A NOTE
ON INCOMPLETE REGULAR TOURNAMENTS
WITH HANDICAP TWO OF ORDER $n \equiv 8 \pmod{16}$

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Abstract. A *d-handicap distance antimagic labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \rightarrow \{1, 2, \ldots, n\}$ with the property that $f(x_i) = i$ and the sequence of weights $w(x_1), w(x_2), \ldots, w(x_n)$ (where $w(x_i) = \sum_{x_i x_j \in E} f(x_j)$) forms an increasing arithmetic progression with common difference d. A graph G is a *d-handicap distance antimagic graph* if it allows a *d*-handicap distance antimagic labeling. We construct a class of *k*-regular 2-handicap distance antimagic graphs for every order $n \equiv 8 \pmod{16}, n \geq 56$ and $6 \leq k \leq n - 50$.

Keywords: incomplete tournaments, handicap tournaments, distance magic labeling, handicap labeling.

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1. MOTIVATION

The notions of distance magic, distance antimagic, and handicap distance antimagic graphs have been motivated by various tournament scheduling needs. When we have a group of teams ranked according to their standings in the previous season, we can schedule an incomplete tournament is many different ways.

It certainly makes difference if team A plays opponents whose previous year's standings were $1, 2, 3$, and 4, and team B plays opponents ranked $3, 4, 5$, and 6. Team A plays the four strongest teams and therefore has more difficult schedule than team B.

Hence, we may want to schedule a tournament in which every team plays the same number of games, say r, and every team meets opponents whose total strength (that is, the sum of their previous year's standings) would be the same for each team. Such a schedule corresponds to finding an r-regular distance magic graph. An example would be a tournament with n teams (where n is even) ranked $1, 2, \ldots, n$ playing $n - 2$ games each in such a way that team ranked i avoids the team ranked $n + 1 - i$. Then

the strength of schedule for every team, that is, the sum of the opponents' rankings, would be

$$S_{n,n-2}(i) = \sum_{j=1}^{n} j - i - (n+1-i) = \sum_{j=2}^{n-1} j = (n-2)(n+1)/2.$$

One can also schedule tournaments which make seemingly little sense, that is, in which the strongest team plays the weakest opponents and vice-versa. While this looks absurd, the complete round-robin tournament (where each team plays all other teams) has exactly this property. Regular tournaments with this property are (for their similarity with the round-robin) called *fair incomplete tournaments*, and tournaments in which all teams have the same strength of schedule are called *equalized tournaments*. An example of a fair incomplete tournament is a tournament with $n = 4k$ teams and $n - 3$ games per team, in which every pair of teams ranked i and $n + 1 - i$ for $i = 1, 2, \ldots, 2k$ avoids teams ranked $2k + 1 - i$ and $2k + i$. The strength of schedule will then be

$$S_{n,n-3}(i) = \sum_{j=1}^{n} j - i - (2k+1-i) - (2k+i)$$

$$= \sum_{j=1}^{n} j - i - n - 1$$

$$= \sum_{j=2}^{n-1} j - i = (n-2)(n+1)/2 - i.$$

One can easily observe that a complement of a fair tournament is equalized. Fair and equalized tournaments were studied by the author and his co-authors in [7] and [1].

However, even an equalized tournament where the strength of schedule is the same for each team does not give the same chance of winning to all teams. In the above example, the strongest and weakest team play exactly the same set of opponents. It is then very likely that the stronger team will win more games than the weaker one. This can be eliminated by having the strongest teams play strong opponents and the weakest teams play weak opponents.

One way of achieving this goal is to schedule *handicap tournaments*. In such tournaments, the strength of schedule of team ranked i would be

$$S_{n,r}(i) = m + di$$

for some positive constants m and d.

Handicap tournaments with handicap $d = 1$ (also called 1-*handicap* or just *handicap*) have been investigated in several papers. An overview of results on regular 1-handicap graphs of even order with additional references was recently accepted for publication [8]. For even-regular 1-handicap graphs of odd order, the results so far are sparse – see [4]. For $d = 2$, the author has studied handicap graphs with $n \equiv 0 \pmod{16}$ vertices [5,6]. In this note, we construct an infinite class of such graphs for $n \equiv 8 \pmod{16}$.

2. DEFINITIONS, TOOLS, AND KNOWN RESULTS

The notions of handicap tournaments and handicap distance antimagic labelings were introduced by the author who originally called the labeling *ordered distance antimagic* in [2]. However, the term "handicap distance antimagic labeling" was coined by Kovarova [12].

Definition 2.1. A *handicap distance d-antimagic labeling* or shortly *d-handicap labeling* of a graph $G = (V, E)$ with n vertices is a bijection $f : V \to \{1, 2, \ldots, n\}$ with induced *weight function*

$$w(x_i) = \sum_{x_j : x_i x_j \in E} f(x_j)$$

such that $f(x_i) = i$ and the sequence of weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an increasing arithmetic progression with difference d. When $d = 1$, the labeling is called just a *handicap labeling*.

A graph G is a *handicap distance d-antimagic graph* (or just *d-handicap graph*) if it allows a handicap distance d-antimagic labeling, and a *handicap distance antimagic graph* or a *handicap graph* when $d = 1$.

In other words, G has a 2-handicap labeling defined by $f(x_i) = i$ if there exists a constant μ such that $w(x_i) = \mu + 2i$ for every $i = 1, 2, \ldots, n$.

The notion of d-handicap labeling is a special case of *distance d-antimagic labeling*. In that labeling the weight of a vertex does not depend on its own label. It is only required that the sequence of weights $w(x_1), w(x_2), \ldots, w(x_n)$ forms an arithmetic progression.

Constructions in this paper are based on properties of magic rectangles, which are a generalization of the well-known notion of magic squares.

Definition 2.2. A *magic rectangle $MR(a, b)$* is an $a \times b$ array whose entries are $1, 2, \ldots, ab$, each appearing once, with all row sums equal to a constant ρ and all column sums equal to a constant σ.

The complete existence result was proved by T. Harmuth [10, 11] more than 130 years ago.

Theorem 2.3 ([10, 11])**.** *A magic rectangle $MR(a, b)$ exists if and only if $a, b > 1$, $ab > 4$, and $a \equiv b \pmod 2$.*

Magic rectangles can be used to construct 1-handicap graphs of order $n = ab$ and regularity $r = (a - 1)(b - 1)$ (see [2]).

Hagedorn also introduced m-dimensional magic rectangles [9]. We define them here just for the 3-dimensional case, as the higher dimensions are not relevant to our results.

Definition 2.4. A 3-*dimensional magic rectangle $3\text{-}MR(a_1, a_2, a_3)$* is an $a_1 \times a_2 \times a_3$ array with entries r_{i_1, i_2, i_3} which are elements of $\{1, 2, \ldots, a_1 a_2 a_3\}$, each appearing

once, such that all sums in the k-th direction are equal to a constant σ_k. That is, we have

$$\sum_{j=1}^{a_1} r_{j,b_2,b_3} = \sigma_1, \sum_{j=1}^{a_2} r_{b_1,j,b_3} = \sigma_2, \sum_{j=1}^{a_3} r_{b_1,b_2,j} = \sigma_3$$

for every selection of indices b_1, b_2, b_3, and $\sigma_k = a_k(a_1 a_2 a_3 + 1)/2$.

An existence result for 3-dimensional magic rectangles of even order was proved by Hagedorn. His construction actually covers also all higher dimensions, but we again state just the relevant case.

Theorem 2.5 ([9])**.** *A 3-dimensional magic rectangle* 3-$MR(a_1, a_2, a_3)$ *of an even order* $n = a_1 a_2 a_3$ *with* $a_1 \leq a_2 \leq a_3$ *exists if and only if* a_1, a_2, a_3 *are all even and* $a_2 \geq 4$.

3-dimensional magic rectangles can be used for a wider spectrum or regularities. However, since the "magic property" in the third direction is never used in constructions of handicap graphs, and on the other hand Hagedorn's result implies that no such rectangle exists unless all parameters a_i are even, the author introduced in [2] a generalization of magic rectangles, called magic rectangle sets.

Definition 2.6. *A magic rectangle set* $MRS(a, b; c)$ *is a collection of* c *arrays* $\mathcal{R} = \{R^1, R^2, \ldots, R^c\}$, *each of size* $a \times b$ *whose entries are elements of* $\{1, 2, \ldots, abc\}$, *each appearing once, with all row sums in every rectangle equal to a constant* ρ *and all column sums in every rectangle equal to a constant* σ.

Existence of 3-dimensional magic rectangles implies existence of some magic rectangle sets, because slicing a 3-dimensional magic rectangle into single layers in any direction produces magic rectangle sets $MRS(a_i, a_j; a_k)$ for any permutation of $\{i, j, k\} = \{1, 2, 3\}$.

The author proved the existence of magic rectangle sets for the remaining cases that are not covered by Theorem 2.5 in [3]. This gives a complete existence characterization of magic rectangle sets of even order.

Theorem 2.7 ([3, 9])**.** *Let* $n = abc$ *be even and* $a \leq b$. *Then a magic rectangle set* $MR(a, b; c)$ *of order* n *exists if and only if* a, b *are both even and* $b \geq 4$, *while* $c \geq 1$ *is arbitrary.*

We denote by $H(n, k, d)$ a k-regular handicap distance d-antimagic graph of order n. A necessary condition for existence of such graphs was recently proved by the author in [6].

Theorem 2.8. *If there exists a* k-regular 2-handicap graph $H(n, k, 2)$ *of an even order* n, *then* $4 \leq k \leq n - 6$.

The following existence result was proved by the author in [5].

Theorem 2.9. *There exists a* k-regular 2-handicap graph $H(16m, k, 2)$ *of order* $16m$ *for every positive* m *and every even* k *satisfying* $4m + 2 \leq k \leq 12m - 2$.

The result was recently strengthened by the author to obtain a full spectrum for regularity in [6].

Theorem 2.10. *There exists a k-regular 2-handicap graph $H(n, k, 2)$ of order $n \equiv 0 \pmod{16}$ if and only if $4 \le k \le n - 6$.*

3. NEW RESULTS FOR $n \equiv 8 \pmod{16}$

First we observe that odd-regular 2-handicap graphs do not exist.

Theorem 3.1. *If G is a k-regular 2-handicap graph, then k is even.*

Proof. The claim is obviously true for n odd. Now let G be a k-regular 2-handicap graph with n vertices, where n is even, and $k = 2z + 1$. By Definition 2.1, there is a constant μ such that a vertex x_l labeled l has weight $w(x_l) = \mu + 2l$. Hence we have

$$\sum_{l=1}^{n} w(x_l) = \sum_{l=1}^{n} (\mu + 2l) = n\mu + n(n+1). \tag{3.1}$$

On the other hand, each label in the above sum is counted k times, so we also have

$$\sum_{l=1}^{n} w(x_l) = kn(n+1)/2. \tag{3.2}$$

Comparing right-hand sides of (3.1) and (3.2), we obtain

$$n\mu + n(n+1) = kn(n+1)/2$$

and after substituting $k = 2z + 1$ we get

$$2n(\mu + (n+1)) = (2z+1)n(n+1),$$

which yields

$$2(\mu + n + 1) = (2z+1)(n+1).$$

However, since n is even, the right-hand side is odd, while the left-hand side is even, a contradiction. □

We base our construction on magic rectangle sets. A similar approach was used in [6] for $n \equiv 0 \pmod{16}$. In that construction, the set always consisted of two rectangles, regardless of the value of n. In this construction we always use c rectangles of size 2×4, where c is odd and at least 3.

For any $c \ge 3$, we define the magic rectangle set $\mathcal{R} = \text{MRS}(2, 4; c)$ as follows. The entries of R^s are r_{ij}^s for $1 \le s \le c, 1 \le i \le 2, 1 \le j \le 4$.

In R^s for $j = 1, 2$ and $s = 1, 2, \ldots, c$ we have

$$r_{i1}^s = \begin{cases} 2s - 1 & \text{for } i = 1, \\ 8c - 2s + 2 & \text{for } i = 2, \end{cases}$$

and

$$r_{i2}^s = \begin{cases} 8c - 2s + 1 & \text{for } i = 1, \\ 2s & \text{for } i = 2. \end{cases}$$

For $j = 3, 4$ and $s = 1, 2, \ldots, c - 1$ we have

$$r_{i3}^s = \begin{cases} 4c + 2s + 2 & \text{for } i = 1, \\ 4c - 2s - 1 & \text{for } i = 2, \end{cases}$$

$$r_{i4}^s = \begin{cases} 4c - 2s & \text{for } i = 1, \\ 4c + 2s + 1 & \text{for } i = 2. \end{cases}$$

Finally,

$$r_{i3}^c = \begin{cases} 4c + 2 & \text{for } i = 1, \\ 4c - 1 & \text{for } i = 2 \end{cases}$$

and

$$r_{i4}^c = \begin{cases} 4c & \text{for } i = 1, \\ 4c + 1 & \text{for } i = 2. \end{cases}$$

A small example is shown in Figure 1.

1	23	16	10
24	2	9	15

3	21	18	8
22	4	7	17

5	19	14	12
20	6	11	13

Fig. 1. MRS(2, 4; 3)

Now we prove existence of 6-regular 2-handicap graphs of order $n \equiv 8 \pmod{16}$.

Theorem 3.2. *There exists a* 6-*regular* 2-*handicap graph* $H(8c, 6, 2)$ *of order* 8c *for every odd c, c $\geq$ 5.*

Proof. We construct a 6-regular graph $H(8c, 6, 2)$ using the magic rectangle set $\mathcal{R} = \text{MRS}(2, 4; c)$ defined above. We denote the vertices x_{ij}^s for $s = 1, 2, \ldots, c, i = 1, 2$ and $j = 1, 2, 3, 4$ and define a 2-handicap labeling as $f(x_{ij}^s) = r_{ij}^s$, where r_{ij}^s is an entry of $\mathcal{R}$.

We construct $H(8c, 6, 2)$ in three steps. First, we create c copies of 3-regular graphs $K_{4,4} - M$, where M is a perfect matching $4K_2$. Let $K^s = (V^s, E^s)$ be the s-th copy with bipartition $V^s = \{x_{1j}^s \mid j = 1, 2, 3, 4\} \cup \{x_{2t}^s \mid t = 1, 2, 3, 4\}$ and edges $x_{1j}^s x_{2t}^s$ for $j, t \in \{1, 2, 3, 4\}$ and $j \neq t$. Then we add edges joining vertices whose labels differ by $4c$, namely $x_{i3}^s x_{i+1\,2}^{s+1}$ and $x_{i4}^s x_{i+1\,1}^{s+1}$ where the superscripts are calculated modulo c and the first subscripts modulo 2.

Observe that the labels in each partite set in both copies of $K_{4,4} - M$ come from one row of one rectangle in $\mathcal{R}$. The sum of all labels is $8c(8c + 1)/2$ and we have

c copies of $K_{4,4} - M$ and hence $2c$ partite sets, each corresponding to one row in a rectangle in $\mathcal{R}$, which means that the sum of the labels in each partite set is equal to $\rho = 2(8c + 1)$.

The temporary weight $w'(x_{ij}^s)$ is now for $j = 1$

$$w'(x_{i1}^s) = \rho - f(x_{i+1\ 1}^s) + f(x_{i+1\ 4}^{s-1}) = \rho - r_{i+1\ 1}^s + r_{i+1\ 4}^{s-1}$$

and similarly for $j = 2$

$$w'(x_{i2}^s) = \rho - f(x_{i+1\ 2}^s) + f(x_{i+1\ 3}^{s-1}) = \rho - r_{i+1\ 2}^s + r_{i+1\ 3}^{s-1},$$

while for $j = 3$ we have

$$w'(x_{i3}^s) = \rho - f(x_{i+1\ 3}^s) + f(x_{i+1\ 2}^{s+1}) = \rho - r_{i+1\ 3}^s + r_{i+1\ 2}^{s+1}$$

and for $j = 4$ it is

$$w'(x_{i4}^s) = \rho - f(x_{i+1\ 4}^s) + f(x_{i+1\ 1}^{s+1}) = \rho - r_{i+1\ 4}^s + r_{i+1\ 1}^{s+1}.$$

When $f(x_{ij}^s) = r_{ij}^s = l \leq 4c$, then the neighbor of x_{ij}^s in K^{s+1} or K^{s-1} is labeled $l + 4c$.

Hence, for $l \leq 4c$ we obtain

$$w'(x_{ij}^s) = 2c(8c + 1) - (8c + 1 - l) + l + 4c = 16c^2 - 2c + 2l - 1$$

and for $l > 4c$ we have

$$w'(x_{ij}^s) = 2c(8c + 1) - (8c + 1 - l) + l - 4c = 16c^2 - 10c + 2l - 1.$$

We observe that when labels of two vertices differ by $4c$, they both have the same temporary weight, since $2(l + 4c) \equiv 2l \pmod{8c}$. Therefore, we now want to add edges so that the resulting weight of the $4c$ higher-labeled vertices would increase by $8c$ more than the weights of the lower-labeled vertices.

We therefore identify pairs of vertices whose labels add up to $4c + 1$, and pairs whose labels add up to $12c + 1$. Then we join each pair of the former ones to a pair of the latter, avoiding using edges that have already been used.

We observe that

$$f(x_{14}^s) + f(x_{11}^{s+1}) = f(x_{23}^s) + f(x_{22}^{s+1}) = 4c + 1$$

and

$$f(x_{13}^s) + f(x_{12}^{s+1}) = f(x_{24}^s) + f(x_{21}^{s+1}) = 12c + 1.$$

Hence, we add edges between lower-labeled pairs $[x_{14}^s, x_{11}^{s+1}]$ and $[x_{23}^{s+2}, x_{22}^{s+3}]$ and also between higher-labeled pairs $[x_{13}^s, x_{12}^{s+1}]$ and $[x_{24}^{s+2}, x_{21}^{s+3}]$.

Now we have for x_{ij}^s with $f(x_{ij}^s) = l \leq 8c$ the total weight

$$w(x_{ij}^s) = w'(x_{ij}^s) + 4c + 1 = (16c^2 - 2c + 2l - 1) + 4c + 1 = 16c^2 + 2c + 2l$$

and for x_{ij}^s with $f(x_{ij}^s) = l > 8c$ we have

$$w(x_{ij}^s) = w'(x_{ij}^s) + 12c + 1 = (16c^2 - 10c + 2l - 1) + 12c + 1 = 16c^2 + 2c + 2l.$$

Denoting $16c^2 + 2c = \mu$ we can see that for a vertex x_{ij}^s with $f(x_{ij}^s) = l$ we have $w(x_{ij}^s) = \mu + 2l$. This means that if we order the vertices in an arithmetic sequence according to the sequence of their labels, say $f(y_m) = m$, then $w(y_m) = \mu + 2m$ as required by the definition of 2-handicap labeling.

This completes the proof. □

Next we construct even-regular 2-handicap graphs of higher degree. This can be achieved by placing edges of graphs $K_{2,2}$ between appropriate pairs of vertices whose labels add up to $8c + 1$.

Before we do that, we need some decomposition lemmas. We recall that the *lexicographic product* of C_m and rK_1 (also called *blown-up cycle*) denoted $C_m \circ rK_1$ is the graph with vertex set $U = \{u_{ij} \mid i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, r\}$ and edge set $F = \{u_{ij}u_{i+1\,t} \mid i = 1, 2, \ldots, m; \ j, t = 1, 2, \ldots, r\}$.

Lemma 3.3. *Every blown-up cycle $G = C_m \circ 4K_1$ allows a one-factorization.*

Proof. For m even, say $m = 2m'$, the proof is obvious. First we decompose G into two 4-factors $m'K_{4,4}$ and then simultaneously one-factorize the complete bipartite graphs $K_{4,4}$.

For $m = 2m' + 1$, we first decompose G into non-isomorphic 2-factors consisting of even cycles of lengths $4m' + 2$ or $8m' + 4$, and then split each such 2-factor into two 1-factors.

The first two 2-factors, G_1 and G_2, are constructed by first taking paths of length $2m'$ and then joining them into a single $(8m' + 4)$-cycle. Factor G_1 consists of paths $u_{1j}, u_{2\,j+2}, u_{3\,j+2}, \ldots, u_{2m'+1\,j+2}$ for $j = 1, 2, 3, 4$. The additional edges completing the $(8m' + 4)$-cycle are $u_{2m'+1\,1}u_{11}, u_{2m'+1\,2}u_{12}, u_{2m'+1\,3}u_{14}, u_{2m'+1\,4}u_{13}$. G_2 consists of paths $u_{1j}, u_{2\,j}, u_{3\,j+2}, u_{4j}, \ldots, u_{2m'+1\,j+2}$. The additional edges are $u_{2m'+1\,1}u_{12}, u_{2m'+1\,2}u_{11}, u_{2m'+1\,3}u_{13}, u_{2m'+1\,4}u_{14}$.

Factor G_3 consists of paths $u_{1j}, u_{2\,j+1}, u_{3j}, u_{4\,j+1}, \ldots, u_{2m'+1\,j}$. Adding edges $u_{2m'+1\,1}u_{14}$ and $u_{2m'+1\,4}u_{11}$ we obtain one $(4m' + 2)$-cycle, and edges $u_{2m'+1\,2}u_{13}$ and $u_{2m'+1\,3}u_{12}$ complete the other $(4m' + 2)$-cycle.

Finally, G_4 consists of paths $u_{1j}, u_{2\,j+3}, u_{3j}, u_{4\,j+3}, \ldots, u_{2m'+1\,j}$. We add edges $u_{2m'+1\,1}u_{13}$ and $u_{2m'+1\,3}u_{11}$ to get one $(4m'+2)$-cycle, and $u_{2m'+1\,2}u_{14}$ and $u_{2m'+1\,4}u_{12}$ to get the other $(4m' + 2)$-cycle. □

Lemma 3.4. *The graph $H = C_m \circ 8K_1$ allows a $K_{2,2}$-factorization into graphs $2mK_{2,2}$.*

Proof. Clearly, $H = G \circ 2K_1$. We blow up each vertex u_{ij} into a pair of vertices (v_{ij}, w_{ij}) and every edge into $K_{2,2}$. Hence, every edge in a 1-factorization of G becomes $K_{2,2}$ as well, providing the required $K_{2,2}$-factorization of H. □

We observe that in $K_{2c'+1}$ for a fixed $p \leq c'$ all edges of the form z_iz_{i+p} with $i = 1, 2, \ldots, 2c' + 1$ form a 2-factor consisting of cycles of an odd length

$(2c' + 1)/\gcd(2c' + 1, p)$. If we now replace $K_{2c'+1}$ by $K_{2c'+1} \circ 8K_1$, each cycle C_m in the 2-factor becomes $C_m \circ 8K_1$ and can be decomposed into $K_{2,2}$-factors.

Theorem 3.5. *There exists a k-regular 2-handicap graph of order n for every positive $n \equiv 8 \pmod{16}, n \geq 56$ and every even k satisfying $6 \leq k \leq n - 50$.*

Proof. We first recall that $f(x_{ij}^s) + f(x_{i+1\ j}^s) = 8c + 1$. Next we need to identify pairs of vertices where no edges have been used in the construction of graph $H(8c, 6, 2)$ leading to Theorem 3.2.

Indeed, we cannot use edges within copies K^s. Now for each pair $[x_{ij}^s, x_{i+1\ j}^s]$ we need to find all "forbidden" pairs $[x_{ij}^t, x_{i+1\ j}^t]$, that is, pairs with the property that at least one edge between the pairs of vertices has already been used.

First we look at $[x_{11}^s, x_{21}^s]$. We used edges $x_{11}^s x_{24}^{s-1}$ and $x_{21}^s x_{14}^{s-1}$ in the first step and $x_{11}^s x_{23}^{s+1}, x_{11}^s x_{22}^{s+2}$ for x_{11}^s and $x_{21}^s x_{13}^{s-3}, x_{21}^s x_{12}^{s-2}$ for x_{21}^s in the second step. Hence, the forbidden pairs are

$$[x_{13}^{s-3}, x_{23}^{s-3}], [x_{12}^{s-2}, x_{22}^{s-2}], [x_{14}^{s-1}, x_{24}^{s-1}], [x_{13}^{s+1}, x_{23}^{s+1}], [x_{12}^{s+2}, x_{22}^{s+2}],$$

total five pairs. Notice that because we have bounded n from below by 56, it follows that $c \geq 7$ and the copies K^{s+2} and K^{s-3} are not identical.

For $[x_{12}^s, x_{22}^s]$, we used edges $x_{12}^s x_{23}^{s-1}$ and $x_{22}^s x_{13}^{s-1}$ in the first step and $x_{12}^s x_{24}^{s+1}, x_{12}^s x_{21}^{s+2}$ for x_{12}^s and $x_{22}^s x_{14}^{s-3}, x_{22}^s x_{11}^{s-2}$ for x_{22}^s in the second step. Hence, there are again five forbidden pairs

$$[x_{14}^{s-3}, x_{24}^{s-3}], [x_{11}^{s-2}, x_{21}^{s-2}], [x_{13}^{s-1}, x_{23}^{s-1}], [x_{14}^{s+1}, x_{24}^{s+1}], [x_{11}^{s+2}, x_{21}^{s+2}].$$

For $[x_{13}^s, x_{23}^s]$, we used edges $x_{13}^s x_{22}^{s+1}$ and $x_{23}^s x_{12}^{s+1}$ in the first step and $x_{13}^s x_{24}^{s+2}, x_{13}^s x_{21}^{s+3}$ for x_{13}^s and $x_{23}^s x_{14}^{s-2}, x_{23}^s x_{11}^{s-1}$ for x_{23}^s in the second step. Hence, the five forbidden pairs are

$$[x_{14}^{s-2}, x_{24}^{s-2}], [x_{11}^{s-1}, x_{21}^{s-1}], [x_{12}^{s+1}, x_{22}^{s+1}], [x_{14}^{s+2}, x_{24}^{s+2}], [x_{11}^{s+3}, x_{21}^{s+3}].$$

Finally, for $[x_{14}^s, x_{24}^s]$, we used edges $x_{14}^s x_{21}^{s+1}$ and $x_{24}^s x_{11}^{s+1}$ in the first step and $x_{14}^s x_{23}^{s+2}, x_{14}^s x_{22}^{s+3}$ for x_{14}^s and $x_{24}^s x_{13}^{s-2}, x_{24}^s x_{12}^{s-1}$ for x_{24}^s in the second step. Hence, five the forbidden pairs are

$$[x_{13}^{s-2}, x_{23}^{s-2}], [x_{12}^{s-1}, x_{22}^{s-1}], [x_{11}^{s+1}, x_{21}^{s+1}], [x_{13}^{s+2}, x_{23}^{s+2}], [x_{12}^{s+3}, x_{22}^{s+3}].$$

It follows that for each pair $[x_{1j}^s, x_{2j}^s]$, there are five forbidden pairs. We observe that the neighborhoods of all pairs are consistent, because each edge is listed twice. We also observe that all edges are of the form $x_{ij}^s x_{i'j'}^{s+q}$, where $q \in \{1, 2, 3\}$. We can now condense all eight vertices in each K_s into a single vertex z_s to obtain $K_c = K_{2c'+1}$. The edges in forbidden pair correspond to edges $z_s z_{s+q}$ of lengths $1, 2$, and 3. As we observed above, all other lengths can be used to produce 2-factors of $K_{2c'+1}$, which in turn can be blown up to graphs $C_m \circ 8K_1$. Each of them can be decomposed into $K_{2,2}$-factors.

Because in $K_{2c'+1}$ we have edges of c' different lengths, and three of them correspond to the forbidden edges in K_{8c}, we can use the decompositions described in Lemmas 3.3 and 3.4 to increase the regularity by any even number up to $16(c' - 3) = n - 56$, which along with six edges guaranteed by Theorem 3.2 gives the desired result. $\square$

While it is obvious that a careful examination of available edges within the pairs K^s, K^{s+q} could increase the regularity, we chose not to try here. The reason is that this method cannot get regularity higher than $n - 12$, which is still too far from the potential maximum of $n - 6$. Hence, a different method needs to be used. One can try methods similar to those used in papers referred to in [8].

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