^{(2)} & \text{if } \alpha \in (0, 1), \\ \lim_{\tau \rightarrow 0^+} S_{1\tau}^{(2,t)} &= \lim_{\tau \rightarrow 0^+} S_{1\tau}^{(2,x)} = M_1^* + \frac{c^2}{4}Q_1 + \frac{c^2}{4}\Lambda_1 + \frac{3}{2}c\Lambda_1^{(1)} + 2\Lambda_1^{(2)} & \text{if } \alpha = 1, \\ \lim_{\tau \rightarrow 0^+} S_{2\tau} &= \Lambda_2, & \lim_{\tau \rightarrow 0^+} S_{2\tau}^{(1,x)} &= \Lambda_2^{(1)}, \\ \lim_{\tau \rightarrow 0^+} S_{2\tau}^{(2,t)} &= H_2^*, & \lim_{\tau \rightarrow 0^+} S_{2\tau}^{(2,x)} &= \Lambda_2^{(2)} & \text{if } \alpha \in (0, 1), \\ \lim_{\tau \rightarrow 0^+} S_{2\tau}^{(2,x)} &= M_2^{(1)*} \left(1 + Q_1^{(1)} + nQ_2^{(1)} \right) + \Lambda_2^{(2)} & \text{if } \alpha = 1, \end{aligned}$$

we may by Assumption $\mathcal{H}_3[Q]$ choose $\tau \in (0, T]$ sufficiently small in order that $S_{i\tau} \leq Q_i, S_{i\tau}^{(j,t)} \leq Q_i^{(j)}, S_{i\tau}^{(j,x)} \leq Q_i^{(j)}, i, j = 1, 2$.

3. INTEGRAL SYSTEM

In this section we give a lemma which will be crucial in our future considerations.

Consider a nonlinear integral system of $(1 + n)$ equations of the form

$$\left\{ \begin{aligned} u(t, x) &= \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4}u(s, y) \right] dy ds \\ &\quad + \frac{1}{2}e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y) dy + \frac{c}{4}e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y) dy \\ &\quad + \frac{1}{2}e^{-\frac{c}{2}t} [u_0(x+t) + u_0(x-t)], & (t, x) \in [0, T] \times \mathbb{R}, \\ v(t, x) &= v_0(x) + \int_0^t g(s, x, u(s, x), v(s, x)) ds, & (t, x) \in [0, T] \times \mathbb{R}. \end{aligned} \right. \tag{3.1}$$

Lemma 3.1. Under Assumptions $\mathcal{H}_1[u_0, v_0, u_1]$ and $\mathcal{H}_2[f, g]$, the differential initial problem (1.1), (1.2) and the integral system (3.1) are equivalent in the sense that any solution $(u, v) \in C^2([0, T] \times \mathbb{R}, \mathbb{R}) \times C^1([0, T] \times \mathbb{R}, \mathbb{R}^n)$ of (1.1), (1.2) is a solution of (3.1) and any solution $(u, v) \in C^1([0, T] \times \mathbb{R}, \mathbb{R}^{1+n})$ of (3.1) belongs to $C^2([0, T] \times \mathbb{R}, \mathbb{R}) \times C^1([0, T] \times \mathbb{R}, \mathbb{R}^n)$ and fulfills (1.1), (1.2).

Proof. The first equation in (1.1) is equivalent to the equation

$$u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u = f(t, x, u, v) + \frac{c^2}{4}u, \quad (t, x) \in [0, T] \times \mathbb{R}.$$

Hence, the ansatz $u = we^{-\frac{c}{2}t}$ transforms problem (1.1), (1.2) to the equivalent differential system

$$\begin{cases} w_{tt} - w_{xx} = e^{\frac{c}{2}t} f(t, x, we^{-\frac{c}{2}t}, v) + \frac{c^2}{4}w, & (t, x) \in [0, T] \times \mathbb{R}, \\ v_t = g(t, x, we^{-\frac{c}{2}t}, v), & (t, x) \in [0, T] \times \mathbb{R} \end{cases} \tag{3.2}$$

with the equivalent initial conditions

$$\begin{cases} w(0, x) = u_0(x), & x \in \mathbb{R}, \\ v(0, x) = v_0(x), & x \in \mathbb{R}, \\ w_t(0, x) = \frac{c}{2}u_0(x) + u_1(x), & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

The continuity of f, g, u_0, v_0, u_1 , the use of Riemann’s method (see [16, p. 196]) for the first equation in (3.2) and integration of the second one imply that any solution $(w, v) \in C^2([0, T] \times \mathbb{R}, \mathbb{R}) \times C^1([0, T] \times \mathbb{R}, \mathbb{R}^n)$ of (3.2), (3.3) is a solution of a nonlinear integral system of $(1 + n)$ equations of the form

$$\begin{cases} w(t, x) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} \left[e^{\frac{c}{2}s} f(s, y, w(s, y), v(s, y)) + \frac{c^2}{4} w(s, y) \right] dy ds \\ \quad + \frac{1}{2} \int_{x-t}^{x+t} \left[\frac{c}{2} u_0(y) + u_1(y) \right] dy \\ \quad + \frac{1}{2} [u_0(x+t) + u_0(x-t)], & (t, x) \in [0, T] \times \mathbb{R}, \\ v(t, x) = v_0(x) + \int_0^t g(s, x, w(s, x), v(s, x)) ds, & (t, x) \in [0, T] \times \mathbb{R}. \end{cases} \tag{3.4}$$

Multiplying the first equation in (3.4) by $e^{-\frac{c}{2}t}$ we have that any solution $(u, v) \in C^2([0, T] \times \mathbb{R}, \mathbb{R}) \times C^1([0, T] \times \mathbb{R}, \mathbb{R}^n)$ of the differential initial problem (1.1), (1.2) is a solution of the integral system (3.1).

On the other hand, let $(u, v) \in C^1([0, T] \times \mathbb{R}, \mathbb{R}^{1+n})$ be a solution of (3.1). Put for simplicity

$$A(t, x, s) = (s, x + (t - s), u(s, x + (t - s)), v(s, x + (t - s))),$$

$$B(t, x, s) = (s, x - (t - s), u(s, x - (t - s)), v(s, x - (t - s))).$$

From the regularity of f, g, u_0, v_0, u_1 and differentiation of the integrals in (3.1) we have

$$\begin{aligned}
u_t(t, x) &= -\frac{c}{4} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] dy ds \\
&\quad + \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t, x, s)) + \frac{c^2}{4} u(s, x + (t-s)) \right] ds \\
&\quad + \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t, x, s)) + \frac{c^2}{4} u(s, x - (t-s)) \right] ds \\
&\quad - \frac{c}{4} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y) dy + \frac{1}{2} e^{-\frac{c}{2}t} [u_1(x+t) + u_1(x-t)] \\
&\quad - \frac{c^2}{8} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y) dy + \frac{1}{2} e^{-\frac{c}{2}t} [(u_0)_x(x+t) - (u_0)_x(x-t)], \\
\\
u_{tt}(t, x) &= \frac{c^2}{8} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] dy ds \\
&\quad - \frac{c}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t, x, s)) + \frac{c^2}{4} u(s, x + (t-s)) \right] ds \\
&\quad - \frac{c}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t, x, s)) + \frac{c^2}{4} u(s, x - (t-s)) \right] ds \\
&\quad + \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f_x(A(t, x, s)) + f_p(A(t, x, s)) u_x(s, x + (t-s)) \right. \\
&\quad \left. + \sum_{i=1}^n f_{r_i}(A(t, x, s)) (v_i)_x(s, x + (t-s)) + \frac{c^2}{4} u_x(s, x + (t-s)) \right] ds
\end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f_x(B(t, x, s)) + f_p(B(t, x, s))u_x(s, x - (t - s)) \right. \\
 & \left. + \sum_{i=1}^n f_{r_i}(B(t, x, s))(v_i)_x(s, x - (t - s)) + \frac{c^2}{4}u_x(s, x - (t - s)) \right] ds \\
 & + f(t, x, u(t, x), v(t, x)) + \frac{c^2}{4}u(t, x) \\
 & + \frac{c^2}{8}e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y)dy - \frac{c}{2}e^{-\frac{c}{2}t} [u_1(x+t) + u_1(x-t)] \\
 & + \frac{1}{2}e^{-\frac{c}{2}t} [(u_1)_x(x+t) - (u_1)_x(x-t)] \\
 & + \frac{c^3}{16}e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y)dy - \frac{c^2}{8}e^{-\frac{c}{2}t} [u_0(x+t) + u_0(x-t)] \\
 & - \frac{c}{4}e^{-\frac{c}{2}t} [(u_0)_x(x+t) - (u_0)_x(x-t)] \\
 & + \frac{1}{2}e^{-\frac{c}{2}t} [(u_0)_{xx}(x+t) + (u_0)_{xx}(x-t)],
 \end{aligned}$$

$$\begin{aligned}
 u_{xx}(t, x) &= \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f_x(A(t, x, s)) + f_p(A(t, x, s))u_x(s, x + (t - s)) \right. \\
 & \left. + \sum_{i=1}^n f_{r_i}(A(t, x, s))(v_i)_x(s, x + (t - s)) + \frac{c^2}{4}u_x(s, x + (t - s)) \right] ds \\
 & - \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f_x(B(t, x, s)) + f_p(B(t, x, s))u_x(s, x - (t - s)) \right. \\
 & \left. + \sum_{i=1}^n f_{r_i}(B(t, x, s))(v_i)_x(s, x - (t - s)) + \frac{c^2}{4}u_x(s, x - (t - s)) \right] ds \\
 & + \frac{1}{2}e^{-\frac{c}{2}t} [(u_1)_x(x+t) - (u_1)_x(x-t)] \\
 & + \frac{c}{4}e^{-\frac{c}{2}t} [(u_0)_x(x+t) - (u_0)_x(x-t)] \\
 & + \frac{1}{2}e^{-\frac{c}{2}t} [(u_0)_{xx}(x+t) + (u_0)_{xx}(x-t)],
 \end{aligned}$$

$$v_t(t, x) = g(t, x, u(t, x), v(t, x)).$$

It is clear that $(u, v) \in C^2([0, T] \times \mathbb{R}, \mathbb{R}) \times C^1([0, T] \times \mathbb{R}, \mathbb{R}^n)$ and fulfills (1.1), (1.2).

□

4. THE MAIN RESULT

Theorem 4.1. *If Assumptions $\mathcal{H}_1[u_0, v_0, u_1]$, $\mathcal{H}_2[f, g]$, $\mathcal{H}_3[Q]$ are satisfied, then there is $\tau \in (0, T]$ such that problem (1.1), (1.2) has an unique solution $(u, v) \in C_{b,\tau}^{1,\alpha}(Q)$ and $u \in C^2([0, \tau] \times \mathbb{R}, \mathbb{R})$.*

Proof. Define the operator $\mathbf{H} = (F, G)$ on $C_{b,\tau}^{1,\alpha}(Q)$ by the formula

$$\begin{aligned}
 F[u, v](t, x) = & \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] dy ds \\
 & + \frac{1}{2} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y) dy + \frac{c}{4} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y) dy
 \end{aligned} \tag{4.1}$$

$$+ \frac{1}{2} e^{-\frac{c}{2}t} [u_0(x+t) + u_0(x-t)],$$

$$G[u, v](t, x) = v_0(x) + \int_0^t g(s, x, u(s, x), v(s, x)) ds, \tag{4.2}$$

where $(u, v) \in C_{b,\tau}^{1,\alpha}(Q)$, $(t, x) \in [0, \tau] \times \mathbb{R}$. It is obvious that $(u, v) \in C_{b,\tau}^{1,\alpha}(Q)$ is a solution of the integral system (3.1) and consequently, under additional assumptions, a solution of the initial differential problem (1.1), (1.2) (see Lemma 3.1) if and only if its a fix-point of \mathbf{H} .

Note that by the Arzeli-Ascola lemma, the set $C_{b,\tau}^{1,\alpha}(Q)$ is closed in the Banach space $(C_b([0, \tau] \times \mathbb{R}, \mathbb{R}^{1+n}), \|\cdot\|_0)$. We will show that there is $\tau \in (0, T]$ such that \mathbf{H} maps $C_{b,\tau}^{1,\alpha}(Q)$ into itself and it is a contraction in the complete metric space $(C_{b,\tau}^{1,\alpha}(Q), d)$ with the metric d generated by the norm $\|\cdot\|_0$. Then we will use the Banach fixed-point theorem.

Put for simplicity

$$A(t, x, s) = (s, x + (t - s), u(s, x + (t - s)), v(s, x + (t - s))),$$

$$B(t, x, s) = (s, x - (t - s), u(s, x - (t - s)), v(s, x - (t - s))),$$

$$C(x, s) = (s, x, u(s, x), v(s, x)).$$

Definitions (4.1), (4.2) and Assumptions $\mathcal{H}_1[u_0, v_0, u_1]$, $\mathcal{H}_2[f, g]$ imply

$$\begin{aligned}
 F_t[u, v](t, x) = & -\frac{c}{4} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] dy ds \\
 & + \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t, x, s)) + \frac{c^2}{4} u(s, x + (t - s)) \right] ds \\
 & + \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t, x, s)) + \frac{c^2}{4} u(s, x - (t - s)) \right] ds \tag{4.3} \\
 & - \frac{c}{4} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y) dy + \frac{1}{2} e^{-\frac{c}{2}t} [u_1(x + t) + u_1(x - t)] \\
 & - \frac{c^2}{8} e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y) dy + \frac{1}{2} e^{-\frac{c}{2}t} [(u_0)_x(x + t) - (u_0)_x(x - t)],
 \end{aligned}$$

$$\begin{aligned}
 F_x[u, v](t, x) = & \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t, x, s)) + \frac{c^2}{4} u(s, x + (t - s)) \right] ds \\
 & - \frac{1}{2} \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t, x, s)) + \frac{c^2}{4} u(s, x - (t - s)) \right] ds \tag{4.4} \\
 & + \frac{1}{2} e^{-\frac{c}{2}t} [u_1(x + t) - u_1(x - t)] \\
 & + \frac{c}{4} e^{-\frac{c}{2}t} [u_0(x + t) - u_0(x - t)] + \frac{1}{2} e^{-\frac{c}{2}t} [(u_0)_x(x + t) + (u_0)_x(x - t)],
 \end{aligned}$$

$$G_t[u, v](t, x) = g(t, x, u(t, x), v(t, x)), \tag{4.5}$$

$$\begin{aligned}
 G_x[u, v](t, x) = & (v_0)_x(x) + \int_0^t [g_x(C(x, s)) + g_p(C(x, s)) u_x(s, x) \\
 & + \sum_{i=1}^n g_{r_i}(C(x, s)) (v_i)_x(s, x)] ds, \tag{4.6}
 \end{aligned}$$

where $(u, v) \in C_{b,\tau}^{1,\alpha}(Q)$, $(t, x) \in [0, \tau] \times \mathbb{R}$.

We prove firstly that for a sufficiently small $\tau \in (0, T]$ the operator \mathbf{H} maps $C_{b,\tau}^{1,\alpha}(Q)$ into itself. Let $(u, v) \in C_{b,\tau}^{1,\alpha}(Q)$ be fixed. Obviously, $\mathbf{H}[u, v] \in C_b^1([0, \tau] \times \mathbb{R}, \mathbb{R}^{1+n})$ and for $x \in \mathbb{R}$

$$F[u, v](0, x) = u_0(x), \quad G[u, v](0, x) = v_0(x), \quad F_t[u, v](0, x) = u_1(x). \tag{4.7}$$

It is enough to show that there exists $\tau \in (0, T]$ such that for each $(t, x) \in [0, \tau] \times \mathbb{R}$

$$|F[u, v](t, x)| \leq Q_1, \quad |F_t[u, v](t, x)| \leq Q_1^{(1)}, \quad |F_x[u, v](t, x)| \leq Q_1^{(1)}, \quad (4.8)$$

$$\|G[u, v](t, x)\| \leq Q_2, \quad \|G_t[u, v](t, x)\| \leq Q_2^{(1)}, \quad \|G_x[u, v](t, x)\| \leq Q_2^{(1)}, \quad (4.9)$$

and for an arbitrarily fixed triangle $\Delta(X, \tau)$, $X \in \mathbb{R}$ and for each two points $(t, x), (\bar{t}, \bar{x}) \in \Delta(X, \tau)$

$$|F_t[u, v](t, x) - F_t[u, v](\bar{t}, \bar{x})| \leq Q_1^{(2)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha, \quad (4.10)$$

$$|F_x[u, v](t, x) - F_x[u, v](\bar{t}, \bar{x})| \leq Q_1^{(2)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha,$$

$$\|G_t[u, v](t, x) - G_t[u, v](\bar{t}, \bar{x})\| \leq Q_2^{(2)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha, \quad (4.11)$$

$$\|G_x[u, v](t, x) - G_x[u, v](\bar{t}, \bar{x})\| \leq Q_2^{(2)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha.$$

From Assumptions $\mathcal{H}_1[u_0, v_0, u_1]$, $\mathcal{H}_2[f, g]$, the properties of integrals and definitions (2.19), (2.20) we have

$$|F[u, v](t, x)| \leq S_{1\tau}, \quad |F_t[u, v](t, x)| \leq S_{1\tau}^{(1,t)}, \quad |F_x[u, v](t, x)| \leq S_{1\tau}^{(1,x)}, \quad (4.12)$$

$$\|G[u, v](t, x)\| \leq S_{2\tau}, \quad \|G_t[u, v](t, x)\| \leq S_{2\tau}^{(1,t)}, \quad \|G_x[u, v](t, x)\| \leq S_{2\tau}^{(1,x)} \quad (4.13)$$

for $(t, x) \in [0, \tau] \times \mathbb{R}$. Let $\Delta(X, \tau)$ be fixed and $(t, x), (\bar{t}, \bar{x}) \in \Delta(X, \tau)$. Using the mean value theorem for the functions:

$$f_1(t, x) = \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{\sigma}{2}(t-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] dy ds,$$

$$f_2(t, x) = \int_0^t e^{-\frac{\sigma}{2}(t-s)} \left[f(A(t, x, s)) + \frac{c^2}{4} u(s, x + (t - s)) \right] ds,$$

$$f_3(t, x) = \int_0^t e^{-\frac{\sigma}{2}(t-s)} \left[f(B(t, x, s)) + \frac{c^2}{4} u(s, x - (t - s)) \right] ds,$$

$$\tilde{u}_1(t, x) = e^{-\frac{\sigma}{2}t} \int_{x-t}^{x+t} u_1(y) dy, \quad \tilde{u}_0(t, x) = e^{-\frac{\sigma}{2}t} \int_{x-t}^{x+t} u_0(y) dy, \quad u_0$$

and the relations: $|t - \bar{t}| = |t - \bar{t}|^{1-\alpha} |t - \bar{t}|^\alpha$, $|x - \bar{x}| = |x - \bar{x}|^{1-\alpha} |x - \bar{x}|^\alpha$, $|t - \bar{t}| \leq \tau$, $|x - \bar{x}| \leq 2\tau$ we obtain

$$\begin{aligned}
 & |F_{\bar{t}}[u, v](t, x) - F_{\bar{t}}[u, v](\bar{t}, \bar{x})| \\
 & \leq \frac{c}{4} \left| \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] \right. \\
 & \quad \left. - \int_0^{\bar{t}} \int_{\bar{x}-(\bar{t}-s)}^{\bar{x}+(\bar{t}-s)} \int e^{-\frac{c}{2}(\bar{t}-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] \right| \\
 & + \frac{1}{2} \left| \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(A(t, x, s)) + \frac{c^2}{4} u(s, x + (t - s)) \right] ds \right. \\
 & \quad \left. - \int_0^{\bar{t}} e^{-\frac{c}{2}(\bar{t}-s)} \left[f(A(\bar{t}, \bar{x}, s)) + \frac{c^2}{4} u(s, \bar{x} + (\bar{t} - s)) \right] ds \right| \\
 & + \frac{1}{2} \left| \int_0^t e^{-\frac{c}{2}(t-s)} \left[f(B(t, x, s)) + \frac{c^2}{4} u(s, x - (t - s)) \right] ds \right. \\
 & \quad \left. - \int_0^{\bar{t}} e^{-\frac{c}{2}(\bar{t}-s)} \left[f(B(\bar{t}, \bar{x}, s)) + \frac{c^2}{4} u(s, \bar{x} - (\bar{t} - s)) \right] ds \right| \tag{4.14} \\
 & + \frac{c}{4} \left| e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_1(y) dy - e^{-\frac{c}{2}\bar{t}} \int_{\bar{x}-\bar{t}}^{\bar{x}+\bar{t}} u_1(y) dy \right| \\
 & + \frac{1}{2} \left| e^{-\frac{c}{2}t} u_1(x+t) - e^{-\frac{c}{2}\bar{t}} u_1(x+t) + e^{-\frac{c}{2}\bar{t}} u_1(x+t) - e^{-\frac{c}{2}\bar{t}} u_1(\bar{x}+\bar{t}) \right| \\
 & + \frac{1}{2} \left| e^{-\frac{c}{2}t} u_1(x-t) - e^{-\frac{c}{2}\bar{t}} u_1(x-t) + e^{-\frac{c}{2}\bar{t}} u_1(x-t) - e^{-\frac{c}{2}\bar{t}} u_1(\bar{x}-\bar{t}) \right| \\
 & + \frac{c^2}{8} \left| e^{-\frac{c}{2}t} \int_{x-t}^{x+t} u_0(y) dy - e^{-\frac{c}{2}\bar{t}} \int_{\bar{x}-\bar{t}}^{\bar{x}+\bar{t}} u_0(y) dy \right| \\
 & + \frac{1}{2} \left| e^{-\frac{c}{2}t} (u_0)_x(x+t) - e^{-\frac{c}{2}\bar{t}} (u_0)_x(x+t) \right. \\
 & \quad \left. + e^{-\frac{c}{2}\bar{t}} (u_0)_x(x+t) - e^{-\frac{c}{2}\bar{t}} (u_0)_x(\bar{x}+\bar{t}) \right| \\
 & + \frac{1}{2} \left| e^{-\frac{c}{2}t} (u_0)_x(x-t) - e^{-\frac{c}{2}\bar{t}} (u_0)_x(x-t) \right. \\
 & \quad \left. + e^{-\frac{c}{2}\bar{t}} (u_0)_x(x-t) - e^{-\frac{c}{2}\bar{t}} (u_0)_x(\bar{x}-\bar{t}) \right|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{c}{4} \left| -\frac{c}{2} \int_0^{t_1} \int_{x_1-(t_1-s)}^{x_1+(t_1-s)} e^{-\frac{c}{2}(t_1-s)} \left[f(s, y, u(s, y), v(s, y)) + \frac{c^2}{4} u(s, y) \right] ds \right. \\
 &\quad + \int_0^{t_1} e^{-\frac{c}{2}(t_1-s)} \left[f(A(t_1, x_1, s)) + \frac{c^2}{4} u(s, x_1 + (t_1 - s)) \right] ds \\
 &\quad + \left. \int_0^{t_1} e^{-\frac{c}{2}(t_1-s)} \left[f(B(t_1, x_1, s)) + \frac{c^2}{4} u(s, x_1 - (t_1 - s)) \right] ds \right| |t - \bar{t}| \\
 &\quad + \frac{c}{4} \left| \int_0^{t_1} e^{-\frac{c}{2}(t_1-s)} \left[f(A(t_1, x_1, s)) + \frac{c^2}{4} u(s, x_1 + (t_1 - s)) \right] ds \right. \\
 &\quad \left. - \int_0^{t_1} e^{-\frac{c}{2}(t_1-s)} \left[f(B(t_1, x_1, s)) + \frac{c^2}{4} u(s, x_1 - (t_1 - s)) \right] ds \right| |x - \bar{x}| \\
 &\quad + \frac{1}{2} \left| \int_0^{t_2} \left[-\frac{c}{2} e^{-\frac{c}{2}(t_2-s)} \left[f(A(t_2, x_2, s)) + \frac{c^2}{4} u(s, x_2 + (t_2 - s)) \right] \right. \right. \\
 &\quad \left. \left. + e^{-\frac{c}{2}(t_2-s)} \left[f_x(A(t_2, x_2, s)) + f_p(A(t_2, x_2, s)) u_x(s, x_2 + (t_2 - s)) \right. \right. \right. \\
 &\quad \left. \left. + \sum_{i=1}^n f_{r_i}(A(t_2, x_2, s)) (v_i)_x(s, x_2 + (t_2 - s)) \right. \right. \\
 &\quad \left. \left. + \frac{c^2}{4} u_x(s, x_2 + (t_2 - s)) \right] \right] ds \\
 &\quad \left. + f(t_2, x_2, u(t_2, x_2), v(t_2, x_2)) + \frac{c^2}{4} u(t_2, x_2) \right| |t - \bar{t}| \\
 &\quad + \frac{1}{2} \left| \int_0^{t_2} e^{-\frac{c}{2}(t_2-s)} \left[f_x(A(t_2, x_2, s)) + f_p(A(t_2, x_2, s)) u_x(s, x_2 + (t_2 - s)) \right. \right. \\
 &\quad \left. \left. + \sum_{i=1}^n f_{r_i}(A(t_2, x_2, s)) (v_i)_x(s, x_2 + (t_2 - s)) \right. \right. \\
 &\quad \left. \left. + \frac{c^2}{4} u_x(s, x_2 + (t_2 - s)) \right] ds \right| |x - \bar{x}|
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \left| \int_0^{t_3} \left[-\frac{c}{2} e^{-\frac{c}{2}(t_3-s)} \left[f(B(t_3, x_3, s)) + \frac{c^2}{4} u(s, x_3 - (t_3 - s)) \right] \right. \right. \\
 & - e^{-\frac{c}{2}(t_3-s)} [f_x(B(t_3, x_3, s)) + f_p(B(t_3, x_3, s)) u_x(s, x_3 - (t_3 - s)) \\
 & \left. \left. + \sum_{i=1}^n f_{r_i}(B(t_3, x_3, s)) (v_i)_x(s, x_3 - (t_3 - s)) + \frac{c^2}{4} u_x(s, x_3 - (t_3 - s)) \right] \right] ds \\
 & + f(t_3, x_3, u(t_3, x_3), v(t_3, x_3)) + \frac{c^2}{4} u(t_3, x_3) \Big| |t - \bar{t}| \\
 & + \frac{1}{2} \left| \int_0^{t_3} e^{-\frac{c}{2}(t_3-s)} [f_x(B(t_3, x_3, s)) + f_p(B(t_3, x_3, s)) u_x(s, x_3 - (t_3 - s)) \right. \\
 & \left. + \sum_{i=1}^n f_{r_i}(B(t_3, x_3, s)) (v_i)_x(s, x_3 - (t_3 - s)) + \frac{c^2}{4} u_x(s, x_3 - (t_3 - s)) \right] ds \Big| \\
 & \times |x - \bar{x}| \\
 & + \frac{c}{4} \left[\left| -\frac{c}{2} e^{-\frac{c}{2}t_4} \int_{x_4-t_4}^{x_4+t_4} u_1(y) dy + e^{-\frac{c}{2}t_4} [u_1(x_4 + t_4) + u_1(x_4 - t_4)] \right| |t - \bar{t}| \right. \\
 & \left. + e^{-\frac{c}{2}t_4} |u_1(x_4 + t_4) - u_1(x_4 - t_4)| |x - \bar{x}| \right] \\
 & + \frac{1}{2} \left| -\frac{c}{2} e^{-\frac{c}{2}t_5}(t - \bar{t}) \right| |u_1(x + t)| + \frac{1}{2} e^{-\frac{c}{2}\bar{t}} |u_1(x + t) - u_1(\bar{x} + \bar{t})| \\
 & + \frac{1}{2} \left| -\frac{c}{2} e^{-\frac{c}{2}t_5}(t - \bar{t}) \right| |u_1(x - t)| + \frac{1}{2} e^{-\frac{c}{2}\bar{t}} |u_1(x - t) - u_1(\bar{x} - \bar{t})| \\
 & + \frac{c^2}{8} \left[\left| -\frac{c}{2} e^{-\frac{c}{2}t_6} \int_{x_6-t_6}^{x_6+t_6} u_0(y) dy + e^{-\frac{c}{2}t_6} [u_0(x_6 + t_6) + u_0(x_6 - t_6)] \right| |t - \bar{t}| \right. \\
 & \left. + e^{-\frac{c}{2}t_6} |u_0(x_6 + t_6) - u_0(x_6 - t_6)| |x - \bar{x}| \right] \\
 & + \frac{1}{2} \left| -\frac{c}{2} e^{-\frac{c}{2}t_5}(t - \bar{t}) \right| |(u_0)_x(x + t)| + \frac{1}{2} e^{-\frac{c}{2}\bar{t}} |(u_0)_x(x + t) - (u_0)_x(\bar{x} + \bar{t})| \\
 & + \frac{1}{2} \left| -\frac{c}{2} e^{-\frac{c}{2}t_5}(t - \bar{t}) \right| |(u_0)_x(x - t)| + \frac{1}{2} e^{-\frac{c}{2}\bar{t}} |(u_0)_x(x - t) - (u_0)_x(\bar{x} - \bar{t})|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \left\{ \frac{c}{4} \left[\frac{c}{2} \left(M_1^* + \frac{c^2}{4} Q_1 \right) \tau^2 \tau^{1-\alpha} + 2 \left(M_1^* + \frac{c^2}{4} Q_1 \right) \tau \tau^{1-\alpha} \right. \right. \\
 &\quad \left. \left. + 2 \left(M_1^* + \frac{c^2}{4} Q_1 \right) \tau (2\tau)^{1-\alpha} \right] + 2 \cdot \frac{1}{2} \left[\frac{c}{2} \left(M_1^* + \frac{c^2}{4} Q_1 \right) \tau \tau^{1-\alpha} \right. \right. \\
 &\quad \left. \left. + M_1^{(1)*} \left(1 + Q_1^{(1)} + nQ_2^{(1)} \right) \tau \tau^{1-\alpha} + \frac{c^2}{4} Q_1^{(1)} \tau \tau^{1-\alpha} + \left(M_1^* + \frac{c^2}{4} Q_1 \right) \tau^{1-\alpha} \right. \right. \\
 &\quad \left. \left. + M_1^{(1)*} \left(1 + Q_1^{(1)} + nQ_2^{(1)} \right) \tau (2\tau)^{1-\alpha} + \frac{c^2}{4} Q_1^{(1)} \tau (2\tau)^{1-\alpha} \right] \right. \\
 &\quad \left. + \frac{c}{4} \left[\left(\frac{c}{2} \Lambda_1^{(1)} 2\tau + 2\Lambda_1^{(1)} \right) \tau^{1-\alpha} + \Lambda_1^{(2)} (2\tau)^\alpha (2\tau)^{1-\alpha} \right] + \frac{c}{2} \Lambda_1^{(1)} \tau^{1-\alpha} + \Lambda_1^{(2)} \right. \\
 &\quad \left. + \frac{c^2}{8} \left[\left(\frac{c}{2} \Lambda_1 2\tau + 2\Lambda_1 \right) \tau^{1-\alpha} + \Lambda_1^{(1)} 2\tau (2\tau)^{1-\alpha} \right] + \frac{c}{2} \Lambda_1^{(1)} \tau^{1-\alpha} + \Lambda_1^{(2)} \right\} \\
 &\quad \times [|t - \bar{t}| + |x - \bar{x}|]^\alpha \\
 &= S_{1\tau}^{(2,t)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha.
 \end{aligned}$$

The points $(t_i, x_i) \in \Delta(X, \tau)$, $i = 1, \dots, 5$, are intermediate ones. Making similar calculations as above we get

$$\begin{aligned}
 &|F_x[u, v](t, x) - F_x[u, v](\bar{t}, \bar{x})| \\
 &\leq \left\{ \left[\left(\frac{c}{2} M_1^* + M_1^{(1)*} \left(1 + Q_1^{(1)} + nQ_2^{(1)} \right) \right) \tau + M_1^* \right] \tau^{1-\alpha} \right. \\
 &\quad \left. + M_1^{(1)*} \left(1 + Q_1^{(1)} + nQ_2^{(1)} \right) (2\tau)^{1-\alpha} \tau + \frac{c^2}{4} \left[\left(\frac{c}{2} Q_1 + Q_1^{(1)} \right) \tau + Q_1 \right] \tau^{1-\alpha} \right. \\
 &\quad \left. + \frac{c^2}{4} Q_1^{(1)} (2\tau)^{1-\alpha} \tau + \Lambda_1^{(2)} + \frac{c}{2} \Lambda_1^{(1)} \tau^{1-\alpha} + \frac{c}{2} \left[\Lambda_1^{(1)} (3\tau)^{1-\alpha} + \frac{c}{2} \Lambda_1 \tau^{1-\alpha} \right] \right. \\
 &\quad \left. + \Lambda_1^{(2)} + \frac{c}{2} \Lambda_1^{(1)} \tau^{1-\alpha} \right\} [|t - \bar{t}| + |x - \bar{x}|]^\alpha \\
 &= S_{1\tau}^{(2,x)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha,
 \end{aligned} \tag{4.15}$$

$$\begin{aligned}
 \|G_t[u, v](t, x) - G_t[u, v](\bar{t}, \bar{x})\| &\leq \left\{ H_2^* + M_2^{(1)*} \left[(2\tau)^{1-\alpha} + Q_1^{(1)} (\tau^{1-\alpha} + (2\tau)^{1-\alpha}) \right. \right. \\
 &\quad \left. \left. + nQ_2^{(1)} (\tau^{1-\alpha} + (2\tau)^{1-\alpha}) \right] \right\} [|t - \bar{t}| + |x - \bar{x}|]^\alpha \\
 &= S_{2\tau}^{(2,t)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha
 \end{aligned} \tag{4.16}$$

for a fixed $\Delta(X, \tau)$ and $(t, x), (\bar{t}, \bar{x}) \in \Delta(X, \tau)$. Because of the non-sufficient regularity of g we can not use the mean value theorem for $g_1(t, x) = G_x[u, v](t, x)$. But using the additivity of an integral we have

$$\begin{aligned}
 & \|G_x[u, v](t, x) - G_x[u, v](\bar{t}, \bar{x})\| \\
 & \leq \| (v_0)_x(x) - (v_0)_x(\bar{x}) \| + \left\| \int_0^t g_x(C(x, s)) ds - \int_0^{\bar{t}} g_x(C(\bar{x}, s)) ds \right\| \\
 & \quad + \left\| \int_0^t g_p(C(x, s)) u_x(s, x) ds - \int_0^{\bar{t}} g_p(C(\bar{x}, s)) u_x(s, \bar{x}) ds \right\| \\
 & \quad + \sum_{i=1}^n \left\| \int_0^t g_{r_i}(C(x, s)) (v_i)_x(s, x) ds - \int_0^{\bar{t}} g_{r_i}(C(\bar{x}, s)) (v_i)_x(s, \bar{x}) ds \right\| \\
 & \leq \Lambda_2^{(2)} |x - \bar{x}|^\alpha + \left\| \int_0^t g_x(C(x, s)) ds - \int_0^t g_x(C(\bar{x}, s)) ds - \int_t^{\bar{t}} g_x(C(\bar{x}, s)) ds \right\| \\
 & \quad + \left\| \int_0^t g_p(C(x, s)) u_x(s, x) ds - \int_0^t g_p(C(\bar{x}, s)) u_x(s, \bar{x}) ds - \int_t^{\bar{t}} g_p(C(\bar{x}, s)) u_x(s, \bar{x}) ds \right\| \\
 & \quad + \sum_{i=1}^n \left\| \int_0^t g_{r_i}(C(x, s)) (v_i)_x(s, x) ds - \int_0^t g_{r_i}(C(\bar{x}, s)) (v_i)_x(s, \bar{x}) ds \right. \\
 & \quad \quad \left. - \int_t^{\bar{t}} g_{r_i}(C(\bar{x}, s)) (v_i)_x(s, \bar{x}) ds \right\| \\
 & \leq \Lambda_2^{(2)} |x - \bar{x}|^\alpha + \int_0^t \|g_x(C(x, s)) - g_x(C(\bar{x}, s))\| ds + \left| \int_t^{\bar{t}} M_2^{(1)*} ds \right| \\
 & \quad + \int_0^t \|g_p(C(x, s)) u_x(s, x) - g_p(C(x, s)) u_x(s, \bar{x})\| ds \\
 & \quad + \int_0^t \|g_p(C(x, s)) u_x(s, \bar{x}) - g_p(C(\bar{x}, s)) u_x(s, \bar{x})\| ds + \left| \int_t^{\bar{t}} M_2^{(1)*} Q_1^{(1)} ds \right| \\
 & \quad + \sum_{i=1}^n \int_0^t \|g_{r_i}(C(x, s)) (v_i)_x(s, x) - g_{r_i}(C(x, s)) (v_i)_x(s, \bar{x})\| ds \\
 & \quad + \sum_{i=1}^n \int_0^t \|g_{r_i}(C(x, s)) (v_i)_x(s, \bar{x}) - g_{r_i}(C(\bar{x}, s)) (v_i)_x(s, \bar{x})\| ds + \left| \int_t^{\bar{t}} M_2^{(1)*} Q_2^{(1)} ds \right|
 \end{aligned}
 \tag{4.17}$$

$$\begin{aligned}
 &\leq \Lambda_2^{(2)} |x - \bar{x}|^\alpha + \int_0^t H_2^{(1)*} [|x - \bar{x}| + |u(s, x) - u(s, \bar{x})| \\
 &\quad + \|v(s, x) - v(s, \bar{x})\|]^\alpha ds + M_2^{(1)*} |t - \bar{t}| \\
 &\quad + \int_0^t M_2^{(1)*} Q_1^{(2)} |x - \bar{x}|^\alpha ds + \int_0^t H_2^{(1)*} [|x - \bar{x}| + |u(s, x) - u(s, \bar{x})| \\
 &\quad + \|v(s, x) - v(s, \bar{x})\|]^\alpha Q_1^{(1)} ds + M_2^{(1)*} Q_1^{(1)} |t - \bar{t}| \\
 &\quad + \sum_{i=1}^n \left[\int_0^t M_2^{(1)*} Q_2^{(2)} |x - \bar{x}|^\alpha ds \right. \\
 &\quad \quad \left. + \int_0^t H_2^{(1)*} [|x - \bar{x}| + |u(s, x) - u(s, \bar{x})| \right. \\
 &\quad \quad \quad \left. + \|v(s, x) - v(s, \bar{x})\|]^\alpha Q_2^{(1)} ds + M_2^{(1)*} Q_2^{(1)} |t - \bar{t}| \right] \\
 &\leq \Lambda_2^{(2)} |x - \bar{x}|^\alpha + H_2^{(1)*} [|x - \bar{x}| + Q_1^{(1)} |x - \bar{x}| + Q_2^{(1)} |x - \bar{x}|]^\alpha \tau + M_2^{(1)*} |t - \bar{t}| \\
 &\quad + M_2^{(1)*} Q_1^{(2)} |x - \bar{x}|^\alpha \tau + H_2^{(1)*} [|x - \bar{x}| + Q_1^{(1)} |x - \bar{x}| + Q_2^{(1)} |x - \bar{x}|]^\alpha Q_1^{(1)} \tau \\
 &\quad + M_2^{(1)*} Q_1^{(1)*} |t - \bar{t}| \\
 &\quad + n [M_2^{(1)*} Q_2^{(2)} |x - \bar{x}|^\alpha \tau + H_2^{(1)*} [|x - \bar{x}| + Q_1^{(1)} |x - \bar{x}| + Q_2^{(1)} |x - \bar{x}|]^\alpha Q_2^{(1)} \tau \\
 &\quad \quad + M_2^{(1)*} Q_2^{(1)*} |t - \bar{t}|] \\
 &\leq \left\{ \Lambda_2^{(2)} + H_2^{(1)*} (1 + Q_1^{(1)} + Q_2^{(1)})^\alpha \tau + M_2^{(1)*} \tau^{1-\alpha} \right. \\
 &\quad + M_2^{(1)*} Q_1^{(2)} \tau + H_2^{(1)*} (1 + Q_1^{(1)} + Q_2^{(1)})^\alpha Q_1^{(1)} \tau + M_2^{(1)*} Q_1^{(1)} \tau^{1-\alpha} \\
 &\quad \left. + n [M_2^{(1)*} Q_2^{(2)} \tau + H_2^{(1)*} (1 + Q_1^{(1)} + Q_2^{(1)})^\alpha Q_2^{(1)} \tau + M_2^{(1)*} Q_2^{(1)} \tau^{1-\alpha}] \right\} \\
 &\quad \times [|t - \bar{t}| + |x - \bar{x}|]^\alpha \\
 &= S_{2\tau}^{(2,x)} [|t - \bar{t}| + |x - \bar{x}|]^\alpha
 \end{aligned}$$

for a fixed $\Delta(X, \tau)$ and $(t, x), (\bar{t}, \bar{x}) \in \Delta(X, \tau)$. The inequalities (4.12)–(4.17) and Remark 2.4 imply the existence of $\tau \in (0, T]$ such that (4.8)–(4.11) are true.

Now we prove that for a sufficiently small $\tau \in (0, T]$ the operator \mathbf{H} is a contraction. Indeed, if $(u, v), (\bar{u}, \bar{v}) \in C_{b,\tau}^{1,\alpha}(Q)$ and $(t, x) \in [0, \tau] \times \mathbb{R}$, then using $\mathcal{H}_2[f, g]$, the mean value theorem for f and the theorem on the estimate of an increment of g we have

$$\begin{aligned}
 |F[u, v](t, x) - F[\bar{u}, \bar{v}](t, x)| &\leq \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} |f(s, y, u(s, y), v(s, y)) \\
 &\quad - f(s, y, \bar{u}(s, y), \bar{v}(s, y))| dy ds \\
 &\quad + \frac{c^2}{8} \int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-\frac{c}{2}(t-s)} |u(s, y) - \bar{u}(s, y)| dy ds \\
 &\leq \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} \left[|f_p(P_1)| |u(s, y) - \bar{u}(s, y)| \right. \\
 &\quad \left. + \sum_{i=1}^n |f_{r_i}(P_1)| |v_i(s, y) - \bar{v}_i(s, y)| \right] dy ds \\
 &\quad + \frac{c^2}{8} \int_0^t \int_{x-(t-s)}^{x+(t-s)} |u(s, y) - \bar{u}(s, y)| dy ds \\
 &\leq \frac{1}{2} \left[(1+n)M_1^{(1)*} + \frac{c^2}{4} \right] \tau^2 \|(u - \bar{u}, v - \bar{v})\|_0,
 \end{aligned} \tag{4.18}$$

$$\begin{aligned}
 &\|G[u, v](t, x) - G[\bar{u}, \bar{v}](t, x)\| \\
 &\leq \int_0^t \|g(s, x, u(s, x), v(s, x)) - g(s, x, \bar{u}(s, x), \bar{v}(s, x))\| ds \\
 &\leq \int_0^t \left[\|g_p(P_2)\| |u(s, x) - \bar{u}(s, x)| + \sum_{i=1}^n \|g_{r_i}(P_2)\| |v_i(s, x) - \bar{v}_i(s, x)| \right] ds \\
 &\leq (1+n)M_2^{(1)*} \tau \|(u - \bar{u}, v - \bar{v})\|_0,
 \end{aligned} \tag{4.19}$$

where $P_1, P_2 \in [0, \tau] \times \mathbb{R} \times [-Q_1, Q_1] \times [-Q_2, Q_2]^n$ are intermediate points. Inequalities (4.18), (4.19) imply

$$\|\mathbf{H}[u, v] - \mathbf{H}(\bar{u}, \bar{v})\|_0 \leq D_\tau \|(u - \bar{u}, v - \bar{v})\|_0, \tag{4.20}$$

where

$$D_\tau = \max \left\{ \frac{1}{2} \left[(1+n)M_1^{(1)*} + \frac{c^2}{4} \right] \tau^2, (1+n)M_2^{(1)*} \tau \right\}. \tag{4.21}$$

It is obvious that there is $\tau \in (0, T]$ such that

$$D_\tau < 1. \quad (4.22)$$

The use of the Banach fixed-point theorem and Lemma 3.1 finishes the proof. \square

Remark 4.2. It follows from the proof of Theorem 4.1 that the set $[0, \tau] \times \mathbb{R}$ of the existence and uniqueness of solutions to (1.1), (1.2) can be found by solving the inequalities $S_{i\tau} \leq Q_i$, $S_{i\tau}^{(j,t)} \leq Q_i^{(j)}$, $S_{i\tau}^{(j,x)} \leq Q_i^{(j)}$, $i, j = 1, 2$, (see Remark 2.4) and inequality (4.22).

Remark 4.3. If additionally, in Theorem 4.1, $u_0 \in C_b^2(\mathbb{R}, \mathbb{R})$ and $u_1 \in C_b^1(\mathbb{R}, \mathbb{R})$, then $u \in C_b^2([0, \tau] \times \mathbb{R}, \mathbb{R})$.

Remark 4.4. If the functions f, g generating the system (1.1) do not depend on t, x and f is of C^1 class and g is of C^2 class on \mathbb{R}^{1+n} , then all the assumptions of Theorem 4.1 are satisfied. It is true especially for the equations with any polynomial right-hand sides with respect to p, r . Add that Theorem 4.1 works if we consider the first equation in (1.1) only (the nonlinear wave equation for $c = 0$ and the nonlinear telegraph equation for $c > 0$). The examples of such equations are the forced dissipative sine-Gordon equation with a variable coefficient

$$u_{tt} - u_{xx} + cu_t + a(t, x) \sin u = h(t, x) \quad (4.23)$$

and the superlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t + du^m = h(t, x), \quad (4.24)$$

where $d \in \mathbb{R}$, $m \in \mathbb{N}$ and a, a_x, h, h_x are continuous and bounded on $[0, T] \times \mathbb{R}$ (see [12]).

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