ON THE UNIFORM PERFECTNESS OF EQUIVARIANT DIFFEOMORPHISM GROUPS FOR PRINCIPAL \( G \) MANIFOLDS

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Communicated by P.A. Cojuhari

Abstract. We proved in [K. Abe, K. Fukui, On commutators of equivariant diffeomorphisms, Proc. Japan Acad. 54 (1978), 52–54] that the identity component \( \text{Diff}_{r,c}^G(M)_0 \) of the group of equivariant \( C^r \)-diffeomorphisms of a principal \( G \) bundle \( M \) over a manifold \( B \) is perfect for a compact connected Lie group \( G \) and \( 1 \leq r \leq \infty \) \((r \neq \dim B + 1)\). In this paper, we study the uniform perfectness of the group of equivariant \( C^r \)-diffeomorphisms for a principal \( G \) bundle \( M \) over a manifold \( B \) by relating it to the uniform perfectness of the group of \( C^r \)-diffeomorphisms of \( B \) and show that under a certain condition, \( \text{Diff}_{r,c}^G(M)_0 \) is uniformly perfect if \( B \) belongs to a certain wide class of manifolds. We characterize the uniform perfectness of the group of equivariant \( C^r \)-diffeomorphisms for principal \( G \) bundles over closed manifolds of dimension less than or equal to 3, and in particular we prove the uniform perfectness of the group for the 3-dimensional case and \( r \neq 4 \).

Keywords: uniform perfectness, principal \( G \) manifold, equivariant diffeomorphism.

Mathematics Subject Classification: 58D05, 57R30.

1. INTRODUCTION

For a \( C^r \)-manifold \( M \), let \( \text{Diff}_r^c(M) \) denote the group of \( C^r \)-diffeomorphisms of \( M \) with compact support\((1 \leq r \leq \infty)\). Let \( \text{Diff}_c^r(M)_0 \) be the identity component of \( \text{Diff}_c^r(M) \) equipped with the compact open \( C^r \)-topology. Thurston ([9]) and Mather ([8]) proved that \( \text{Diff}_c^r(M)_0 \) is perfect if \( 1 \leq r \leq \infty \) and \( r \neq \dim M + 1 \), that is, it coincides with its commutator subgroup.

Let \( G \) be a compact connected Lie group and \( M \) be the total space of a principal \( G \) bundle \( M \) over a smooth manifold \( B \). Then we have a canonical smooth free \( G \) action on \( M \) and every smooth free \( G \) action on \( M \) induces a principal \( G \) bundle \( M \) over a smooth manifold \( B \). Let \( \text{Diff}_{r,c}^G(M) \) denote the group of equivariant \( C^r \)-diffeomorphisms of \( M \) with compact support and with the relative topology.
as a subspace of $\text{Diff}^r_c(M)$. Let $\text{Diff}^r_{G,c}(M)_0$ be the identity component of $\text{Diff}^r_{G,c}(M)$.

Abe and the author proved in [1] (and also Banyaga in [3]) using the results of Thurston and Mather that $\text{Diff}^r_{G,c}(M)_0$ is perfect if $1 \leq r \leq \infty$, $r \neq \dim M - \dim G + 1$ and $\dim M - \dim G \geq 1$.

Burago, Ivanov and Polterovich ([4]) and Tsuboi ([10, 11]) studied the uniform perfectness of $\text{Diff}^r_{G,c}(M)_0$, where a group is uniformly perfect if any element in it can be represented by a product of a bounded number of commutators of its elements. Indeed, Tsuboi has proved that $\text{Diff}^r_c(M)_0$ is uniformly perfect if $1 \leq r \leq \infty$ and $r \neq \dim M + 1$ and $M$ belongs to a wide class $\mathcal{C}$ of manifolds (see §3 for $\mathcal{C}$).

In this paper we study the uniform perfectness of $\text{Diff}^r_{G,c}(M)_0$ for a principal $G$ bundle $M$ over a manifold $B$ by relating it to the uniform perfectness of the group of $C^r$-diffeomorphisms of $B$ and show that under a certain condition, the necessary and sufficient condition for $\text{Diff}^r_{G,c}(M)_0$ to be uniformly perfect is that $\text{Diff}^r_r(B)_0$ is uniformly perfect. As corollaries, (i) we have by the results of Tsuboi ([10, 11]) that for $1 \leq r \leq \infty$, $r \neq \dim B + 1$, $\text{Diff}^r_{G,c}(M)_0$ is uniformly perfect if $\dim B \geq 3$, $G = T^n$ and $B \in \mathcal{C}$, and (ii) we characterize the uniform perfectness of the group of equivariant $C^r$-diffeomorphisms for principal $G$ bundles over closed manifolds of dimension $\leq 3$, and in particular we prove the uniform perfectness of the group for the 3-dimensional case and $r \neq 4$.

2. EQUIVARIANT Diffeomorphisms of a Manifold With Trivial $G$ Action

Let $M$ be a smooth manifold without boundary on which a compact connected Lie group $G$ acts smoothly and freely. Then the orbit map $\pi : M \to M/G$ is a principal $G$ bundle over a smooth manifold $B = M/G$. Let $\text{Diff}^r_{G,c}(M)_0$ denote the group of equivariant $C^r$-diffeomorphisms of $M$ with compact support, which are $G$-isotopic to the identity through equivariant $C^r$-diffeomorphisms with compact support.

By using the results of Thurston ([9]) and Mather ([8]), Abe and the author in [1] (and also Banyaga in [3]) proved the following.

**Theorem 2.1.** If $1 \leq r \leq \infty$, $r \neq \dim M - \dim G + 1$ and $\dim M - \dim G \geq 1$, then $\text{Diff}^r_{G,c}(M)_0$ is perfect.

In this section we consider the uniform perfectness of $\text{Diff}^r_{G,c}(M)_0$ for the case $M = \mathbb{R}^m \times G$. Let $\pi : \mathbb{R}^m \times G \to \mathbb{R}^m$ be the projection, which induces the group epimorphism $P : \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \to \text{Diff}^r_c(\mathbb{R}^m)_0$ defined by $P(f) = \tilde{f}$, where $\tilde{f}(x, g) = (\tilde{f}(x), h(x, g))$ for $x \in \mathbb{R}^m$ and $g \in G$.

**Theorem 2.2.**

1. If $1 \leq r \leq \infty$, $r \neq m + 1$ and $m \geq 1$, then $\text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0$ is uniformly perfect. In fact, any $f \in \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0$ can be represented by a product of two commutators of elements in $\text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0$.

2. If $1 \leq r \leq \infty$ and $m \geq 1$, then any $f \in \ker P$ can be represented by a product of two commutators of elements in $\ker P$ and $\text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0$. 
Proof. (1) The proof follows from the proof of [10, Theorem 2.1] of Tsuboi but we write the proof for the completeness. Take \( f \in \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \). By Theorem 2.1, \( f \) can be represented by a product of commutators as

\[
f = \prod_{i=1}^{k} [a_i, b_i], \quad \text{where } a_i, b_i \in \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0.
\]

Let \( U \) be an bounded open set of \( \mathbb{R}^m \) satisfying that \( \pi^{-1}(U) \) contains the supports of \( a_i \) and \( b_i \). Take \( \tilde{\phi} \in \text{Diff}^r_{c}(\mathbb{R}^m)_0 \) satisfying that \( \{ \tilde{\phi}^i(U) \}_{i=1}^k \) are disjoint. Define \( \phi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G \) by \( \phi(x, g) = (\tilde{\phi}(x), g) \) for \( (x, g) \in \mathbb{R}^m \times G \). Then \( \phi \in \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \). We put

\[
F = \prod_{j=1}^{k} \phi^j \left( \prod_{i=j}^{k} [a_i, b_i] \right) \phi^{-j}
\]

which is in \( \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \). Then we have

\[
\phi^{-1} \circ F \circ \phi \circ F^{-1} = f \circ \left( \prod_{j=1}^{k} \phi^j [a_j, b_j]^{-1} \phi^{-j} \right) = f \circ \left[ \prod_{j=1}^{k} \phi^j b_j \phi^{-j}, \prod_{j=1}^{k} \phi^j a_j \phi^{-j} \right].
\]

Thus we have

\[
f = [\phi^{-1}, F] \circ \left[ \prod_{j=1}^{k} \phi^j a_j \phi^{-j}, \prod_{j=1}^{k} \phi^j b_j \phi^{-j} \right].
\]

That is, any \( f \in \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \) can be represented by two commutators of elements in \( \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \).

(2) By Proposition 6 of [1], any \( f \in \ker P \) can be represented by a product of commutators of elements in \( \ker P \) and \( \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \) as

\[
f = \prod_{i=1}^{k} [c_i, d_i], \quad \text{where } c_i \in \ker P \text{ and } d_i \in \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0.
\]

Note that it also holds for \( r = m + 1 \). By the similar way as in (1), we can prove that \( f \in \ker P \) is represented by two commutators of elements in \( \ker P \) and \( \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \). This completes the proof. \( \square \)

3. UNIFORM PERFECTNESS OF \( \text{Diff}^r_{G,c}(M)_0 \)

Let \( G \) be a compact connected Lie group and \( \pi : M \to B \) be a principal \( G \) bundle over an \( m \)-dimensional closed \( C^r \) manifold \( B \) (\( m \geq 1 \)), where “closed” means “compact and without boundary”. Let \( P : \text{Diff}^r_G(M)_0 \to \text{Diff}^r(B)_0 \) be the map defined by \( P(f)(x) = \pi(f(\hat{x})) \) for \( f \in \text{Diff}^r_G(M)_0 \) and \( x \in B, \hat{x} \in M \) with \( \pi(\hat{x}) = x \). Curtis in [5] proved that \( P \) is a surjective homomorphism and a local trivial fibration.

In this section we study the uniform perfectness of \( \text{Diff}^r_G(M)_0 \) by relating it to the uniform perfectness of \( \text{Diff}^r(B)_0 \). Then we have the following.
Theorem 3.1.

1. If $$\text{Diff}^r_G(M)_0$$ is uniformly perfect, then $$\text{Diff}^r(B)_0$$ is uniformly perfect.
2. If the number of connected components of $$\ker P$$ is finite and $$\text{Diff}^r(B)_0$$ is uniformly perfect, then $$\text{Diff}^r_G(M)_0$$ is uniformly perfect.

Proof. (1) Take any $$\bar{f} \in \text{Diff}^r(B)_0$$. Then from the result of Curtis ([5]), we have $$f \in \text{Diff}^r_G(M)_0$$ satisfying $$P(f) = \bar{f}$$. From the assumption, $$f$$ can be represented as a product of a bounded number, say $$k$$, of commutators;

$$f = \prod_{j=1}^{k} [g_j, h_j]$$, where $$g_j, h_j \in \text{Diff}^r_G(M)_0$$.

Then we have

$$\bar{f} = P(f) = P\left( \prod_{j=1}^{k} [g_j, h_j] \right) = \prod_{j=1}^{k} [P(g_j), P(h_j)].$$

(2) Take any $$f \in \text{Diff}^r_G(M)_0$$. Then from the assumption, we have $$\bar{f} = P(f) = \prod_{j=1}^{k} [g_j, h_j]$$, where $$g_j, h_j \in \text{Diff}^r(B)_0$$ and $$k$$ is a bounded number. By using the result of Curtis ([5]) again, we can take $$g_j$$ and $$h_j$$ in $$\text{Diff}^r_G(M)_0$$ satisfying $$P(g_j) = \bar{g}_j$$ and $$P(h_j) = \bar{h}_j$$. Then we have $$\left( \prod_{j=1}^{k} [g_j, h_j] \right)^{-1} \circ f \in \ker P$$.

First we consider the case that $$\psi = \left( \prod_{j=1}^{k} [g_j, h_j] \right)^{-1} \circ f$$ is G-isotopic to the identity in $$\ker P$$. We have $$f = \prod_{j=1}^{k} [g_j, h_j] \circ \psi$$ and $$\psi \in \ker P$$.

Let $$\{U_i\}_{i=1}^{\ell+1}$$ be an open covering of $$B$$ such that each $$U_i$$ is a disjoint union of open balls, where $$\ell$$ is the category number of $$B (\ell \leq m)$$. Let $$\{\lambda_i\}_{i=1}^{\ell+1}$$ be a partition of unity subordinate to the covering $$\{U_i\}_{i=1}^{\ell+1}$$. Let $$\psi_t (0 \leq t \leq 1)$$ be an isotopy in $$\ker P$$ from $$\psi_0 = \text{identity}$$ to $$\psi_1 = \psi$$. Define $$h_i \in \ker P$$ ($$i = 1, 2, \ldots, \ell + 1$$) as follows:

$$h_1(p) = \psi_1 \circ \pi(p)(p) \text{ for } p \in M,$$

$$h_2(p) = h_1^{-1} \circ \psi_1 \circ \pi(p) + h_2 \circ \pi(p)(p) \text{ for } p \in M,$$

and in general

$$h_i(p) = (h_1 \circ \ldots \circ h_{i-1})^{-1} \circ \psi \sum_{j=1}^{i} \lambda_j \circ \pi(p)(p) \text{ for } p \in M \text{ (} i = 3, \ldots, \ell + 1 \text{).}$$

Then we have the support of $$h_i$$ is contained in $$U_i$$ ($$i = 1, 2, \ldots, \ell + 1$$) and $$h_i \in \ker P$$. For, any element $$\psi \in \ker P$$ has locally (say, on $$\pi^{-1}(U)$$ for an open ball $$U$$ in $$B$$) the form of $$\psi(x,g) = (x, g \cdot L(\psi)(x))$$, where $$L : \ker P \to C^r(U, G_0)$$ is the map defined by $$\langle x, L(\psi)(x) \rangle = \psi(x, e)$$ (see [1]). Thus the isotopy $$\psi_t (0 \leq t \leq 1)$$ has the form $$(x, g \cdot L(\psi)_t(x))$$, where $$L(\psi)_t(x)(0 \leq t \leq 1)$$ is a homotopy in $$C^r(U, G)$$ from $$L(\psi)_0(x) = e$$ to $$L(\psi)_1(x) = L(\psi)(x)$$. Hence each $$h_i$$ is in $$\ker P$$. Furthermore we have $$\psi = h_1 \circ h_2 \circ \ldots \circ h_{\ell+1}$$.

As $$U_i$$ is a disjoint union of open balls diffeomorphic to the unit open ball int$$D^m$$, we have only to prove the case that $$U_i$$ is int$$D^m$$ in order to prove Theorem 3.1(2). Since $$\pi$$ is trivial over $$U_i$$, $$\pi^{-1}(U_i)$$ is $$G$$-diffeomorphic to $$U_i \times G$$. Thus we may assume that each
$h_i$ is contained in $\ker P$ for the homomorphism $P : \text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0 \to \text{Diff}^r_c(\mathbb{R}^m)_0$ in §2. From Theorem 2.2(2), each $h_i$ can be represented by a product of two commutators of elements in $\ker P$ and $\text{Diff}^r_{G,c}(\mathbb{R}^m \times G)_0$ if $1 \leq r \leq \infty$. Thus $\psi$ can be represented by a product of $2(\ell+1)$ commutators of elements in $\ker P$ and $\text{Diff}^r_G(M)_0$. Hence $f$ can be represented by a product of $k + 2(\ell+1)$ commutators of elements in $\text{Diff}^r_G(M)_0$, where $k$ and $\ell$ are bounded numbers.

Next we consider the case that $\psi$ is not connected to the identity in $\ker P$. Let $a$ be the number of the connected components of $\ker P$. Take elements, say $g_1, \ldots, g_a$, from each connected component of $\ker P$ and fix them. Then from Theorem of [1], each $g_i$ can be written by $t_i$ commutators of elements in $\text{Diff}^r_G(M)_0$. Put $t = \max\{t_1, \ldots, t_a\}$. For any element $g \in \ker P$, there exists some $i$ ($i = 1, \ldots, a$) satisfying that $g$ and $g_i$ are in the same connected component of $\ker P$. Since $g \circ (g_i)^{-1}$ is in the identity component of $\ker P$, $g$ can be written by at most $2(\ell+1) + t$ commutators. Thus for any element $f \in \text{Diff}^r_G(M)_0$, above $\psi$ can be written by $2(\ell+1) + t$ commutators. Hence $f \in \text{Diff}^r_G(M)_0$ can be written by $k + 2(\ell+1) + t$ commutators of elements in $\text{Diff}^r_G(M)_0$. Since $k, \ell$ and $t$ are bounded numbers, this completes the proof.

The fibration map $P : \text{Diff}^r_G(M)_0 \to \text{Diff}^r(B)_0$ induces the homomorphism between the fundamental groups $P_* : \pi_1(\text{Diff}^r_G(M)_0, 1) \to \pi_1(\text{Diff}^r(B)_0, 1)$.

**Corollary 3.2.** Suppose that the cokernel of the homomorphism

$$P_* : \pi_1(\text{Diff}^r_G(M)_0, 1) \to \pi_1(\text{Diff}^r(B)_0, 1)$$

is finite. Then $\text{Diff}^r_G(M)_0$ is uniformly perfect if $\text{Diff}^r(B)_0$ is uniformly perfect.

**Proof.** The fibration map $P : \text{Diff}^r_G(M)_0 \to \text{Diff}^r(B)_0$ induces the following exact sequence of homotopy groups:

$$\ldots \to \pi_1(\text{Diff}^r_G(M)_0, 1) \to \pi_1(\text{Diff}^r(B)_0, 1) \to \pi_0(\ker P) \to \pi_0(\text{Diff}^r_G(M)_0) = 1.$$

From the assumption $P_* : \pi_1(\text{Diff}^r_G(M)_0, 1) \to \pi_1(\text{Diff}^r(B)_0, 1)$ has finite cokernel. Thus $\pi_0(\ker P)$ is finite, that is, the connected components of $\ker P$ is finite. The proof follows from Theorem 3.1(2).

4. UNIFORM PERFECTNESS OF $\text{Diff}^r_{T^n}(M)_0$

In this section we study the uniform perfectness of $\text{Diff}^r_{T^n}(M)_0$ for principal $T^n$-bundles over closed manifolds $B$. Then we have the following.

**Theorem 4.1.** Suppose that $\dim B \geq 3$. Then $\text{Diff}^r_{T^n}(M)_0$ is uniformly perfect if $\text{Diff}^r(B)_0$ is uniformly perfect.

**Proof.** Take any $f \in \text{Diff}^r_{T^n}(M)_0$. Then from the assumption, we have $\tilde{f} = P(f) = \prod_{j=1}^k [g_j, h_j]$, where $g_j, h_j \in \text{Diff}^r(B)_0$ and $k$ is a bounded number. By using the result of Curtis ([5]) again, we can take $g_j$ and $h_j$ in $\text{Diff}^r_{T^n}(M)_0$ satisfying $P(g_j) = \tilde{g}_j$ and $P(h_j) = \tilde{h}_j$. Then we have $\psi = (\prod_{j=1}^k [g_j, h_j])^{-1} \circ f \in \ker P$. 


Let \( \{U_i\}_{i=1}^{\ell+1} \) and \( \{V_i\}_{i=1}^{\ell+1} \) be open coverings of \( B \) such that each \( U_i \) and \( V_i \) are disjoint unions of open balls and \( U_i \subset V_i \), where \( \ell \) is the category number of \( B (\ell \leq m) \).

Since \( \pi \) is trivial over \( V_i \), \( \pi_j(T^n) = 1 \) \( (j \geq 2) \) and \( m \geq 3 \), we can deform \( \psi \) over \( V_1 \) to \( \psi_1 \in \ker P \) satisfying that \( \psi_1 = \psi \) on \( U_1 \) and \( \psi_1 \) is the identity near the boundary of \( V_1 \). For, \( V_1 - U_1 \) is homeomorphic to \( S^{m-1} \times [0,1] \) and \( \psi \mid_{\partial(U_1)} (x,\cdot) : \partial(U_1) \to T^n \) is homotopic to the constant map \( e \) because \( \pi_j(T^n) = 1 \) \( (j \geq 2) \) and \( m \geq 3 \). Hence \( \psi \) can be deformed in \( V_1 \) to the identity near the boundary of \( V_1 \) fixing \( \psi \) on \( U_1 \) (see the proof of Theorem 3.1(2)).

Next we get \( \psi_2 \in \ker P \) satisfying that \( \psi_2 = \psi_1 \) on \( U_2 \) and \( \psi_2 \) is the identity near the boundary of \( V_2 \) by performing the same procedure as above for \( (\psi_1)^{-1} \circ \psi \) and \( V_2 \). After \( \ell + 1 \) times procedures, we have \( \psi_1, \ldots, \psi_{\ell+1}(\in \ker P) \) satisfying that \( \psi = \psi_1 \circ \ldots \circ \psi_{\ell+1} \) and each \( \psi_i \) is supported in \( V_i \). Since each \( \psi_i \) is in \( \ker P \), we have from Theorem 2(2) that \( \psi_i \) can be represented by a product of two commutators of elements in \( \ker P \) and \( \text{Diff}_{T^n}^r(\mathbb{R}^m \times T^n)_0 \) if \( 1 \leq r \leq \infty \). Thus \( \psi \) can be represented by a product of \( 2(\ell + 1) \) commutators of elements in \( \ker P \) and \( \text{Diff}_{T^n}^r(M)_0 \). Hence \( f \) can be represented by a product of \( k + 2(\ell + 1) \) commutators of elements in \( \text{Diff}_{T^n}^r(M)_0 \), where \( k \) and \( \ell \) are bounded numbers. This completes the proof.

Since \( \pi_2(G) = 0 \) for any Lie group \( G \) and \( \text{Diff}^r(B)_0(r \neq 4) \) is uniformly perfect when \( B \) is a 3 dimensional closed manifold ([4,10]), the above proof induces the following.

**Corollary 4.2.** Suppose that \( B \) is a 3 dimensional closed manifold. Then \( \text{Diff}_{T^n}^r(G)_0 \) is uniformly perfect for \( r \neq 4 \).

We say that a manifold \( B \) belongs to a class \( C \) if \( B \) is one of the following:

1. an \( m \) dimensional closed manifold \( (m \neq 2,4) \) and
2. an \( m \) dimensional closed manifold which has a handle decomposition without handles of the middle index \( (m = 2,4) \).

Then Tuboi ([10,11]) proved the following.

**Theorem 4.3.** If \( B \in C \) and \( 1 \leq r \leq \infty \), \( r \neq \dim B + 1 \), then \( \text{Diff}_{T^n}^r(B)_0 \) is uniformly perfect.

**Corollary 4.4.** Let \( \pi : M \to B \) be a principal \( T^n \) bundle over an \( m \)-dimensional closed manifold \( B \). Suppose that \( m \geq 3 \) and \( B \) belongs to the class \( C \). If \( 1 \leq r \leq \infty \), \( r \neq m+1 \), then \( \text{Diff}_{T^n}^r(M)_0 \) is uniformly perfect.

**Proof.** The proof follows from Theorem 4.1 and Theorem 4.3.

**Corollary 4.5.** Let \( M \) be a closed \( T^n \)-manifold with one orbit type. Suppose that the orbit manifold \( M/G \) belongs to the class \( C \). If \( 1 \leq r \leq \infty \), \( r \neq \dim M - \dim G + 1 \) and \( \dim M - \dim G \geq 3 \), then \( \text{Diff}_{T^n}^r(M)_0 \) is uniformly perfect.

**Proof.** The proof follows from Corollary of [1] and Corollary 4.4.
5. UNIFORM PERFECTNESS OF $\text{Diff}^r_G(M)_0$ FOR PRINCIPAL $G$-BUNDLES OVER LOW DIMENSIONAL CLOSED MANIFOLDS

In this section we consider the uniform perfectness of $\text{Diff}^r_G(M)_0$ for principal $G$-bundles over closed manifolds $B$ of dimension $\leq 2$.

First we consider the case of $\text{Diff}^r_G(M)_0$ for principal $G$-bundles over $S^1$. Since any principal $G$-bundle over $S^1$ is trivial, $\ker P$ is connected for a compact connected Lie group $G$. Furthermore, since $\text{Diff}^r(S^1)_0$ is uniformly perfect ($r \neq 2$), we have the following from Theorem 3.1(2).

**Theorem 5.1.** Let $\pi : M \to S^1$ be a principal $G$ bundle over $S^1$. Then $\text{Diff}^r_G(M)_0$ is uniformly perfect for $r \neq 2$.

Next we study the uniform perfectness of $\text{Diff}^r_G(M)_0$ for principal $G$-bundles over closed orientable surfaces not homeomorphic to $T^2$. Then we have the following.

**Theorem 5.2.** Let $\pi : M \to B$ be a principal $G$ bundle over a 2 dimensional closed orientable manifold $B$.

1. When $B$ is the 2-sphere $S^2$, $\text{Diff}^r_G(M)_0$ is uniformly perfect for $r \neq 3$.
2. When $B$ is a closed orientable surface not homeomorphic to $S^2, T^2$, $\text{Diff}^r_G(M)_0$ is uniformly perfect if and only if $\text{Diff}^r(B)_0$ is uniformly perfect.

**Proof.** (1) For $B = S^2$, we have $\pi_1(\text{Diff}^r(B)_0, 1) \cong \pi_1(SO(3), 1) \cong \mathbb{Z}_2$. Then the connected components of $\ker P$ are at most two. Thus (1) follows from Theorem 3.1(2) and the uniform perfectness of $\text{Diff}^r(S^2)_0$ ($r \neq 3$).

(2) When $B$ is a closed surface not homeomorphic to $S^2, T^2$, $\text{Diff}^r(B)_0$ is contractible. Thus the fibration $P : \text{Diff}^r_G(M)_0 \to \text{Diff}^r(B)_0$ is trivial. Then $\ker P$ is connected. Hence (2) follows from Theorem 3.1(2). \hfill \Box

Finally we have the following problem.

**Problem 5.3.** Discuss the uniform perfectness for the case $B = T^2$.

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Received: August 31, 2016.
Accepted: November 16, 2016.