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ON THE CHROMATIC NUMBER OF $(P_5, \mathbf{windmill})$-FREE GRAPHS

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Abstract. In this paper we study the chromatic number of $(P_5, \mathit{windmill})$-free graphs. For integers $r, p \geq 2$ the *windmill graph* $W_{r+1}^p = K_1 \vee pK_r$ is the graph obtained by joining a single vertex (the center) to the vertices of p disjoint copies of a complete graph K_r. Our main result is that every $(P_5, \mathit{windmill})$-free graph G admits a polynomial χ-binding function. Moreover, we will present polynomial χ-binding functions for several other subclasses of P_5-free graphs.

Keywords: vertex colouring, perfect graphs, χ-binding function, forbidden induced subgraph.

Mathematics Subject Classification: 05C15, 05C17.

1. INTRODUCTION

We consider finite, simple, and undirected graphs, and use standard terminology and notation.

Let G be a graph. An *induced subgraph* of G is a graph H such that $V(H) \subseteq V(G)$, and $uv \in E(H)$ if and only if $uv \in E(G)$ for all $u, v \in V(H)$. Given graphs G and F we say that G *contains* F if F is isomorphic to an induced subgraph of G. We say that a graph G is F-free, if it does not contain F. For two graphs G, H we denote by $G + H$ the disjoint union and by $G \vee H$ the join of G and H, respectively.

A graph G is called *k-colourable*, if its vertices can be coloured with k colours so that adjacent vertices obtain distinct colours. The smallest k such that a given graph G is k-colourable is called its *chromatic number*, denoted by $\chi(G)$. It is well-known that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$ for any graph G, where $\omega(G)$ denotes its clique number and $\Delta(G)$ its maximum degree. A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

A family $\mathcal{G}$ of graphs is called χ-bound with binding function f if $\chi(G') \leq f(\omega(G'))$ holds whenever $G \in \mathcal{G}$ and G' is an induced subgraph of G. For a fixed graph H let $\mathcal{G}(H)$ denote the family of graphs which are H-free.

The following theorems are well known in chromatic graph theory.

Theorem 1.1 (Erdős [5]). *For any positive integers $k, l \geq 3$ there exists a graph G with girth $g(G) \geq l$ and chromatic number $\chi(G) \geq k$.*

Theorem 1.2 (The Strong Perfect Graph Theorem [4]). *A graph is perfect if and only if it contains neither an induced odd cycle of length at least five nor its complement.*

In this paper we study the chromatic number of P_5-free graphs. Our work was motivated by the following conjecture of Gyárfás.

Conjecture 1.3 (Gyárfás' Conjecture [8]). *Let T be any tree (or forest). Then there is a function f_T such that every T-free graph G satisfies $\chi(G) \leq f_T(\omega(G))$.*

Gyárfás [8] proved this conjecture when T is a path P_k for all $k \geq 4$ by showing

Theorem 1.4. *Let G be a P_k-free graph for $k \geq 4$ with clique number $\omega(G) \geq 2$. Then*

$$\frac{R(\omega+1, \lceil \frac{k}{2} \rceil) - 1}{\lceil \frac{k}{2} \rceil - 1} \leq f_k(\omega) \leq (k-1)^{\omega(G)-1},$$

where $R(s,t)$ is the Ramsey number.

Note that P_4-free graphs are perfect. The currently best known upper bound for P_5-free graphs is due to Esperet, Lemoine, Maffray, and Morel [6].

Theorem 1.5. *Let G be a P_5-free graph with clique number $\omega(G) \geq 3$. Then $\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$.*

One may wonder whether this exponential bound can be improved. In particular:

Question 1.6. *Are there polynomial (χ-binding) functions f_k for $k \geq 5$ such that every P_k-free graph G satisfies $\chi(G) \leq f_k(\omega(G))$?*

If there would be a polynomial (χ-binding) function f_k for some $k \geq 5$, then it would imply the Erdős-Hajnal conjecture for P_k-free graphs. The Erdős-Hajnal conjecture states that for every graph H, there exists a constant $\delta(H) > 0$ such that every graph G with no induced subgraph isomorphic to H has either a clique or a stable set of size at least $|V(G)|^{\delta(H)}$. However, the Erdős-Hajnal conjecture is still open for P_k-free graphs for all $k \geq 5$ (cf. [3] for a survey).

2. GENERAL GRAPHS

One of the earliest results is due to Wagon, who has considered graphs without induced matchings.

Theorem 2.1 ([10]). *Let G be a $2K_2$-free graph with clique number $\omega(G)$. Then $\chi(G) \leq \binom{\omega(G)+1}{2}$.*

This theorem admits a nice generalization as follows.

Theorem 2.2. *Let H be a graph such that $\mathcal{G}(H)$ has an $O(\omega^t)$ χ-binding function for some $t \geq 1$, and let G be a $K_2 + H$-free graph with clique number $\omega(G)$. Then G has an $O(\omega^{2+t})$ χ-binding function.*

The proof for Theorem 2.2 will be given after the proof of Theorem 3.6.

Theorem 2.3 ([10])**.** *The family $\mathcal{G}(pK_2)$ has an $O(\omega^{2p-2})$ χ-binding function for all $p \geq 1$.*

Note that the statement of Theorem 2.3 can be made more precise as follows. In [10] a χ-binding function $f_p(\omega)$ for the class of pK_2-free graphs was defined by $f_1(\omega) = 1, f_{p+1}(\omega) = \binom{\omega}{2} f_p(\omega) + \omega$. From this one can deduce that $f_p(\omega) \leq (\omega+1)\frac{\omega^{2p-3}}{2^{p-1}}$ for all $p \geq 1$.

Like Theorem 2.2 for Theorem 2.1, Theorem 2.3 has the following counterpart.

Theorem 2.4. *Let H be a graph such that $\mathcal{G}(H)$ has an $O(\omega^t)$ χ-binding function for some $t \geq 1$, and let G be a $pK_2 + H$-free graph with clique number $\omega(G)$. Then G has an $O(\omega^{2p-2+t})$ χ-binding function.*

3. P_k-FREE GRAPHS

In this section we will consider P_k-free graphs. Since P_4-free graphs are perfect graphs, we may assume $k \geq 5$. For the presentation of our results we will need several forbidden induced subgraphs, which are presented in Figure 1 and Figure 2. The following results have been shown for P_5-free graphs.

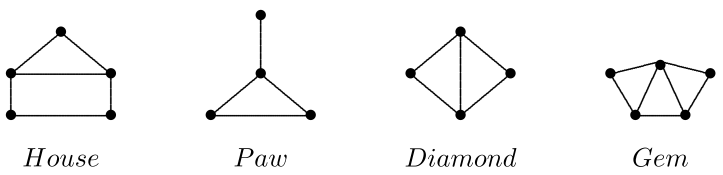

Fig. 1. The graphs *House*, *Paw*, *Diamond*, and *Gem*.

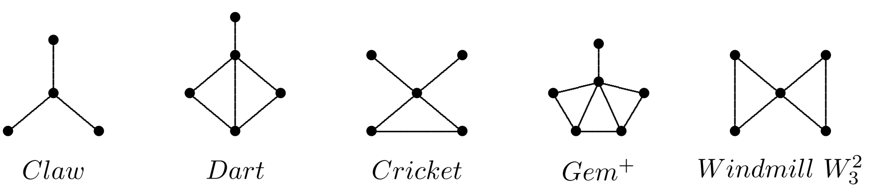

Fig. 2. The graphs *Claw*, *Dart*, *Cricket*, *Gem*$^+$, and *Windmill*

Theorem 3.1 ([7]). *Let G be a connected $(P_5, House)$-free graph of order n and clique number $\omega(G)$. Then $\chi(G) \leq \binom{\omega(G)+1}{2}$.*

Theorem 3.2 ([2]). *Let G be a connected (P_5, Gem)-free graph of order n and clique number $\omega(G)$. Then $\chi(G) \leq 6\omega(G)$.*

Since the Paw and the Diamond are both induced subgraphs of the Gem, we obtain the following corollary.

Corollary 3.3. *Let G be a connected (P_5, H)-free graph of order n and clique number $\omega(G)$, where $H \in \{Paw, Diamond\}$. Then $\chi(G) \leq 6\omega(G)$.*

In [9] the subgraph *Gem* was replaced by the supergraph $Gem^+ = K_1 \vee (K_1 + P_4)$.

Theorem 3.4. *Let G be a (P_5, Gem^+)-free graph of order n and clique number $\omega(G)$. Then $\chi(G) \leq \omega^2(G)$.*

Since the Claw, the Dart and the Cricket are induced subgraphs of the Gem^+, we obtain the following corollary.

Corollary 3.5. *Let G be a connected (P_5, H)-free graph of order n and clique number $\omega(G)$, where $H \in \{Claw, Dart, Cricket\}$. Then $\chi(G) \leq \omega^2(G)$.*

We start by proving a generalization of Theorem 2.1 for P_5-free graphs.

Theorem 3.6. *Let G be a $(P_k, K_{n_1} + K_{n_2})$-free graph for some $n_1 \geq n_2 \geq 2$. Then $\chi(G) \leq c(n_1) \cdot \omega^{n_1}$ for a constant $c(n_1)$.*

Proof. Let $\omega = \omega(G)$ and let F be a complete subgraph of G with $|V(F)| = \omega$. For a subset $T \subset V(F)$ with $|T| = t$ and $1 \leq t \leq n_1 - 1$, let

$$M(T) = \{v \in V(G) \setminus F \mid N(v) \cap V(F) = V(F) \setminus T\}.$$

Then $G[M(T)]$ is K_{t+1}-free, since otherwise there would be a complete subgraph of G of order at least $(\omega - t) + (t+1) = \omega + 1$, a contradiction. Hence $\chi(G[M(T)]) \leq f_{P_k}(t)$. For a subset $T \subset V(F)$ with $|T| = n_1$ let

$$M(T) = \{v \in V(G) \setminus F \mid N(v) \cap V(F) \subseteq V(F) \setminus T\}.$$

Then $G[M(T)]$ is K_{n_2}-free, since G is $(K_{n_1} + K_{n_2})$-free. Hence we have $\chi(G[M(T)]) \leq f_{P_5}(n_2 - 1)$.

We now colour the vertices of G as follows. The vertices of F obtain ω distinct colours. For every vertex $w \in V(F)$, the set $w \cup M(\{w\})$ is independent. So all vertices of $M(\{w\})$ obtain the same colour as w. Next for every subset $T \subset V(F)$ with $2 \leq t \leq n_1$ we choose a private set of $f_{P_k}(t)$ colours. For every t with $2 \leq t \leq n_1$ there are $\binom{\omega}{t}$ subsets $T \subset V(F)$ with $|T| = t$. So we obtain

$$\chi(G) \leq \omega + \sum_{t=2}^{n_1} \binom{\omega}{t} f_{P_k}(t),$$

which is a polynomial of degree n_1 in ω. □

Proof of Theorem 2.2. We can follow the proof of Theorem 3.6 with the following modification. With $n_1 = 2$ and K_{n_2} replaced by H we obtain

$$\chi(G) \leq \omega + \binom{\omega}{2} O(\omega^t).$$

Hence G has an $O(\omega^{2+t})$ χ-binding function. □

Theorem 3.6 can be generalized to $(P_k, K_{n_1} + K_{n_2} + \ldots + K_{n_p})$-free graphs for $p \geq 3$ and $n_1 \geq n_2 \geq \ldots \geq n_p$ as follows. We use the proof above as an induction step with the following modification. For a subset $T \subset V(F)$ with $|T| = n_1$ the subgraph $G[M(T)]$ is $(P_k, K_{n_2} + \ldots + K_{n_p})$-free. Hence

$$\chi(G[M(T)]) \leq c(n_2, \ldots, n_p) \cdot \omega^{\sum_{i=2}^{p-1} n_i},$$

which leads to

$$\chi(G) \leq \omega + \sum_{t=2}^{n_1-1} \binom{\omega}{t} f_{P_k}(t) + \binom{\omega}{n_1} \cdot c(n_2, \ldots, n_p) \cdot \omega^{\sum_{i=2}^{p-1} n_i},$$

which is a polynomial of degree $\sum_{i=1}^{p-1} n_i$ in ω. So we obtain the following result.

Theorem 3.7. *Let G be a $(P_k, K_{n_1} + K_{n_2} + \ldots + K_{n_p})$-free graph for some $p \geq 2$ and $n_1 \geq n_2 \geq \ldots \geq n_p \geq 2$. Then*

$$\chi(G) \leq c(n_1, \ldots, n_p) \cdot \omega^{\sum_{i=1}^{p-1} n_i}$$

for a constant $c(n_1, \ldots, n_p)$.

Theorem 3.6 also leads to the following variation.

Theorem 3.8. *Let G be a $(P_k, K_{n_1} + P_4)$-free graph for some $n_1 \geq 2$. Then $\chi(G) \leq c(n_1) \cdot \omega^{n_1+1}$ for a constant $c(n_1)$.*

Proof. We follow the proof of Theorem 3.6 with the following modification:
For a subset $T \subset V(F)$ with $|T| = n_1$ let

$$M(T) = \{v \in V(G) \setminus F \mid N(v) \cap V(F) \subseteq V(F) \setminus T\}.$$

Then $G[M(T)]$ is P_4-free, since G is $(K_{n_1} + P_4)$-free. Hence $\chi(G[M(T)]) \leq \omega$, since P_4-free graphs are perfect graphs. So we obtain

$$\chi(G) \leq \omega + \sum_{t=2}^{n_1-1} \binom{\omega}{t} f_{P_k}(t) + \binom{\omega}{n_1} \omega,$$

which is a polynomial of degree $n_1 + 1$ in ω. □

The counterpart of Theorem 2.2 for P_k-free graphs is the following.

Theorem 3.9. *Let H be a graph such that $\mathcal{G}(H)$ has an $O(\omega^t)$ χ-binding function for some $t \geq 1$, and let G be a $(P_k, K_{n_1} + H)$-free graph for some $n_1 \geq 2$. Then $\chi(G) \leq c(n_1, H) \cdot \omega^{n_1+t}$ for a constant $c(n_1, H)$.*

4. $(P_5, windmill)$-FREE GRAPHS

For integers $r, p \geq 2$ the *windmill graph* $W_{r+1}^p = K_1 \vee pK_r$ is the graph obtained by joining a single vertex (the center) to the vertices of p disjoint copies of a complete graph K_r (the Windmill W_3^2 is shown in Figure 2). For integers $n_1 \geq n_2 \geq \ldots \geq n_p \geq 2$, the *generalized windmill graph* $W(n_1, n_2, \ldots, n_p) = K_1 \vee (K_{n_1} + K_{n_2} + \ldots + K_{n_p})$ is the graph obtained by joining a single vertex (the center) to the vertices of p disjoint complete graphs $K_{n_1}, \ldots, K_{n_p}$.

We start with a structural result for connected P_5-free graphs.

Theorem 4.1 (Bacsó and Tuza [1]). *Every connected P_5-free graph contains a dominating clique or a dominating P_3.*

This admits the following result for P_5-free graphs.

Theorem 4.2. *Let H be a graph such that $\mathcal{G}(H)$ has an $O(\omega^t)$ χ-binding function for some $t \geq 1$, and let G be a connected $(P_5, K_1 \vee H)$-free graph with clique number $\omega(G)$. Then G has an $O(\omega^{t+1})$ χ-binding function.*

Proof. Let $D = \{w_1, w_2, \ldots, w_d\}$ be a dominating set as in Theorem 4.1 with $d = |D|$. We may assume $\omega \geq 3$, since otherwise $\chi(G) \leq 3$ by Theorem 1.4. For $1 \leq i \leq d$, let $G_i = \{w_i\} \cup (N(w_i) \cap (V(G) - D))$. Then $\chi(G) \leq \sum_{i=1}^{d} \chi(G_i)$. Moreover, if $G_i - w_i$ is H-free, then G_i is $(K_1 \vee H)$-free. Then G has an $O(\omega^{t+1})$ χ-binding function, since H has an $O(\omega^t)$ χ-binding function for some $t \geq 1$ and $d \leq \omega$. □

So we can apply Theorem 4.2 to obtain the following results for $(P_5, windmill)$-free graphs.

Theorem 4.3. *Let G be a $(P_5, W(n_1, n_2))$-free graph for some $n_1 \geq n_2 \geq 2$. Then $\chi(G) \leq c(n_1) \cdot \omega^{n_1+1}$ for a constant $c(n_1)$.*

Theorem 4.4. *Let G be a $(P_5, W(n_1, n_2, \ldots, n_p))$-free graph for some $p \geq 2$ and $n_1 \geq n_2 \geq \ldots \geq n_p \geq 2$. Then*

$$\chi(G) \leq c(n_1, \ldots, n_p) \cdot \omega^{1+\sum_{i=1}^{p-1} n_i}$$

for a constant $c(n_1, \ldots, n_p)$.

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