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ON TRIANGULAR (D_n)-ACTIONS ON CYCLIC p-GONAL RIEMANN SURFACES

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Abstract. A compact Riemann surface X of genus $g > 1$ which has a conformal automorphism ρ of prime order p such that the orbit space $X/\langle\rho\rangle$ is the Riemann sphere is called cyclic p-gonal. Exceptional points in the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g are unique surface classes whose full group of conformal automorphisms acts with a triangular signature. We study symmetries of exceptional points in the cyclic p-gonal locus in $\mathcal{M}_g$ for which $\mathrm{Aut}(X)/\langle\rho\rangle$ is a dihedral group D_n.

Keywords: Riemann surface, symmetry, triangle group, Fuchsian group, NEC group.

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1. INTRODUCTION

A Riemann surface is a connected and compact surface equipped with an analytic structure. A conformal automorphism of a Riemann surface X of genus $g > 1$ is a homeomorphism $\rho : X \to X$ which is analytic in local coordinates. The group $\mathrm{Aut}(X)$ of all such automorphisms is called the full automorphism group. By Theorem of Hurwitz [10], the order of $\mathrm{Aut}(X)$ is bounded by $84(g-1)$, and this bound is attained for infinitely many values of g as was shown by Macbeath in [14].

A finite group G is said to act on a Riemann surface X if it is a subgroup of $\mathrm{Aut}(X)$. Then there exist a Fuchsian group Λ and an epimorphism $\theta : \Lambda \to G$ whose kernel Γ is a surface Fuchsian group. In this case X is the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of Γ. We say that θ is a *smooth epimorphism* and that G acts with the signature $\sigma(\Lambda)$ associated with Λ. Such a signature determines an algebraic and geometric structure of Λ. Two actions of finite groups G and G' on X are conformally equivalent if G and G' are conjugate in $\mathrm{Aut}(X)$.

A Riemann surface X of genus $g > 1$ which can be realized as a p-sheeted covering of the Riemann sphere for some prime p is called *p-gonal*. If there is an automorphism ρ of order p which permutes the sheets, then X is *cyclic p-gonal*. The Castelnuevo and Severi theorem [6] asserts that for $g > (p-1)^2$ the group generated by such an automorphism is unique in $\mathrm{Aut}(X)$ as was mentioned by Accola in [2]. The subgroup $\langle\rho\rangle$ is called a *p-gonality subgroup*.

Topological classification of conformal actions on a cyclic p-gonal Riemann surface X of genus $g > (p-1)^2$ is given in [20], where six types of actions are distinguished according to if the spherical group $\mathrm{Aut}(X)/\langle\rho\rangle$ is trivial, $Z_N, D_{N/2}, A_4, S_4$ or A_5 (N is the index of $\langle\rho\rangle$ in G). When $\mathrm{Aut}(X)$ acts with a triangular signature and $\mathrm{Aut}(X)/\langle\rho\rangle$ is a dihedral group D_n we say that the action of $\mathrm{Aut}(X)$ is *triangular* (D_n)*-action*, and in this case we denote the surface X by $X_{p,n,g}$. For example, the surface $X_{p,n,g}$ for $p = 2$ and $n = 2g + 2$ is the well known *Accola-Maclachlan surface* whose automorphism group has order $8(g + 1)$. This number is the least value of the size of the automorphism group of a Riemann surface of genus $g > 1$ on condition that the group is nontrivial. It was found independently by Colin Maclachlan in [15] and Robert Accola in [1]. The first author constructed a smooth epimorphism from a triangle Fuchsian group with periods $2, 4$ and $2g + 2$ onto a finite group G of order $8(g + 1)$. The orbit space of the hyperbolic plane under the action of the kernel of the epimorphism is a Riemann surface on which G acts as the automorphism group. In [19] David Singerman showed that this surface is a two-sheeted cover of the sphere branched over the vertices of a regular $(2g + 2)$-gon. Such a cover was constructed independently by Accola as an example of a Riemann surface whose automorphism group consists of $8g+8$ automorphisms arising from $4g+4$ symmetries of the polygon. Singerman proved that the Accola-Maclachlan surface is platonic, hyperelliptic and symmetric, and he calculated the number of all its symmetries. By a slight modification of Maclachlan's construction we can find a surface $X_{p,n,g}$ for $p > 2$, $g \equiv 0 \ (\frac{p-1}{2})$ and $n = 2 + 2g/(p-1)$ which is a p-sheeted cover of the sphere branched over the vertices of a regular n-gon. This allows us to enlarge the notion of an Accola-Maclachlan surface to a cyclic *p-gonal Accola-Maclachlan surface* for any prime $p \geq 2$. In this case n is the number of fixed points of a p-gonal automorphism. A cyclic p-gonal Accola-Maclachlan surface will be denoted by $AM_{p,g}$. We prove that the automorphism group of $X_{p,n,g}$ has order $8(g + 1)$ if and only if $X_{p,n,g}$ is $AM_{2,g}$ or $AM_{p,p-1}$ for $p > 2$.

A *symmetry* of a Riemann surface is an antiholomorphic involution; a surface is *symmetric* if it admits a symmetry. It is known that projective complex algebraic curves correspond to compact Riemann surfaces. Under this correspondence, the fact that a surface X is symmetric means that the corresponding curve is definable over R. In the group of conformal and anticonformal automorphisms of X, nonconjugate symmetries correspond bijectively to real curves which are nonisomorphic (over R), and whose complexifications are birationally equivalent to X. If X has genus g, and δ is a symmetry of X, then the set of fixed points $\mathrm{Fix}(\delta)$ of δ consists of k disjoint Jordan curves called *ovals*, where $0 \leq k \leq g + 1$, by a theorem of Harnack [9].

Exceptional points in the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g are unique surface classes whose full group of conformal automorphisms acts with a triangular signature. Their defining equations as algebraic curves have coefficients in a number field [3]. Determining the exceptional points in $\mathcal{M}_g$ is not a simple matter. Although there are just finitely many possible triangular signatures satisfying the

Riemann-Hurwitz relation in genus g, not all of them correspond to a group action. Furthermore, several distinct groups may act with the same signature, or one group may have topologically distinct actions with the same signature. So the problem can be attacked piecemeal by restricting attention to certain subloci in $\mathcal{M}_g$. The *q-hyperelliptic locus* $\mathcal{M}_g^q \subseteq \mathcal{M}_g$ consists of surfaces admitting a conformal involution (the *q-hyperelliptic involution*), with a quotient surface of genus q. When $q = 0$, these are the classical hyperelliptic surfaces. When $q = 1$, these are the *elliptic-hyperelliptic surfaces*. Exceptional points in hyperelliptic locus and elliptic-hyperelliptic locus were studied in [22] and [21], respectively. In order to determine the cyclic *p-gonal locus* in $\mathcal{M}_g$ we must consider six types of conformal actions on p-gonal Riemann surfaces. In the paper we deal only with (D_n)-actions, the remaining will be studied later. By a result of Singerman characterizing symmetric exceptional points [16], $X_{p,n,g}$ is symmetric. We give a presentation of its full group of conformal and anticonformal automorphisms. By a formula of Gromadzki [8], we check that a symmetry of $X_{p,n,g}$ with fixed points has 1 or p ovals. The case $p = 2$ corresponds to the hyperelliptic locus. Symmetry types of hyperelliptic Riemann surfaces and Accola-Maclachlan surfaces were studied in [5] and [4], respectively, however we do not omit this case for completeness of the paper.

We prove that for any prime $p > 2$ and even $q \geq 2$, there exists a symmetric exceptional point in the cyclic p-gonal locus of $\mathcal{M}_g$ with $g = (q-1)(p-1)$ admitting q symmetries, and every symmetry has p ovals.

2. PRELIMINARY

2.1. NEC GROUPS

Every compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of a discrete, torsion-free group Γ, called a *surface group of genus g*, consisting of orientation-preserving isometries of $\mathcal{H}$, and isomorphic to the fundamental group of X. Any group of conformal and anticonformal automorphisms of $X = \mathcal{H}/\Gamma$ can be represented as Λ/Γ, where Λ is a *non-euclidean crystallographic (NEC) group* containing Γ as a normal subgroup. An NEC group is a co-compact discrete subgroup of the full group $\mathcal{G}$ of isometries (including those which reverse orientation) of $\mathcal{H}$. Let $\mathcal{G}^+$ denote a subgroup of $\mathcal{G}$ consisting of orientation-preserving isometries. An NEC group is called a *Fuchsian group* if it is contained in $\mathcal{G}^+$, and a *proper NEC group* otherwise.

Wilkie [23] and Macbeath [12] associated to every NEC group Λ a *signature* which determines its algebraic and geometric structure. It has the form

$$\sigma(\Lambda) = (g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}), \qquad (2.1)$$

where the numbers $m_i \geq 2$ are called the *proper periods*, the brackets ( ) (which may be empty) are called the *period cycles*, the numbers $n_{ij} \geq 2$ are called the *link periods*, and $g \geq 0$ is the *orbit genus*. An NEC group with a signature of the form

$(g;\pm;[-];\{(-),\dots,(-)\})$ is called a *surface NEC group of genus g*. A Fuchsian group is an NEC group with a signature of the form

$$(g;+;[m_1,\dots,m_r];\{-\}). \tag{2.2}$$

In the particular case $g=0$ we shall write briefly $[m_1,\dots,m_r]$. A group with a signature $[m_1,m_2,m_3]$ is called a *triangle group*, and the signature is called *triangular*. If Λ is a proper NEC group with the signature (2.1), its *canonical Fuchsian subgroup* $\Lambda^+=\Lambda\cap\mathcal{G}^+$ has the signature

$$(\gamma;+;[m_1,m_1,\dots,m_r,m_r,n_{11},\dots n_{1s_1},\dots,n_{k1},\dots,n_{ks_k}];\{-\}), \tag{2.3}$$

where $\gamma=\alpha g+k-1$ and $\alpha=2$ if the sign is $+$ and $\alpha=1$ otherwise. In the paper we shall use only NEC groups with the signature (2.1) where the sign is $+$. In this case the group has a presentation given by generators:

(i)	$x_i,\quad i=1,\dots,r,$	(elliptic generators)
(ii)	$c_{ij},\quad i=1,\dots,k; j=0,\dots s_i,$	(reflection generators)
(iii)	$e_i,\quad i=1,\dots,k,$	(boundary generators)
(iv)	$a_i,b_i,\quad i=1,\dots g$ if the sign is $+$,	(hyperbolic generators)

and relations

(1) $x_i^{m_i}=1,\quad i=1,\dots,r,$
(2) $c_{is_i}=e_i^{-1}c_{i0}e_i,\quad i=1,\dots,k,$
(3) $c_{ij-1}^2=c_{ij}^2=(c_{ij-1}c_{ij})^{n_{ij}}=1,\quad i=1,\dots,k; j=1,\dots,s_i,$
(4) $x_1\dots x_re_1\dots e_ka_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}=1.$

Any system of generators of an NEC group satisfying the above relations will be called *a canonical system* of generators.

Every NEC group has a fundamental region, whose hyperbolic area is given by

$$\mu(\Lambda)=2\pi\Big(\alpha g+k-2+\sum_{i=1}^{r}(1-1/m_i)+1/2\sum_{i=1}^{k}\sum_{i=1}^{s_i}(1-1/n_{ij})\Big), \tag{2.4}$$

where α is defined as in (2.3). It is known that an abstract group with the presentation given by the generators (i)–(iv) and the relations (1)–(4) can be realized as an NEC group with the signature (2.1) if and only if the right-hand side of (2.4) is positive. If Γ is a subgroup of finite index in an NEC group Λ then it is an NEC group itself and the *Riemann-Hurwitz relation* is

$$[\Lambda:\Gamma]=\mu(\Gamma)/\mu(\Lambda). \tag{2.5}$$

The number of fixed points of an automorphism of a Riemann surface can be calculated by Macbeath's theorem [13].

Theorem 2.1. *Let $X = H/\Gamma$ be a Riemann surface with the automorphism group $G = \Lambda/\Gamma$ and let $x_1, \ldots, x_r$ be elliptic canonical generators of Λ with periods $m_1, \ldots, m_r$, respectively. Let $\theta : \Lambda \to G$ be the canonical epimorphism and for $1 \neq g \in G$ let $\varepsilon_i(g)$ be 1 or 0 according as g is or is not conjugate to a power of $\theta(x_i)$. Then the number $F(g)$ of points of X fixed by g is given by the formula*

$$F(g) = |N_G(\langle g \rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i, \tag{2.6}$$

where $N_G(\langle g \rangle)$ denotes the normalizer in G of the subgroup $\langle g \rangle$.

2.2. SYMMETRIES OF BELYI SURFACES

The famous theorem of Belyi states that a compact Riemann surface X can be defined over the number field if and only if X can be uniformized by a finite index subgroup Γ of a Fuchsian triangle group Λ. As a result nowadays such surfaces are called *Belyi surfaces*. The existence of a symmetry on X means that Λ is the canonical Fuchsian group of a proper NEC group $\widetilde{\Lambda}$, containing Λ with index 2, and containing Γ as a normal subgroup. By (2.3), there are two possibilities for the signature of $\widetilde{\Lambda}$: $(0; +; [-]; (k, l, m))$, and, if $k = l$, $(0; +; [k]; \{(m)\})$. We shall call X a symmetric surface of the *first type* or of the *second type* respectively.

Let $\mathcal{A} \cong \widetilde{\Lambda}/\Gamma$ be the full group of automorphisms (conformal and anticonformal) of X, let $\tilde{\theta}$ be the canonical epimorphism $\widetilde{\Lambda} \to \mathcal{A}$ with kernel Γ. A symmetry $\phi \in \mathcal{A}$ is the image under $\widetilde{\theta}$ of an element d from the subset $\widetilde{\Lambda} \setminus \Lambda$ of orientation-reversing elements of $\widetilde{\Lambda}$. If d cannot be chosen as a reflection then ϕ has no ovals. Otherwise, d is conjugate to one of the reflection generators in the canonical system of generators of $\widetilde{\Lambda}$. The number of ovals $\|\phi\|$ is the number of empty period cycles in the group $\widetilde{\Gamma} = \widetilde{\theta}^{-1}(\langle \phi \rangle)$. A formula for $\|\phi\|$ is given in [8] in terms of orders of centralizers:

$$\|\phi\| = \sum |C(\mathcal{A}, \widetilde{\theta}(c_i))|/|\widetilde{\theta}(C(\widetilde{\Lambda}, c_i))|, \tag{2.7}$$

where c_i runs over pairwise non-conjugate canonical reflection generators in $\widetilde{\Lambda}$ whose images are conjugate to ϕ, and $C(\mathcal{A}, a)$ denotes the centralizer of the element a in the group $\mathcal{A}$. In [17] (see also [16]) it is proved that the centralizer of a reflection c in an NEC group $\widetilde{\Lambda}$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$ if the associated period cycle in $\widetilde{\Lambda}$ is empty or consists of odd periods only; otherwise it is isomorphic to $\mathbb{Z}_2 \oplus (\mathbb{Z} * \mathbb{Z})$, where $*$ denotes the free product.

Using these results we obtain the following classification of the centralizers of reflections in an NEC group whose canonical Fuchsian group is a triangle group. The notation $c_1 \sim c_2$ denotes conjugacy in $\widetilde{\Lambda}$.

Lemma 2.2. (a) *Let* $\widetilde{\Lambda}$ *be an NEC group with signature* $(0;+;[-];(k',l',m'))$ *and let* c_0, c_1, c_2 *be the canonical system of generators of* $\widetilde{\Lambda}$. *Then*

(i) *for* $k'=2k, l'=2l, m'=2m$
$C(\widetilde{\Lambda}, c_0) = \langle c_0\rangle \oplus (\langle (c_0c_1)^k\rangle * \langle (c_0c_2)^m\rangle)$,
$C(\widetilde{\Lambda}, c_1) = \langle c_1\rangle \oplus (\langle (c_0c_1)^k\rangle * \langle (c_1c_2)^l\rangle)$,
$C(\widetilde{\Lambda}, c_2) = \langle c_2\rangle \oplus (\langle (c_0c_2)^m\rangle * \langle (c_1c_2)^l\rangle)$;

(ii) *for* $k'=2k, l'=2l+1, m'=2m, c_1 \sim c_2$ *and*
$C(\widetilde{\Lambda}, c_0) = \langle c_0\rangle \oplus (\langle (c_0c_1)^k\rangle * \langle (c_2c_0)^m\rangle)$,
$C(\widetilde{\Lambda}, c_1) = \langle c_1\rangle \oplus (\langle (c_1c_0)^k\rangle * \langle (c_2c_1)^l(c_2c_0)^m(c_1c_2)^l\rangle)$.

(b) *Let* $\widetilde{\Lambda}$ *be an NEC group with signature* $(0;+;[k];\{(m)\})$ *and let* x, e, c_0, c_1 *be a canonical system of generators of* $\widetilde{\Lambda}$. *Then* $c_0 \sim c_1$ *and*

$$C(\widetilde{\Lambda}, c_0) = \begin{cases} \langle c_0\rangle \oplus \langle (c_0c_1)^{m/2}\rangle * \langle e(c_0c_1)^{m/2}e^{-1}\rangle & \text{if } m \text{ is even,} \\ \langle c_0\rangle \oplus (\langle e(c_0c_1)^{(m-1)/2}\rangle) & \text{if } m \text{ is odd.} \end{cases}$$

3. TRIANGULAR (D_n)-ACTIONS ON CYCLIC p-GONAL RIEMANN SURFACES

Let G be a finite group acting on a Riemann surface of genus $g>1$ such that the canonical projection $X \to X/G$ is ramified at r points with multiplicities $m_1, \ldots, m_r$ and the genus of X/G is zero. Then the vector of numbers $(m_1, \ldots, m_r)$ is called the branching data of G on X. A sequence $(g_1, \ldots, g_r)$ of generators of G such that $g_i^{m_i} = 1$ for $i = 1, \ldots, r$, $g_1g_2\ldots g_r = 1$ and $2g-2 = |G|(r-2-\frac{1}{m_1}-\ldots-\frac{1}{m_r})$ is called a *generating* $(m_1, \ldots, m_r)$*-vector*.

A cyclic p-gonal Riemann surface of genus $g>(p-1)^2$ for which the action of the automorphism group is a triangular (D_n)-action is denoted by $X_{p,n,g}$. Finite group actions on such surfaces are determined by cases $(C.e)-(C.h)$ of Theorem 3.7 in [20] for $p>2$. For the convenience of the reader we repeat the arguments from the proof of this Theorem because we shall need them later in the paper, and we explain why the cases $(C.a)-(C.d)$ must be excluded if we restrict ourselves to triangular signatures.

Theorem 3.1. *Let* (p,n,g) *be a triple of integers such that* $g, n > 1$, *and* p *is a prime. Then there exists a surface* $X_{p,n,g}$ *if and only if* $g = [n(1+a_3)/2 + a_2 - 1](p-1)$ *for some* $a_2, a_3 \in \{0,1\}$, *and* n *satisfies the condition in the last column of Table 1. An action of the full automorphism group of* $X_{p,n,g}$ *is given by a presentation* $(a)-(f)$ *listed below and* $(2p, n\varepsilon_2, 2\varepsilon_3)$*-generating vector* $v_G^r = (R^r, SR^{1-r}, (RS)^{-1})$, *where* $r=1$ *in all cases but* (a) *and* (f), r *is an odd integer different from* p *in range* $1 \le r \le 2p-1$ *in two exceptional cases,* $r \ne 1$ *in* (f), $n \equiv 0\ (p)$ *in* (a) *for* $r \ne 1$, *and* $\varepsilon_i = 1$ *or* p *according to if* $a_i = 0$ *or* 1 *for* $i = 2, 3$.

Table 1.

Case	(a_2, a_3)	Presentation of $\mathrm{Aut}(X_{p,n,g})$	Conditions
(a)	$(0,1)$	$A^n_{1,p} = \langle R, S : R^{2p}, S^n, (RS)^2R^{-2}\rangle$	none
(b)	$(1,1)$	$B^n_{p,p} = \langle R, S : R^4, S^nR^{-2}, (RS)^2R^{-2}\rangle$	$p = 2$
(c)	$(0,0)$	$C^n_{1,1} = \langle R, S : R^{2p}, S^n, (RS)^2, SR^2S^{-1}R^{-2}\rangle$	$n \equiv 0\ (p)$
(d)	$(0,0)$	$D^n_{1,1} = \langle R, S : R^{2p}, S^n, (RS)^2, SR^2S^{-1}R^2\rangle$	$n \equiv 0\ (2)$,$p > 2$
(e)	$(1,0)$	$E^{n,\delta}_{p,1} = \langle R, S : R^{2p}, (RS)^2, S^nR^{-2\delta}\rangle$	$2\delta + n \equiv 0\ (p)$
			g.c.d.$(\delta, p) = 1$
(f)	$(1,1)$	$F^n_{p,p} = \langle R, S : R^{2p}, S^n, (RS)^2R^{-2}\rangle$	$p > 2, n \not\equiv 0\ (p)$

Proof. Assume that the action of a finite group G on a p-gonal Riemann surface X of genus $g > 1$ is a triangular (D_n)-action. Then there exist a Fuchsian group Λ and an epimorphism $\theta : \Lambda \to G$ with a surface kernel Γ of orbit genus g. In this case $X = \mathcal{H}/\Gamma$, and by Lemma 3.1 in [20], $\sigma(\Lambda) = [\varepsilon_1 2, \varepsilon_2 n, \varepsilon_3 2]$ for some $\varepsilon_i \in \{1, p\}$, where at least one of ε_i is equal to p. Since θ preserves the orders of canonical generators of Λ, it follows that G is generated by two elements $R = \theta(x_1)$ and $S = \theta(x_2)$ of orders $2\varepsilon_1$, and $n\varepsilon_2$ respectively whose product has order $2\varepsilon_3$. First suppose that $\varepsilon_1 = p$. Then the only p-gonality subgroup $H \leq G$ is generated by R^2. Since H is a normal subgroup, it follows that $SR^2S^{-1} = R^{2\alpha}$, $S^n = R^{2\delta}$ and $(RS)^2 = R^{2\gamma}$ for some integers $\alpha, \delta, \gamma \in \{0, \ldots, p-1\}$, where δ and γ are zero if and only if $\varepsilon_2 = 1$ and $\varepsilon_3 = 1$ respectively. By the equations $R^2 = (RS)^2R^2(RS)^{-2} = R^{2\alpha^2}$, $\alpha^2 \equiv 1\ (p)$ and so $\alpha = 1$ or $\alpha = -1$.

If $\varepsilon_3 = p$ then $\gamma \neq 0$ and $\alpha = 1$. Since $(RS)^2 = R^{2\gamma}$, it follows that $RSR^{-1} = S^{-1}R^{2(\gamma-1)}$. By raising the last equation to the n-th power we get $S^{2n} = R^{2n(\gamma-1)}$ and so $R^{2(2\delta+n(1-\gamma))} = 1$. Thus

$$2\delta + n(1-\gamma) \equiv 0\ (p). \tag{3.1}$$

For $p = 2$, $\gamma = 1$ and G has the presentation (a) or (b) according to if $\varepsilon_2 = 1$ or p. So assume that $p > 2$. If $\gamma = 1$ then $\delta = 0$ and G has the presentation (a). Otherwise, let $\varepsilon = (1-r)(p+1)/2$ for an integer r such that $r\gamma \equiv 1\ (p)$. Then by (3.1),

$$2(r\delta - \varepsilon n) \equiv 0\ (p). \tag{3.2}$$

Let $h = (RS)^2$, $R' = Rh^{\varepsilon}$ and $S' = Sh^{-\varepsilon}$. Then $h^r = (RS)^{2r} = R^{2r\gamma} = R^2$ and so $R'^2 = R^2h^{1-r} = h = (R'S')^2$. Furthermore, by (3.2), $S'^n = S^nh^{-\varepsilon n} = R^{2\delta}h^{-\varepsilon n} = h^{r\delta-\varepsilon n} = 1$. Thus, G has the presentation $\langle S', R' : R'^{2p} = 1, S'^n = 1, (R'S')^2 = R'^2\rangle$. By the last relation, $R'S'R'^{-1} = S'^{-1}$ and so $R'S'^kR'^{-1} = S'^{-k}$ for any integer k. Since $R = R'^r$ and $S = S'R'^{1-r}$ if r is odd, and $R = R'^{p+r}$ and $S = S'R'^{1-(p+r)}$ if r is even, we can assume that a generating vector $(\theta(x_1), \theta(x_2), \theta(x_3))$ of G has the form $(R'^r, S'R'^{1-r}, (R'S')^{-1})$ for some odd r different from p in the range $1 \leq r < 2p$. In the proof of Lemma 3.4 [20], it was justified that any two such generating vectors corresponding to different values of r are not equivalent. By (3.2), $\delta \equiv \gamma\varepsilon n\ (p)$. Thus, $\varepsilon_2 = 1$ if $n \equiv 0\ (p)$ and $\varepsilon_2 = p$ otherwise.

If $\varepsilon_3 = 1$, then $G = \langle R, S : R^{2p} = 1, S^n = R^{2\delta}, (RS)^2 = 1, SR^2S^{-1} = R^{2\alpha}\rangle$. Thus, for $\varepsilon_2 = 1$, G has the presentation (c) or (d) according to $\alpha = 1$ or -1, and for $\varepsilon_2 = p$ it has the presentation (e). Since $(RS)^2 = 1$, it follows that $RSR^{-1} = S^{-1}R^{-2}$. So if $\alpha = 1$, then for any integer k, $RS^k = S^{-k}R^{1-2k}$. In particular, for $k = n$, $1 = S^{2n}R^{2n} = R^{2(2\delta+n)}$, and so $2\delta + n \equiv 0\ (p)$. If $\alpha = -1$ then $RS^2R^{-1} = (S^{-1}R^{-2})^2 = S^{-2}$. Thus, $RS^{2k}R^{-1} = S^{-2k}$ and $RS^{2k+1}R^{-1} = S^{-(2k+1)}R^{-2}$ for any integer k. Similarly, $SR^{2k}S^{-1} = R^{-2k}$ and $SR^{2k+1}S^{-1} = R^{-(2k+1)}S^{-2}$. If n is odd, then for $k = (n-1)/2$, the relation $RS^{2k+1}R^{-1} = S^{-(2k+1)}R^{-2}$ gives $R^2 = 1$, a contradiction. Thus n must be even.

For $\varepsilon_1 = 1$, then there is only one triple of parameters $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ for which the action of G is not equivalent to any action discussed above. In this case $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, p, 1)$, and G is the dihedral group D_{np}.

In all cases there is an exact sequence of homomorphisms $1 \to H \to G \to D_n \to 1$ for the unique p-gonality subgroup $H \leq G$ and so G has order $2pn$. Moreover, G is the full automorphism group of X by Theorem 4.1 in [20]. The p-gonality subgroup H is isomorphic to Δ/Γ for some Fuchsian subgroup $\Delta < \Lambda$ containing Γ as a normal subgroup of index p. By (2.5), Δ has the signature $[p, \overset{t}{\ldots}, p]$ for $t = 2 + 2g/(p-1)$. On the other hand, t is the number of fixed points of R^2 which by (2.6) is equal to $2pn(a_1/2\varepsilon_1 + a_2/n\varepsilon_2 + a_3/2\varepsilon_3)$, where $a_i = 1$ or $a_i = 0$ according to if $\varepsilon_i = p$ or $\varepsilon_i = 1$. Thus, $g = [n(a_1 + a_3)/2 + a_2 - 1](p-1)$. Since $g = 0$ for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, p, 1)$, the signature $[2, np, 2]$ must be ruled out and so we can assume that $a_1 = 1$.

Conversely, if (p, n, g) is a triple of integers such that p is prime, $g = [n(1+a_3)/2 + a_2 - 1](p-1) > 1$ for some $a_2, a_3 \in \{0, 1\}$ and n satisfies the condition in the last column of Table 1, then there exist a triangular (D_n)-action determined by a Fuchsian group Λ with the signature $[2p, n\varepsilon_2, 2\varepsilon_3]$, where $\varepsilon_i = 1$ or p according to $a_i = 0$ or 1, the group G with the presentation given in Table 1 corresponding to (a_2, a_3), and the generating vector v_G^r. Indeed, v_G^r defines an epimorphism $\theta : \Lambda \to G$ such that $\Gamma = \ker\theta$ is a surface Fuchsian group of orbit genus g, and $H = \langle R^2\rangle$ is a p-gonality subgroup of G with the quotient $G/\langle R^2\rangle = D_n$. □

Corollary 3.2. *Let* $G = \langle R, S\rangle$ *be a group with a presentation* (a)–(f) *listed in Table* 1. *Then for any integer* k

$$RS^k = S^{-k}R \text{ in } (a), (b) \text{ and} (f), \tag{3.3}$$

$$RS^k = S^{-k}R^{1-2k} \text{ in } (c) \text{ and } (e), \tag{3.4}$$

$$\begin{aligned} &RS^{2k} = S^{-2k}R,\ RS^{2k+1} = S^{-(2k+1)}R^{-1}, \\ &SR^{2k} = R^{-2k}S,\ SR^{2k+1} = R^{-(2k+1)}S^{-1} \text{ in } (d). \end{aligned} \tag{3.5}$$

Corollary 3.3. *In cases* (c) *and* (d)*,* $X_{p,n,g}$ *is a* p*-sheeted cover of the sphere ramified over the vertices of a regular* n*-gon, and in* (e) $X_{p,n,g}$ *is the cover of the sphere branched over* n *vertices of the dihedron and two points in the poles of the sphere.*

Proof. We use the same notation as in the proof of Theorem 3.1. In cases (c), (d) and (e) the normal subgroup $\Delta \leq \Lambda$ corresponds to a regular map of type $\{\varepsilon_2 n, 2p\}$ on the sphere. The action of R on the H-cosets is a product of n two-cycles and the action of S is a product of 2 n-cycles. Thus, the map must be a dihedron since it has n vertices of valency 2 and 2 faces of valency n. In (c) and (d), $X_{p,n,g}$ is a p-sheeted cover of the sphere with n branched points being the vertices of the dihedron, and in (e) the cover is branched over two additional points in the poles of the sphere coming from period $\varepsilon_2 n$ in the signature of Λ. □

The surface $X_{p,n,g}$ whose automorphism group has presentation (c) or (d) is called a *p-gonal Accola-Maclachlan surface*, and is denoted by $AM_{p,g}$.

Corollary 3.4. *Let $G = \mathrm{Aut}(X_{p,n,g})$. Then for $p = 2$, $n = 1+g, g, 2g+2$ or $2g$ and $G = A_{1,2}^{1+g}$, $B_{2,2}^{g}, C_{1,1}^{2+2g}$ or $E_{2,1}^{2g,1}$ respectively. For $p > 2$, $g \equiv 0$ $(\frac{p-1}{2})$ and the pair (n, G) is one of those listed below:*

n	G	*Conditions*
$1+\frac{g}{p-1}$	$A_{1,p}^{n}$	*any*
$2+\frac{2g}{p-1}$	$C_{1,1}^{n}$	$n \equiv 0\ (p)$
$2+\frac{2g}{p-1}$	$D_{1,1}^{n}$	$n \equiv 0\ (2)$
$\frac{2g}{p-1}$	$E_{p,1}^{n,\delta}$	$2\delta + n \equiv 0\ (p)$
$\frac{g}{p-1}$	$F_{p,p}^{n}$	$n \not\equiv 0\ (p)$

Furthermore, the order of G has the minimum size $8(g+1)$ if and only if $X_{p,n,g}$ is $AM_{2,g}$ and $G = C_{1,1}^{2+2g}$, or $X_{p,n,g}$ is $AM_{p,p-1}$ for $p > 2$ and $G = D_{1,1}^{4}$ or $C_{1,1}^{4}$.

Proof. The first part is obvious and we justify only the second one. Since

$$g = [n(1+a_3)/2 + a_2 - 1](p-1)$$

for some $a_2, a_3 \in \{0,1\}$ and $|G| = 2pn$, it follows that the equation $8(g+1) = |G|$ is only satisfied for $(a_2, a_3, n) = (0, 0, 2g+2)$ if $p = 2$, and $(0, 0, 4)$ if $p > 2$. In the first case $X_{p,n,g}$ is the surface $AM_{2,g}$ with the automorphism group $G = C_{1,1}^{2g+2}$, in the other $X_{p,n,g}$ is a surface $AM_{p,p-1}$ with $G = D_{1,1}^{4}$ or $C_{1,1}^{4}$. □

In the next section we shall need the following lemma.

Lemma 3.5. *Assume that $G = \langle R, S\rangle$ is a group with a presentation $(a)-(f)$ in Table 1. Then there are 2 or 4 pairs of integers (α, β) in range $0 \le \alpha < n$ and $0 \le \beta < 2p$ such that $S^{2\alpha}R^{2\beta} = 1$ according to if n is odd or even. In the first case $(\alpha, \beta) = (0, ip)$, and in the other $(\alpha, \beta) = (0, ip)$ or $(\frac{n}{2}, ip - \delta)$, where $i = 0, 1$ and $\delta = 0$ in $(a), (c), (d), (f)$, $\delta = 1$ in (b), and δ is an integer co-prime with p such that $2\delta + n \equiv 0$ (p) in (e).*

4. TRIANGULAR SYMMETRIC (D_n)-ACTIONS

In this chapter we study the existence of symmetries of a Riemann surface whose full group of conformal automorphisms has a triangular (D_n)-action. G. Jones, D. Singerman and P. Watson in [11] gave the necessary and sufficient conditions on the existence of a symmetry of a quasiplatonic Riemann surface. Since, we deal only with full automorphism groups we shall use the following simpler version of their theorem which is close to the original in [16].

Theorem 4.1. *Let X be a quasiplatonic Riemann surface, uniformised by a normal subgroup Γ of finite index in a co-compact triangle group Λ, and let g_1, g_2 and g_3 be the images in $G = \Lambda/\Gamma$ of a canonical generating triple x_1, x_2, x_3 for Λ. Then X is symmetric if and only if either*

1. *G has an automorphism $\alpha : g_1 \mapsto g_1^{-1}$, $g_2 \mapsto g_2^{-1}$ or*
2. *G has an automorphism $\beta : g_1 \mapsto g_2^{-1}, g_2 \mapsto g_1^{-1}$ (possibly after a cyclic permutation of the canonical generators).*

We shall call the action of G a *triangular symmetric action of the first type* or of the *second type* according to if the case (1) or (2) holds. In the particular case when G acts on $X_{p,n,g}$ we shall write $X^I_{p,n,g}$ or $X^{II}_{p,n,g}$ respectively.

Theorem 4.2. *The topological type of symmetric action on $X^I_{p,n,g}$ is determined by a finite group $G = A^n_{1,p}$, $B^n_{p,p}$, $C^n_{1,1}$, $D^n_{1,1}$, $E^{n,\delta}_{p,1}$ or $F^n_{p,p}$, a Fuchsian group Λ with the signature $[2p, n\varepsilon_2, 2\varepsilon_3]$ and a generating vector v^r_G, where the presentation of G and v^r_G are given in Theorem 3.1, and $\varepsilon_2, \varepsilon_3$ are the lower indices in the symbol of the group G. The full group $\mathcal{A}$ of conformal and anticonformal automorphisms of $X_{p,n,g}$ is a semidirect product $G \rtimes \langle T : T^2\rangle$ with respect to the action $TRT = R^{-1}, TST = S^{-1}$, where T is a symmetry of X.*

Proof. Suppose that $G = \Lambda/\Gamma$ is the full automorphism group of a symmetric surface $X^I_{p,n,g} = \mathcal{H}/\Gamma$. Then Λ has a signature $\sigma(\Lambda) = [2p, \varepsilon_2 n, \varepsilon_3 2]$ for some $\varepsilon_2, \varepsilon_3 \in \{1, p\}$ and there exists an NEC group $\tilde{\Lambda}$ with the signature $(0; +; [-]; \{(2p, \varepsilon_2 n, \varepsilon_3 2)\})$ containing Λ as a subgroup of index 2 and Γ as a normal subgroup. Let c_0, c_1 and c_2 be a system of canonical generators of $\tilde{\Lambda}$. Then $x_1 = c_0c_1$, $x_2 = c_1c_2$ and $x_3 = c_2c_0$ can be chosen as a system of canonical generators of Λ. Let g_1, g_2 and T be the images of x_1, x_2 and c_1 in $\mathcal{A} = \tilde{\Lambda}/\Gamma$. Since $c_1x_1c_1 = x_1^{-1}$ and $c_1x_2c_1 = x_2^{-1}$, it follows that $\mathcal{A}$ is a semidirect product $G \rtimes \langle T : T^2\rangle$ with respect to the action $Tg_1T = g_1^{-1}$ and $Tg_2T = g_2^{-1}$, where T is a symmetry of X. By Theorem 3.1, G is generated by two elements R and S which satisfy the relations given in Table 1. It is easy to check that in each case $(a), \ldots, (f)$, the assignment $R \mapsto R^{-1}, S \mapsto S^{-1}$ induces an isomorphism of G, and so the action of G is a symmetric triangular action of the first type. □

Lemma 4.3. *Let $v^r_G = (R^r, SR^{1-r}, (RS)^{-1})$ be a generating vector of the group G with a presentation $(a), \ldots, (f)$ listed in Table 1, and let $\mathcal{A}$ be the semidirect product $G \rtimes \langle T : T^2\rangle$ with respect to the action $TRT = R^{-1}, TST = S^{-1}$. For $a \in \mathcal{A}$, let t_a denote the order of the centralizer of a in $\mathcal{A}$. Then $t_{R^rT} = 4n$, $t_T = 8$ or 4 according to n even or odd. For $p > 2$, in all cases but (d) $t_{TSR^{1-r}} = 8$ or 4 according to n even or odd, and in the exceptional case $t_{TSR^{1-r}} = 8p$. For $p = 2$, $t_{TSR^{1-r}} = 8$ in (e), 16 in (c), and 8 or 4 in $(a), (b)$ according to n even or odd.*

Proof. Any element $g \in \mathcal{A}$ can be written in the form $g = S^kR^lT^m$ for a unique triple of integers (k, l, m) in the range $0 \le k \le n-1$, $0 \le l \le 2p-1$ and $m = 0, 1$. If $m = 0$, then by the relations listed in Corollary 3.2, the equation $gR^rTg^{-1} = R^rT$ can be transformed into $R^{2l} = 1$ in cases (a), (b), (f), $R^{2(l-k)} = 1$ in $(c), (e)$, and $R^{2l} = 1$ or $R^{-2(l+1)}$ in (d) according to k even or odd. If $m = 1$ then we get the equation $R^{2(l-r)} = 1$ in $(a), (b), (f)$, $R^{2(l-k-1)} = 1$ in $(c), (e)$, and $R^{2(l-1)} = 1$ or $R^{-2l} = 1$ in (d) according to k even or odd. Thus t_{R^rT} is the number of triples:

$$
\begin{array}{ll}
(a), (b), (f): & (k, ip, 0), (k, r + ip, 1), \\
(c), (e): & (k, k + ip, 0), (k, k + 1 + ip, 1), \\
(d): & (2t, ip, 0), (2t + 1, ip - 1, 0), (2t, ip + 1, 1), (2t + 1, ip, 1),
\end{array}
$$

for $i = 0, 1$, $k = 0, \dots, n - 1$, $t = 0, \dots, \frac{n}{2} - 1$, and so $t_{R^rT} = 4n$.

The equation $gTg^{-1}T = 1$ can be transformed into $R^{(-1)^k 2l}S^{2k} = 1$ in (d) and $S^{2k}R^{2l} = 1$ in other cases. Thus, by Lemma 3.5, $t_T = 4$ or 8 according to n odd or even.

Finally, by the equation $g(TSR^{1-r})g^{-1}(TSR^{1-r})^{-1} = 1$ we get

$$S^kR^lS^{-1}R^lS^{k+1} = 1 \quad \text{or} \quad S^kR^lSR^lS^{k+1}R^{2(1-r)} = 1$$

according to is $m = 0$ or 1. If l is even, then by Corollary 3.2, in all cases but (d) we get $R^{2l}S^{2k} = 1$ or $R^{2(l+1-r)}S^{2(k+1)} = 1$, and $S^{2k} = 1$ or $S^{2(k+1)} = 1$ in the exceptional case. For l odd, we get $R^{2l}S^{2(k+1)} = 1$ or $R^{2(l+1-r)}S^{2k} = 1$ in (a), (b), (f), $R^{2(l+1)}S^{2(k+1)} = 1$ or $R^{2(l-1)}S^{2k} = 1$ in (c), (e), and $S^{2(k+1)} = 1$ or $S^{2k} = 1$ in (d). By Lemma 3.5, it is easy to determine all possible triples (k, l, m) for which the above equations are satisfied, and the number of such triples is equal to the order of TSR^{1-r}. □

Theorem 4.4. *There are two or three conjugacy classes of symmetries with fixed points of a surface $X^I_{p,n,g}$ according to n odd or even. Let us denote the number of their ovals by (k_1, k_2) (if n is odd) and (l_1, l_2, l_3) if n is even. Then for $p > 2$, in all cases but (d) $(k_1, k_2) = (1, 1)$ and $(l_1, l_2, l_3) = (1, 1, 1)$, in the exceptional case $(l_1, l_2, l_3) = (1, 1, 1)$ or $(1, 1, p)$ according to $n \equiv 2\ (4)$ or $n \equiv 0\ (4)$. For $p = 2$ and odd n, $(k_1, k_2) = (g + 1, 1)$ in (a) and $(g, 2)$ in (b). For even n: $(l_1, l_2, l_3) = (g + 1, 1, 1)$ in (a), $(g, 2, 2)$ in (b), $(g, 2, 1)$ in (e), and $(g + 1, 1, 1)$ or $(g + 1, 1, 2)$ in (c) according to $n \equiv 2\ (4)$ or $n \equiv 0\ (4)$.*

Proof. Assume that $G = \Lambda/\Gamma$ is one of the groups listed in Theorem 4.2 which acts on $X^I_{p,n,g} = \mathcal{H}/\Gamma$ with the signature $\sigma(\Lambda) = [2p, \varepsilon_2 n, \varepsilon_3 2]$ for $\varepsilon_2, \varepsilon_3 \in \{1, p\}$. Then the group $\mathcal{A}$ of conformal and anticonformal automorphisms of $X^I_{p,n,g}$ is a semidirect product $G \rtimes Z_2 = \langle R, S \rangle \rtimes \langle T \rangle$ with respect to the action $TRT = R^{-1}$ and $TST = S^{-1}$, where T is a symmetry of $X_{p,n,g}$. Let $\tilde{\Lambda}$ be a NEC group with the signature $(0; +; [-]; \{(2p, \varepsilon_2 n, \varepsilon_3 2)\})$ containing Λ and Γ as normal subgroups. Then Γ is the kernel of an epimorphism $\tilde{\theta} : \tilde{\Lambda} \to \mathcal{A}$ defined by $\tilde{\theta}(c_0) = R^rT$, $\tilde{\theta}(c_1) = T$ and $\tilde{\theta}(c_2) = TSR^{1-r}$, where $r = 1$ in all cases but (a) and (f), r is an odd integer different from p in range $1 \le r < 2p$ in two exceptional cases, $r \ne 1$ in (f), and $n \equiv 0\ (p)$ for $r \ne 1$ in (a). A symmetry of X is a $\tilde{\theta}$-image of order 2 of a reversing orientation element $\tilde{\lambda} \in \tilde{\Lambda}$, and it has fixed points if and only if $\tilde{\lambda}$ is conjugate to a reflection generator of $\tilde{\Lambda}$. For even n, the elements R^rT, T and TSR^{1-r} are not pairwise conjugate, and for odd n, T is conjugate to TSR^{1-r} via $S^{\frac{n-1}{2}}R^{\frac{r-1}{2}}$ in $(a),(c),(f)$, and via $S^{\frac{n-1}{2}}R^{-\delta}$ in $(b),(e)$. Thus, there are three or two conjugacy classes of symmetries with fixed points according to n even or odd. The number of ovals of a symmetry can be calculated by (2.7). The orders of centralizers of R^rT, T and TSR^{r-1} are given in Lemma 4.3, so it suffices to calculate the orders t_i of $\tilde{\theta}(C(\tilde{\Lambda}, c_i))$ for reflection generators c_i of $\tilde{\Lambda}$.

First assume that all periods in the signature of $\tilde{\Lambda}$ are even. This requires even n, except case (b) where n can be arbitrary. By Lemma 2.2, $t_i = 4\sharp(g_i)$, where $\sharp(g_i)$ denotes the order of g_i for

$$g_0 = R^p(RS)^{\varepsilon_3}, g_1 = R^p(SR^{1-r})^{\frac{n\varepsilon_2}{2}} \text{ and } g_2 = (RS)^{\varepsilon_3}(SR^{1-r})^{\frac{n\varepsilon_2}{2}}.$$

Thus, for $p = 2$ we have

Case	t_0	t_1	t_2
(a)	4	8	8
(b)	4	4	4
(c)	8	8	$\begin{cases} 8 & \text{if } n \equiv 0\ (4) \\ 16 & \text{if } n \equiv 2\ (4) \end{cases}$
(e)	8	4	8

Next assume that $p > 2$. Then $g_0 = R^{p+1}S$ for $\varepsilon_3 = 1$. Since R^2 is central in (c) and (e), it follows that $g_0^n = R^n$ and $R^{n+2\delta}$ respectively, where $n \equiv 0\ (2p)$ in the first case and $n + 2\delta \equiv 0\ (2p)$ in the other. In case (d), by relations (3.5) we have

$$g_0^n = ((R^{p+1}S)^2)^{\frac{n}{2}} = S^n = 1.$$

Since $\varepsilon_3 = p$ and R^2 is central in cases (a) and (f), it follows that

$$g_0 = R^{p+1}S(RS)^{p-1} = R^{p+1}S(R^2)^{\frac{p-1}{2}} = S.$$

So in all cases g_0 has order n. Similarly, by using the relations listed in Corollary 3.2 and assumptions on n given in the last column of Table 1, it is easy to check that g_1 has order 2.

If $\varepsilon_3 = p$, then

$$g_2 = (RS)^p(SR^{r-1})^{\frac{\varepsilon_2 n}{2}} = (R^2)^{\frac{p-1}{2}}RS^{1+\frac{\varepsilon_2 n}{2}}R^{(r-1)\frac{\varepsilon_2 n}{2}} = R^pS^{1+\frac{\varepsilon_2 n}{2}}R^{(r-1)\frac{\varepsilon_2 n}{2}}.$$

Thus,

$$g_2^2 = RS^{1+\frac{\varepsilon_2 n}{2}}R^{-1}S^{1+\frac{\varepsilon_2 n}{2}}R^{(r-1)\varepsilon_2 n} = 1$$

by assumptions on r. If $\varepsilon_3 = 1$, then $g_2 = RS^{1+\frac{n\varepsilon_2}{2}}$. Thus, by (3.4), $g_2^2 = R^{-\varepsilon_2 n} = 1$ in cases (c) and (e). In (d), $g_2^2 = 1$ or R^{-2} according to $n \equiv 0\ (4)$ or $n \equiv 2\ (4)$ and so g_2 has order 2 or p respectively.

Summing up, in all cases but (d) with $n \equiv 2\ (4)$, $t_0 = 4n$, $t_1 = 8$ and $t_2 = 8$. In the exceptional case t_0 and t_1 are as above and $t_2 = 8p$.

Next assume that $\varepsilon_2 n$ is odd. For $p = 2$ it is possible only in case (a), and for $p > 2$ in all cases but (d) on condition that n is odd. By item (ii) of Lemma 2.2, there are two conjugacy classes of reflections in $\tilde{\Lambda}$ represented by c_0 and c_1 for which $t_0 = 4\sharp(g_0)$ and $t_1 = 4\sharp(g_1)$, where

$$g_0 = R^p(RS)^{-\varepsilon_3} \quad \text{and} \quad g_1 = R^p(SR^{1-r})^{\frac{1-n\varepsilon_2}{2}}(RS)^{\varepsilon_3}(SR^{1-r})^{\frac{n\varepsilon_2-1}{2}}.$$

For $p = 2$, $t_0 = t_1 = 4$. If $p > 2$ then in (a) and (f):

$$t_0 = 4\sharp(R^p[(RS)^2]^{\frac{1-p}{2}}(RS)^{-1}) = 4\sharp(RS^{-1}R^{-1}) = 4n$$

and

$$t_1 = 4\sharp(R^pS^{\frac{1-n}{2}}[(RS)^2]^{\frac{p-1}{2}}RS^{1+\frac{n-1}{2}}) = 4\sharp(R^pS^{\frac{1-n}{2}}R^pS^{\frac{n+1}{2}}) = 4.$$

In (c) and (e), $n \equiv 0$ (p) and $n + 2\delta \equiv 0$ (p) respectively. Thus, $t_0 = 4\sharp(R^{p+1}S) = 4n$ and

$$g_1 = 4\sharp(R^pS^{\frac{1-n\varepsilon_2}{2}}RS^{\frac{\varepsilon_2 n+1}{2}}) = 4\sharp(R^{p-\varepsilon_2 n}) = 4. \qquad \square$$

Theorem 4.5. *A surface $X^I_{p,n,g}$ has k symmetries with ovals, where $k = p + pn$ in all cases but (d) with $p > 2$ and (c) with $p = 2$, and $k = p\frac{n}{2} + p + \frac{n}{2}$ in two exceptional cases. Furthermore, $X^I_{p,n,g}$ has l symmetries without fixed points, where for $p > 2$ in all cases but (d), $l = p$ or 0 according to if n is even or odd, and in the exceptional case $l = 1$ for $n \equiv 2$ (4) and $l = p$ for $n \equiv 0$ (4). If $p = 2$, then $l = 2$ or 0 in $(a), (e)$ according to g odd or even, $l = 2$ in (b) for any g, $l = 1 + g$ or $3 + g$ in (c) according to g even or odd.*

Proof. The group $\mathcal{A}$ of conformal and anticonformal automorphisms of $X^I_{p,n,g}$ is a semidirect product $G \rtimes \langle T\rangle$, where T is a symmetry of X which acts on generators R, S of G by $TRT = R^{-1}$ and $TST = S^{-1}$. Any element $g \in \mathcal{A}$ can be identified with the unique triple of integers (k, l, m) in the range $0 \le k \le n-1$, $0 \le l \le 2p-1$ and $m = 0, 1$ for which $g = S^kR^lT^m$. In particular, g is a symmetry if $m = 1$ and $S^kR^lS^{-k}R^{-l} = 1$. In case (d), by relation (3.5) we get $R^{l((-1)^k-1)} = 1$ or $R^{l((-1)^k-1)}S^{(-1)^{l+1}2k} = 1$ according to if l is even or odd. Thus there are the following symmetries: $S^{2r}R^{2t}T$, $S^{2r+1}T$ and $R^{2t+1}T$ for $r = 0, \ldots, \frac{n}{2} - 1$, $t = 0, \ldots, p - 1$. In addition, there is p symmetries: $S^{\frac{n}{2}}R^{2t+1}T$ if $n \equiv 0$ (4) or one $S^{\frac{n}{2}}R^pT$ if $n \equiv 2$ (4).

In all cases but (d), there is pn symmetries $S^kR^{2t}T$ for $t = 0, \ldots, p - 1$ and $k = 0, \ldots, n-1$. By Corollary 3.2, the second power of $g = S^kR^lT$ with l odd is equal to S^{2k} in (a), (f) and (b), and to $S^{2k}R^{2k}$ in (c), (e). Thus in (a), and (f) there is p symmetries of the form $R^{2t+1}T$ if n is odd or $2p$ symmetries $S^{i\frac{n}{2}}R^{2t+1}T$ if n is even, and in (b) there are 4 symmetries $S^{in}R^{2t+1}T$ for $i, t = 0, 1$. For $p > 2$, in cases (c) and (e) we get the same symmetries as in case (a), since $n \equiv 0$ (p) and $2\delta + n \equiv 0$ (p) respectively. If $p = 2$ in (c) or (e), then $S^{2k}R^{2k} = 1$ for $k = \frac{n}{2}$ on condition that $n \equiv 0$ (4) or $n \equiv 2$ (4) respectively, and hence there are 4 or 2 reflections according to whether this congruence is satisfied or not.

By Theorem 4.4, there are three or two conjugacy classes of symmetries with fixed points in $\mathcal{A}$ according to n odd or even which are represented by R^rT, T, TSR^{1-r} or by R^rT, T respectively. The number of elements in a class is the quotient of $|\mathcal{A}| = 4pn$ by the order of the centralizer of a representative of the class given in Lemma 4.3. Let k be the sum of elements in all classes. Then the number of symmetries without fixed points is the difference between the number of all symmetries and k. $\square$

Theorem 4.6. *The topological type of symmetric action on $X^{II}_{p,n,g}$ is determined by a finite group $G = \langle R, S\rangle$, a Fuchsian group Λ with the signature $[2p, \varepsilon_2 n, \varepsilon_3 2]$ where two periods are equal and a generating vector $v^r_G = (R^r, SR^{1-r}, (RS)^{-1})$. The full group $\mathcal{A}$ of conformal and anticonformal automorphisms of $X^{II}_{p,n,g}$ is a semidirect product $G \rtimes \langle T\rangle$, where T is a symmetry of $X^{II}_{p,n,g}$. The group G, the parameter r and the action of T on G are listed in Table 2.*

Table 2.

G	r	Action of T	Conditions
$B^n_{2,2}$, $A^n_{1,p}$	1	$TRT = RS$, $TST = S^{-1}$	$\varepsilon_3 = p$
$A^n_{1,p}$, $F^n_{p,p}$	-1	$TRT = (RS)^{-1}$, $TST = S^{-1}$	$\varepsilon_3 = p$
$D^{2p}_{1,1}$, $B^2_{2,2}$	1	$TRT = S^{-1}$, $TST = R^{-1}$	$\varepsilon_2 n = 2p$
$F^2_{p,p}$	$\frac{p+1}{2}$	$TRT = SR^{-1-p}$, $TST = R^{-p}$	$\varepsilon_2 n = 2p$, $p \equiv 1\ (4)$
$F^2_{p,p}$	$\frac{p-1}{2}$	$TRT = SR^{-1-p}$, $TST = R^{-p}$	$\varepsilon_2 n = 2p$, $p \equiv 3\ (4)$

Proof. Assume that $G = \Lambda/\Gamma$ is the group of conformal automorphisms of a symmetric Riemann surface $X = \mathcal{H}/\Gamma$ of the second type, where $\sigma(\Lambda) = [2p, \varepsilon_2 n, \varepsilon_3 2]$ with two equal periods. Then there exists a NEC group $\tilde{\Lambda}$ with the signature

$$\sigma_1 = (0; +; [2p]; \{(\varepsilon_2 n)\}) \quad \text{or} \quad \sigma_2 = (0; +; [2p]; \{(\varepsilon_3 2)\}) \tag{4.1}$$

according to whether $\varepsilon_3 = p$ or $2p = \varepsilon_2 n$, which contains Λ as a subgroup with index 2 and Γ as a normal subgroup. Let x, e, c_0 and c_1 be a system of canonical generators of $\tilde{\Lambda}$. If $\sigma(\tilde{\Lambda}) = \sigma_1$ then $x_1 = x$, $x_2 = c_0 c_1$ and $x_3 = c_1 x^{-1} c_1$ can be chosen as a system of canonical generators of Λ. Let g_1, g_2, g_3 and T be images of x_1, x_2, x_3 and c_1 in the group $\mathcal{A} = \tilde{\Lambda}/\Gamma$. Then (g_1, g_2, g_3) is a generating vector of G, and $\mathcal{A}$ is a semidirect product $G \rtimes \langle T\rangle$, where T is a symmetry of X which acts on generators of G by $Tg_1T = g_3^{-1}$ and $Tg_3T = g_1^{-1}$.

By Theorem 3.1, a finite group acting with the signature $[2p, n\varepsilon_2, 2p]$ is one of the groups $A^n_{1,p}, B^n_{2,2}$, $F^n_{p,p}$, and its generating vector has the form $v^r_G = (R^r, SR^{1-r}, (RS)^{-1})$ for some odd integer r different from p. The assignment $g_1 \to g_3^{-1}$ and $g_3 \to g_1^{-1}$ induces an automorphism of such a group if and only if $r = -1$ for $G = F^n_{p,p}$, $r = 1$ for $G = B^n_{2,2}$, and $r = 1$ or -1 for $G = A^n_{1,p}$. Then $TST = S^{-1}$ and $TRT = RS$ or $(RS)^{-1}$ according to whether $r = 1$ or $r = -1$ respectively.

If $\sigma(\tilde{\Lambda}) = \sigma_2$ then we can chose $x_1 = x, x_2 = c_0 x^{-1} c_0$ and $x_3 = c_0 c_1$ as a system of canonical generators of Λ. Let g_1, g_2, g_3 and T be images of x_1, x_2, x_3 and c_0 in $\mathcal{A}$. Then (g_1, g_2, g_3) is a generating vector of G and $\mathcal{A}$ is a semidirect product $\mathcal{A} = G \rtimes \langle T\rangle$, where T is a symmetry of X which acts on generators of G by $Tg_1T = g_2^{-1}$ and $Tg_2T = g_1^{-1}$. The assignment $g_1 \mapsto g_2^{-1}$ and $g_2 \mapsto g_1^{-1}$ induces an automorphism of a group G in Table 1 acting with the signature $[2p, 2p, \varepsilon_3 2]$ if and only if $G = B^2_{2,2}$, $D^{2p}_{1,1}$, $F^2_{p,p}$, and $r = 1$ for two first groups, and $r = \frac{p+1}{2}$ or $r = \frac{1-p}{2}$ for the last, according to if $p \equiv 1$ (4) or $p \equiv 3$ (4) respectively. Then $TRT = S^{-1}$ and $TST = R^{-1}$ for two first groups, and $TRT = SR^{-p-1}$ and $TST = R^{-p}$ for the last. $\square$

Theorem 4.7. *Let $\mathcal{A} = G \rtimes \langle T : T^2 \rangle$ be the group of conformal and anticonformal automorphisms of $X^{II}_{p,n,g}$. If G acts with the signature $[2p, 2p, \varepsilon_3 2]$ then $X^{II}_{p,n,g}$ has $2p$ symmetries and all of them have 1 oval. If G acts with the signature $[2p, \varepsilon_2 n, 2p]$ then for any G but $A^n_{1,p}$ with $r = 1$, $X^{II}_{p,n,g}$ has pn symmetries with 1 oval, and if n is odd then it has p additional symmetries without fixed points. In the exceptional case $X^{II}_{p,n,g}$ has n symmetries with 1 or p ovals according to n odd or even, and it has n symmetries without fixed points for $p = 2$ or 1 such a symmetry if $p > 2$ and n is odd.*

Proof. If G acts with the signature $[2p, \varepsilon_3 n, 2p]$, then there exists a NEC group $\tilde{\Lambda}$ with the signature σ_1, see (4.1), containing Γ and Λ as normal subgroups. The group $\mathcal{A}$ of conformal and anticonformal automorphisms of $X^{II}_{p,n,g}$ is the semidirect product $\mathcal{A} = G \rtimes \langle T \rangle$, where T is a symmetry of $X^{II}_{p,n,g}$ which acts on generators of G by $TST = S^{-1}$ and $TRT = RT$ or $TRT = (RS)^{-1}$ according to whether $r = 1$ or -1. It is easy to check that Γ is the kernel of an epimorphism $\tilde{\theta} : \tilde{\Lambda} \to \mathcal{A}$ defined by $\tilde{\theta}(x) = R^r, \tilde{\theta}(e) = R^{-r}, \tilde{\theta}(c_0) = SR^{1-r}T$ and $\tilde{\theta}(c_1) = T$. Any symmetry with fixed points is conjugate to T and we can calculate the number of its ovals by formula (2.7). For, we must find the order of the centralizer $C(\mathcal{A}, T)$. If $r = 1$ then an element $g = S^k R^l T^j \in \mathcal{A}$ commutes with T iff $S^k R^l (RS)^{-l} S^k = 1$. Thus $S^{2k} = 1$ or $S^{2k+1} = 1$ according to n even or odd. For $G = A^n_{1,p}$, $(k, l) = (i\frac{n}{2}, 2t)$ in the first case and $(k, l) = (0, 2t)$ or $(\frac{n-1}{2}, 2t + 1)$ in the other, where $t = 1, \ldots, p$ and $i = 0, 1$ and so $C(\mathcal{A}, T)$ has order $4p$. For $B^n_{2,2}$ $(k, l) = (0, 2t)$ and so $t_T = 4$.

If $r = -1$, then $S^k R^l T^j \in C(\mathcal{A}, T)$ on condition that $S^k R^l (RS)^l S^k = 1$. The last equation can be transformed to $R^{2l} S^{2k} = 1$ or $R^{2l} S^{2k+1}$ according to l even or odd. Thus $(k, l) = (i\frac{n}{2}, 0)$ for even n, and $(k, l) = (0, 0)$ or $(\frac{n-1}{2}, p)$ for odd n. In both cases the order of $C(\mathcal{A}, T)$ is equal to 4.

Let t_0 be the order of $\tilde{\theta}(C(\tilde{\Lambda}, c_0))$. Then by Lemma 2.2,

$$t_0 = 4\sharp((SR^{1-r})^{\frac{\varepsilon_2 n}{2}} R^{-r} (SR^{1-r})^{\frac{\varepsilon_2 n}{2}} R^r) \quad \text{or} \quad t_0 = 2\sharp(R^{-r} (SR^{1-r})^{\frac{\varepsilon_2 n - 1}{2}})$$

according to $\varepsilon_2 n$ even or odd. For $r = 1$ we get $t_0 = 4$ or $4p$ respectively, and for $r = -1$ $t_0 = 4$ for any n.

Thus, by (2.7), a symmetry with fixed points has 1 oval except for the case $G = A^n_{1,p}$ with even n and $r = 1$ at which it has p ovals. The number of all symmetries with fixed points is the quotient $|\mathcal{A}|/t_T$, and we get n such symmetries for $G = A^n_{1,p}$ with $r = 1$, and pn in the remaining cases.

The total number of symmetries is the number of elements $S^k R^l T \in \mathcal{A}$ of order 2. Thus for $r = 1$, k and l satisfy the equation $R^{2l} = 1$ or $S^{2k+1} R^{2l} = 1$ according to l even or odd. In particular, for $p = 2$, there is $2n$ symmetries of the form $S^k R^{2i} T$ for $k = 0, \ldots, n-1$, $i = 0, 1$, and for $G = B^n_{2,2}$ with odd n, there is 1 additional symmetry $S^{\frac{n-1}{2}} R^{-1} T$. For $p > 2$, we have n symmetries of the form $S^k T$ for $k = 1, \ldots, n$, and one more $S^{\frac{n-1}{2}} R^p T$ if n is odd.

For $r = -1$, we get the equation $S^k R^l S^{-k} (RS)^{-l} = 1$. Thus symmetries have the form $S^k R^{2t} T$ and $S^{\frac{\varepsilon_2 n - 1}{2}} R^{2t+1} T$ if n is odd. Summing up, there are np symmetries if n is even, and all of them have fixed points. If n is odd then there are $np + p$ symmetries and p of them have no fixed points.

If $\sigma(\Lambda) = [2p, 2p, 2\varepsilon_3]$, then there exists a NEC group $\tilde{\Lambda}$ with the signature σ_2 containing Λ and Γ as normal subgroups. The group $\mathcal{A}$ of conformal and anticonformal automorphisms of $X^{II}_{p,n,g}$ is one of the groups $D^{2p}_{1,1}$, $B^2_{2,2}$ or $F^2_{p,p}$ and $r = 1$ for the two first groups, $r = (p+1)/2$ or $(p-1)/2$ for the last group. Now Γ is the kernel of an epimorphism $\tilde{\theta} : \tilde{\Lambda} \to \mathcal{A}$ defined by $\tilde{\theta}(x) = R^r, \tilde{\theta}(e) = R^{-r}, \tilde{\theta}(c_0) = T$ and $\tilde{\theta}(c_1) = T(RS)^{-1}$. By Lemma 2.2, the order of $\tilde{\theta}(C(\tilde{\Lambda}, c_0))$ is equal to

$$t_0 = 4\sharp((RS)^{-\varepsilon_3} R^{-1} (RS)^{-\varepsilon_3} R).$$

Thus, $t_0 = 4p$ for $\varepsilon_3 = 1$ and $t_0 = 4$ for $\varepsilon_3 = p$. In the first case $t_T = 4p$, and $t_T = 4$ in the other. So there is $2p$ symmetries in the conjugacy class of T and by (2.7), any of them has one oval. The set of all symmetries consists of $2p$ elements: $S^k R^k T$ for $G = D^{2p}_{1,1}$, R^2T, T, RST, SR^3T for $G = B^2_{2,2}$, and $R^{2t}T, SR^{2t+1}$ for $G = F^2_{p,p}$, where $t = 0, \ldots, p$. There is no symmetry without fixed points. □

Corollary 4.8. *For any prime $p > 2$ and $a \equiv 2$ (4), there exists a symmetric Riemann surface X of the second type of genus $g = a(p-1)/2$ such that every symmetry of X has exactly p ovals.*

Proof. By Theorem 3.3, there exists a p-gonal Riemann surface X of genus $g = a(p-1)/2$ with an automorphism group $A^n_{1,p}$ for $n = 1 + a/2$. X is symmetric of the second type, and according to Theorem 4.7, any symmetry of X has p ovals. □

Corollary 4.9. *For any prime $p > 2$ and $a \equiv 2$ (4), there exists a symmetric Riemann surface of the first type of genus $g = a(p-1)/2$ such that any symmetry of X with fixed points has either 1 or p ovals.*

Proof. By Theorem 3.3, there exists a Riemann surface X of genus $g = a(p-1)/2$ with an automorphism group $D^n_{1,1}$ for $n = 2 + a$. X is symmetric of the first type, and according to Theorem 4.4, X has three conjugacy classes of symmetries with fixed points whose numbers of ovals are $1, 1$ and p. □

Corollary 4.10. *A p-gonal Riemann surface of genus g with a symmetric triangular D_n-action of the automorphism group G admits a symmetry with the maximal number $g+1$ of ovals iff $p > 2$ and $G = A^2_{1,p}$ or $D^4_{1,1}$, or $p = 2$ and $G = A^{g+1}_{1,2}$ or $C^{2g+2}_{1,1}$.*

5. EXCEPTIONAL POINTS WITH TRIANGULAR SYMMETRIC (D_n)-ACTIONS

An exceptional point in the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g is a unique surface class whose full group of conformal automorphisms acts with a triangular signature. The cyclic *p-gonal locus* $\mathcal{M}^p_g \subseteq \mathcal{M}_g$ consists of surfaces admitting a conformal automorphism (a p-gonal automorphism), with the quotient the Riemann sphere. By results of previous section there are nine non-equivalent symmetric triangular (D_n)-actions on exceptional points in $\mathcal{M}^p_g$ listed in the next theorem.

Theorem 5.1. *Given prime $p > 2$, the genus g of an exceptional point $X \in \mathcal{M}_g^p$ is equal to $a\frac{p-1}{2}$ for some integer $a \geq 2$. The full group $\mathcal{A}$ of conformal and anticonformal automorphisms of X is a semidirect product $G \rtimes \langle T : T^2 \rangle$, where G is a group listed in Table 3, and T is a symmetry of X whose action on G is given below. X has k symmetries with fixed points belonging to one, two or three conjugacy classes, and l symmetries without fixed points.*

Table 3.

Case	G	TRT	TST	k	l	Ovals	Conditions
(1)	$A_{1,p}^{1+\frac{a}{2}}$	R^{-1}	S^{-1}	$p(2+\frac{a}{2})$	0	$1,1$	$a \equiv 0\ (4)$
					p	$1,1,1$	$a \equiv 2\ (4)$
(2)	$A_{1,p}^{1+\frac{a}{2}}$	RS	S^{-1}	$1+\frac{a}{2}$	1	1	$a \equiv 0\ (4)$
					0	p	$a \equiv 2\ (4)$
(3)	$A_{1,p}^{1+\frac{a}{2}}$	$(RS)^{-1}$	S^{-1}	$p(1+\frac{a}{2})$	0	1	$a \equiv 2\ (4)$
					p	1	$a \equiv 0\ (4)$
(4)	$C_{1,1}^{2+a}$	R^{-1}	S^{-1}	$p(3+a)$	0	$1,1$	$a \equiv 1\ (2)$
					p	$1,1,1$	$a \equiv 0\ (2)$
(5)	$D_{1,1}^{2+a}$	R^{-1}	S^{-1}	$p(2+\frac{a}{2})+1+\frac{a}{2}$	p	$1,1,p$	$a \equiv 2\ (4)$
					1	$1,1,1$	$a \equiv 0\ (4)$
(6)	$D_{1,1}^{2+a}$	S^{-1}	R^{-1}	$2+a$	0	1	$a = 2p-2$
(7)	$E_{p,1}^{a}$	R^{-1}	S^{-1}	$p(a+1)$	0	$1,1$	$a \equiv 1\ (2)$
					p	$1,1,1$	$a \equiv 0\ (2)$
(8)	$F_{p,p}^{\frac{a}{2}}$	$(RS)^{-1}$	S^{-1}	$p\frac{a}{2}$	0	1	$a \equiv 0\ (4)$
					p	1	$a \equiv 2\ (4)$
(9)	$F_{p,p}^{\frac{a}{2}}$	SR^{-p-1}	R^{-p}	$2p$	0	1	$a = 4$

By observing the action (2) in Table 3 we get the following corollary.

Corollary 5.2. *For any prime $p > 2$ and even $q \geq 2$, there exists a symmetric exceptional point of the second type in $\mathcal{M}_g^p$, $g = (q-1)(p-1)$, which admits q symmetries, and every symmetry has p ovals.*

Corollary 5.3. *For any prime $p > 2$ and even $q \geq 4$, there exists a symmetric exceptional point of the second type in $\mathcal{M}_g^p$, $g = (q-2)(p-1)$, which admits q symmetries: one without fixed points and others with one oval.*

Theorem 5.4. *There are seven non-equivalent symmetric triangular (D_n)-actions of a finite group G on exceptional points $X \in \mathcal{M}_g^2$. The full group of conformal and anticonformal automorphisms of X is a semidirect product $G \rtimes \langle T : T^2 \rangle$, where T is a symmetry of X whose action on G is given in Table 4.*

Table 4.

Case	G	TRT	TST	k	l	Ovals	Conditions
(1)	$A_{1,2}^{1+g}$	R^{-1}	S^{-1}	$2g+4$	2	$g+1,1,1$	$g \equiv 1\ (2)$
				$2g+4$	0	$g+1,1$	$g \equiv 0\ (2)$
(2)	$A_{1,2}^{1+g}$	RS	S^{-1}	$1+g$	$1+g$	2	$g \equiv 1\ (2)$
				$1+g$	$1+g$	1	$g \equiv 0\ (2)$
(3)	$B_{2,2}^{g}$	R^{-1}	S^{-1}	$2g+2$	2	$g,2$	$g \equiv 1\ (2)$
				$2g+2$	2	$g,2,2$	$g \equiv 0\ (2)$
(4)	$B_{2,2}^{g}$	RS	S^{-1}	g	g	2	any
(5)	$B_{2,2}^{g}$	S^{-1}	R^{-1}	4	0	1	$g=2$
(6)	$C_{1,1}^{2+2g}$	R^{-1}	S^{-1}	$3g+5$	$g+1$	$g+1,1,1$	$g \equiv 0\ (2)$
				$3g+5$	$g+3$	$g+1,1,2$	$g \equiv 1\ (2)$
(7)	$E_{2,1}^{2g,1}$	R^{-1}	S^{-1}	$4g+2$	0	$g,2,1$	$g \equiv 0\ (2)$
				$4g+2$	2	$g,2,1$	$g \equiv 1\ (2)$

Corollary 5.5. *For any prime p, there exists a symmetric $AM_{p,g}$ whose automorphism group has order $8(g+1)$ and which admits $g+1$ symmetries without fixed points. The remaining symmetries of X split into 3 conjugacy classes admitting 2, $g+1$ and $2(g+1)$ elements whose number of ovals is $g+1, 1$ and 1, respectively.*

Proof. For $p=2$ we need to take the symmetric action (6) in Table 4 of the group $C_{1,1}^{2+2g}$ on the Accola-Maclachlan surface of even genus g, and for $p>2$ we can take the symmetric action (5) in Table 3 of the group $D_{1,1}^{4}$ on $AM_{p,g}$ of genus $g=p-1$. □

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REFERENCES

[1] R.D.M. Accola, *On the number of automorphisms of a closed Riemann surface*, Trans. Amer. Math. Soc. **131** (1968), 398–408.

[2] R.D.M. Accola, *On cyclic trigonal Riemann surfaces*, Trans. Amer. Math. Soc. **283** (1984) 2, 423–449.

[3] G.V. Belyi, *On Galois extensions of maximal cyclotomic field*, Math. USSR Izvestiya **14** (1980), 247–256.

[4] S.A. Broughton, E. Bujalance, A.F. Costa, J.M. Gamboa, G. Gromadzki, *Symmetries of Accola-Maclachlan and Kulkarni Surfaces*, Proc. Amer. Math. Soc. **127** (1999), 637–646.

[5] E. Bujalance, F.J. Cirre, J.M. Gamboa, G. Gromadzki, *Symmetry Types of Hyperelliptic Riemann Surfaces*, Mémoires de la Société Mathématique De France (2001).

[6] G. Castelnuevo, *Sulle serie algebriche di gruppi di punti appartenenti ad una curve algebraica*, Rend. R. Academia Lincei Ser. 5 XV (1906), (Memorie scelte p. 509).

[7] H.M. Farkas, I. Kra, *Riemann Surfaces*, Graduate Text in Mathematics, Springer-Verlag, New York, 1980.

[8] G. Gromadzki, *On a Harnack-Natanzon Theorem for the family of real forms of Riemann surfaces*, J. Pure Appl. Algebra **121** (1997), 253–269.

[9] A. Harnack, *Über die Vieltheiligkeit der ebenen algebraischen Kurven*, Math. Ann. **10** (1876), 189–199.

[10] A. Hurwitz, *Über alebraische Gebilde mit eindeutigen Transformationen in sich*, Math. Ann. **41** (1983).

[11] G.A. Jones, D. Singerman, P.D. Watson, *Symmetries of quasiplatonic Riemann surfaces*, Rev. Mat. Iberoam. (to appear).

[12] A.M. Macbeath, *The classification of non-Euclidean plane crystallographic groups*, Canad. J. Math. **19** (1966), 1192–1205.

[13] A.M. Macbeath, *Action of automorphisms of a compact Riemann surface on the first homology group*, Bull. Lond. Math. Soc. **5** (1973), 103–108.

[14] A.M. Macbeath, *On a theorem of Hurwitz*, Proc. Glasgow Math. Assoc. **5** (1961), 90–96.

[15] C. Maclachlan, *Smooth coverings of hyperelliptic surfaces*, Quart. Journal of Math. Oxford **2** (1971), 117–123.

[16] D. Singerman, *Symmetries of Riemann surfaces with large automorphism group*, Math. Ann. **210** (1974), 17–32.

[17] D. Singerman, *Non-Euclidean crystallographic groups and Riemann surfaces*, Ph.D. Thesis, Univ. of Birmingham, 1969.

[18] D. Singerman, *Finitely maximal Fuchsian groups*, J. Lond. Math. Soc. (2) **6** (1972), 29–38.

[19] D. Singerman, *The remarkable Accola-Maclachlan surfaces*, [in:] Contemp. Math., vol. 629, Amer. Math. Soc., Providence, RI, 2014, pp. 315–322.

[20] E. Tyszkowska, *Topological classification of conformal actions on cyclic p-gonal Riemann surfaces*, Journal of Algebra **344** (2011), 296–312.

[21] E. Tyszkowska, A. Weaver, *Exceptional points in the elliptic-hyperelliptic locus*, J. Pure Appl. Algebra **212** (2008), 1415–1426.

[22] A. Weaver, *Hyperelliptic surfaces and their moduli*, Geom. Dedicata **103** (2004), 69–87.

[23] H.C. Wilkie, *On non-Euclidean crystallographic groups*, Math. Z. **91** (1966), 87–102.

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