FAN’S CONDITION ON INDUCED SUBGRAPHS FOR CIRCUMFERENCE AND PANCYCLICITY

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Abstract. Let $\mathcal{H}$ be a family of simple graphs and $k$ be a positive integer. We say that a graph $G$ of order $n \geq k$ satisfies Fan’s condition with respect to $\mathcal{H}$ with constant $k$, if for every induced subgraph $H$ of $G$ isomorphic to any of the graphs from $\mathcal{H}$ the following holds:

$$\forall u, v \in V(H): d_H(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2.$$ 

If $G$ satisfies the above condition, we write $G \in \mathcal{F}(\mathcal{H}, k)$. In this paper we show that if $G$ is 2-connected and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, then $G$ contains a cycle of length at least $k$, and that if $G \in \mathcal{F}(\{K_{1,3}, P_4\}, n)$, then $G$ is pancyclic with some exceptions. As corollaries we obtain the previous results by Fan, Benhocine and Wojda, and Ning.

Keywords: Fan’s condition, circumference, hamiltonian cycle, pancyclicity.

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1. INTRODUCTION

In the paper we consider only simple, finite and connected graphs. For basic terminology not defined here we use [6].

Let $G$ be a graph of order $n$. $G$ is said to be hamiltonian, if it contains a cycle $C_n$, and it is called pancyclic, if it contains cycles of all lengths $k$ for $3 \leq k \leq n$. The length of a longest cycle in $G$ is called the circumference of $G$ and denoted $c(G)$.

For a family of graphs $\mathcal{H}$ we say that $G$ is $\mathcal{H}$-free if for every $H \in \mathcal{H}$ there are no induced copies of $H$ in $G$. If one demands $G$ being $\mathcal{H}$-free, then $\mathcal{H}$ is forbidden in $G$. Pairs of forbidden subgraphs ensuring hamiltonicity or pancyclicity of 2-connected graphs were extensively examined by a number of researchers during the last forty years (e.g., [7,10,14,15]). These results were gathered by Bedrossian in his Ph.D. thesis. Together with the results he obtained, they can be formulated in the following way (graphs $Z_i$, $B$, $W$ and $N$ are represented on Figure 1; the “only if” parts of the below Theorems are due to Faudree and Gould).
Theorem 1.1 (Bedrossian [1]; Faudree and Gould [12]). Let $R$ and $S$ be connected graphs with $R, S \neq P_3$ and let $G$ be a 2-connected graph. Then $G$ being $\{R, S\}$-free implies $G$ is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or $W$.

Theorem 1.2 (Bedrossian [1]; Faudree and Gould [12]). Let $R$ and $S$ be connected graphs with $R, S \neq P_3$ and let $G$ be a 2-connected graph which is not a cycle. Then $G$ being $\{R, S\}$-free implies $G$ is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or $Z_2$.

The reason for the path $P_3$ being excluded from the assumptions of the above Theorems is that the only 2-connected $P_3$-free graph is a complete graph, which is obviously both hamiltonian and pancyclic.

Another popular approach to the problem of existence of cycles in graphs is the one involving degree conditions. A classical result in this field is due to Fan.

Theorem 1.3 (Fan [11]). Let $G$ be a 2-connected graph with $n$ vertices and let $3 \leq k \leq n$. If

$$d_G(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2$$

for each pair of vertices $u$ and $v$ in $G$, then $c(G) \geq k$.

In the particular case when $k = n$, Fan’s Theorem implies hamiltonicity of $G$. It was later shown by Benhocine and Wojda that this condition ensures in fact pancyclicity, besides three exceptional graphs (for the graph $F_{4r}$ which consists of a clique of order $2r$ joined via perfect matching with $r$ disjoint copies of a path with two vertices, see Figure 2).
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Theorem 1.4 (Benhocine and Wojda [3]). Let $G$ be a 2-connected graph with $n \geq 3$. If
\[d_G(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq n/2\]
for each pair of vertices $u$ and $v$ in $G$, then $G$ is pancyclic unless $n = 4r$, $r \geq 2$ and $G$ is $F_{4r}$, or $n$ is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.

A natural way of relaxing forbidden subgraphs conditions is to allow these subgraphs to be present in a graph but with a Fan-type degree conditions imposed on them. This idea was explored by many researchers, using various terminology and notations. Before we state their results, we introduce a notion that encapsulates these different notations.

Definition 1.5. Let $\mathcal{H}$ be a family of graphs and $k$ be a positive integer. We say that a graph $G$ satisfies Fan's condition with respect to $\mathcal{H}$ with constant $k$, if for every induced subgraph $H$ of $G$ isomorphic to any of the graphs from $\mathcal{H}$ the following holds:
\[
\forall u, v \in V(H): d_H(u, v) = 2 \Rightarrow \max\{d_G(u), d_G(v)\} \geq k/2.
\]

By $\mathcal{F}(\mathcal{H}, k)$ we denote the family of graphs satisfying the Fan's condition with respect to $\mathcal{H}$ with constant $k$. If $\mathcal{H}$ consists of one element, say $H$, we write $\mathcal{F}(H, k)$ instead of $\mathcal{F}(\{H\}, k)$.

Given a family of graphs $\mathcal{H}$ and a constant $k$, every $\mathcal{H}$-free graph satisfies Fan’s condition with respect to $\mathcal{H}$ with constant $k$. It is also clear, that if $G \in \mathcal{F}(P_3, k)$, then $G \in \mathcal{F}(\mathcal{H}, k)$. The authors of [2] imposed the Fan’s condition on one of the pairs of subgraphs that appear in Theorems 1.1 and 1.2 and obtained the following results.

Theorem 1.6 (Bedrossian, Chen, and Schelp [2]). Let $G$ be a 2-connected graph with $n$ vertices. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, Z_1\}, k)$, then $\chi(G) \geq k$.

Theorem 1.7 (Bedrossian, Chen, and Schelp [2]). Let $G$ be a 2-connected graph of order $n \geq 3$ which is not a cycle. If $G \in \mathcal{F}(\{K_{1,3}, Z_1\}, n)$, then $G$ is pancyclic unless $n = 4r$, $r \geq 2$ and $G$ is $F_{4r}$, or $n$ is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$. 

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{fan_graph}
\caption{Fan’s graph $F_{4r}$}
\end{figure}
Theorems 1.1–1.7 were the main motivation for our research. Similarly to Theorems 1.6 and 1.7, in this paper we prove the following.

**Theorem 1.8.** Let $G$ be a 2-connected graph with $n$ vertices. If $3 \leq k \leq n$ and $G \in F(\{K_{1,3}, P_4\}, k)$, then $c(G) \geq k$.

**Theorem 1.9.** Let $G$ be a 2-connected graph of order $n \geq 3$. If $G \in F(\{K_{1,3}, P_4\}, n)$, then $G$ is pancyclic unless $n = 4r$, $r \geq 2$ and $G$ is $F_{4r}$, or $n$ is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.

It is easy to see that Theorem 1.3 is a corollary from Theorem 1.8 and that Theorem 1.4 follows from Theorem 1.9. Note also that from the exceptional non-pancyclic graphs mentioned in Theorem 1.9 only the cycle $K_{2,2}$ satisfies Fan’s condition with respect to $\{K_{1,3}, P_4\}$ with constant $n + 1$. Hence, the following result also can be deduced from Theorem 1.9.

**Theorem 1.10** (Ning [18]). Let $G$ be a 2-connected graph with $n$ vertices other than $K_{2,2}$. If $G \in F(\{K_{1,3}, P_4\}, n + 1)$, then $G$ is pancyclic.

The above Theorem is one of the many results connecting the Fan-type condition with hamiltonicity and pancyclicity. Some of these results fully extend Theorems 1.1 and 1.2.

**Theorem 1.11.** Let $R$ and $S$ be connected graphs with $R, S \neq P_3$ and let $G$ be a 2-connected graph of order $n$. Then $G \in F(\{R, S\}, n)$ implies $G$ is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S$ is one of the following:

- $P_4, P_5, P_6$ (Chen, Wei, and Zhang [8]),
- $Z_1$ (Bedrossian, Chen, and Schelp [2]),
- $B$ (Li, Wei and Gao [17]),
- $N$ (Chen, Wei and Zhang [9]),
- $Z_2, W$ (Ning and Zhang [19]).

**Theorem 1.12.** Let $R$ and $S$ be connected graphs with $R, S \neq P_3$ and let $G$ be a 2-connected graph of order $n$ which is not a cycle. Then $G \in F(\{R, S\}, n + 1)$ implies $G$ is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S$ is one of the following:

- $Z_1$ (Bedrossian, Chen, and Schelp [2]),
- $Z_2, P_4$ (Ning [18]),
- $P_5$ (Widel [21]).

To close this section we propose the following two conjectures, which seem to be justified in view of the results presented so far.

**Conjecture 1.13.** Let $R$ and $S$ be connected graphs with $R, S \neq P_3$ and let $G$ be a 2-connected graph with $n$ vertices. If $3 \leq k \leq n$, then $G \in F(\{R, S\}, k)$ implies $c(G) \geq k$ if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or $W$. 
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Conjecture 1.14. Let $R$ and $S$ be connected graphs with $R, S \neq P_3$ and let $G$ be a 2-connected graph of order $n$ with $G \notin \{C_n, F_{4(n/4)}, K_{n/2, n/2} - e\}$. Then $G \in \mathcal{F}(\{R, S\}, n)$ implies $G$ is pancyclic if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, Z_1$ or $Z_2$.

In the next section we introduce terminology and notation used throughout the rest of the paper. The proofs of Theorems 1.8 and 1.9 are presented in Sections 3 and 4, respectively.

2. PRELIMINARIES

For a vertex $v \in V(G)$, we denote by $N_G(v)$ the neighbourhood of $v$, i.e., the set of vertices adjacent to $v$. For $A \subseteq V(G)$, we denote $G[A]$ the subgraph of $G$ induced by the vertex set $A$. The neighbourhood of $v$ in $G[A]$, namely $N_G(v) \cap A$, is denoted by $N_A(v)$ and the closed neighbourhood of $v$ in $G[A]$, namely $N_A(v) \cup \{v\}$, is denoted by $N_A[v]$.

The complete bipartite graph $K_{1,3}$ is called a claw. The vertex of degree three of a claw is called its center vertex and the other vertices are its end vertices. Let $A = \{v_1, v_2, v_3, v_4\}$. If $G[A]$ is isomorphic to $K_{1,3}$, with $v_1$ being a center vertex of the claw and $v_2, v_3$ and $v_4$ being its end vertices, we say that $\{v_1; v_2, v_3, v_4\}$ induces $K_{1,3}$ (or induces a claw).

For a cycle $C$ we distinguish one of two possible orientations of $C$. We write $xC^+y$ for the path from $x \in V(C)$ to $y \in V(C)$ following the orientation of $C$, and $xC^-y$ denotes the path from $x$ to $y$ opposite to the direction of $C$. By $d_C(x, y)$ we denote the length of the shorter of the paths $xC^+y$ and $xC^-y$. Similarly, for a path $P = v_1 \ldots v_m$ and two vertices $v_i, v_j \in V(P)$ with $i < j$, we write $v_iP^+v_j$ for the path $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ and $v_jP^-v_i$ for the path $v_j, v_{j-1}, \ldots, v_{i+1}, v_i$. For two positive integers $k$ and $m$, where $k \leq m$, we say that $G$ contains $[k, m]$-cycles if there are cycles $C_k, C_{k+1}, \ldots, C_m$ in $G$.

Let $G$ be a graph of order $n$. Vertex $v \in V(G)$ is called heavy if $d_G(v) \geq n/2$.

Let $A, B \subset V(G)$ be subsets of vertices of $G$. By

$$e(A, B) = |\{e = uv \in E(G): u \in A, v \in B\}|$$

we denote the total number of edges between $A$ and $B$. If both $A$ and $B$ consist of one element, say $A = \{v_A\}$ and $B = \{v_B\}$, we write $e(v_A, v_B)$ instead of $e(\{v_A\}, \{v_B\})$.

3. PROOF OF THEOREM 1.8

We utilize the general idea of the proof of Theorem 1.6. Before we begin the proof, we need to state the following.

**Theorem 3.1** (Bondy [4]). Let $G$ be a 2-connected graph of order $|V(G)| \geq k$ and let $P = v_1 \ldots v_m$ be a path of maximum length in $G$. If $d_G(v_1) + d_G(v_m) \geq k$, then $c(G) \geq k$. 

For the convenience of the reader, we restate Theorem 1.8 below.

**Theorem 1.8.** Let $G$ be a 2-connected graph with $n$ vertices. If $3 \leq k \leq n$ and $G \in \mathcal{F}(\{K_{1,3}, P_4\}, k)$, then $c(G) \geq k$.

**Proof of Theorem 1.8.** Suppose $c(G)$ is less than $k$. It will be shown that this leads to the existence of a longest path $P = v_1 \ldots v_m$ in $G$ such that $d_G(v_1) + d_G(v_m) \geq k$, a contradiction to Theorem 3.1.

For a given longest path $P = v_1 \ldots v_m$ in $G$ let $v_{l_P}$ be the last neighbour of $v_1$ along $P$, i.e., $l_P = \max\{i: v_iv_i \in E(G)\}$, and let $v_{n_P}$ be the last nonneighbour of $v_1$ preceding $v_{l_P}$, that is $n_P = \max\{i: i < l_P$ and $v_iv_{n_P} \notin E(G)\}$.

Clearly, $l_P > 2$. Furthermore, it follows from 2-connectivity of $G$ that $l_P < m$, since otherwise there would be either a hamiltonian cycle or a path longer than $P$ in $G$. Next observe that there exists a longest path $P$ with $n_P > 2$. If this is not the case and $n_P = 1$, let $Q$ be a path from $v_i$ to $v_j$, $i \leq l_P - 1$, $j \geq l_P + 1$, such that $V(P) \cap V(Q) = \{v_i, v_j\}$. Then form the path $P' = v_{j-1}P^{-v_{i+1}}v_{i+1}P^+v_iQ^+v_jP^+v_m$, which is a longest path with $l_{P'} \geq j > l_P$, a contradiction when $P$ is chosen to have the largest $l_P$ value.

Fix a longest path $P = v_1 \ldots v_m$ with $n_P$ of largest possible value. With the above observations it will next be shown that there exists a longest path with one of its endvertices being $v_m$ and the other having degree at least $k/2$. To do this, suppose that $d_G(v_1) < k/2$. Note that, since $n_P > 2$, we have $d_G(v_{n_P}) < k/2$, since otherwise the path $v_{n_P}P^{-v_1v_{n_P+1}}P^+v_m$ is a longest path with $d_G(v_{n_P}) \geq k/2$. Since $G \in \mathcal{F}(K_{1,3}, k)$, it follows that $\{v_{n_P+1}, v_1, v_{n_P}, v_{n_P+2}\}$ can not induce a claw. Thus $v_{n_P+2}$ is adjacent to at least one of the vertices $v_1$ and $v_{n_P}$. Before the proof divides into subcases, we note that $d_G(v_{n_P+1}) < k/2$, since by the previous observation at least one of the paths $v_{n_P+1}P^{-v_1v_{n_P+2}}P^+v_m$ and $v_{n_P+1}P^+v_{n_P}v_{n_P+2}P^+v_m$ is a longest path in $G$ beginning with $v_{n_P+1}$.

Throughout the proof, whenever we declare a contradiction due to a discovered induced subgraph of $G$ isomorphic to the claw or the path $P_4$, it is because the subgraph does not satisfy Fan’s condition with constant $k$.

**Case 1.** $v_1v_{n_P+2} \in E(G), v_{n_P}v_{n_P+2} \notin E(G)$.

Note that under the assumptions of this case we have $m \geq n_P + 3$. We begin with crucial pieces of information regarding the degree of the vertex $v_{n_P+2}$ and the adjacency structure of its neighbourhood.

**Claim 3.2.** $v_{n_P+3}v_{n_P}, v_{n_P+3}v_{n_P+1} \notin E(G)$ and $d_G(v_{n_P+2}) \geq k/2$.

**Proof.** Note that if $v_{n_P+3}$ is adjacent to $v_{n_P}$, then under the assumptions of this case the path $P' = v_{n_P}P^{-v_1v_{n_P+1}}P^+v_m$ is a longest path in $G$ with $n_P' \geq n_P + 2$, contradicting the choice of $P$. Similarly, if $v_{n_P+3}v_{n_P+1} \in E(G)$, then $P' = v_{n_P}P^{-v_1v_{n_P+2}}v_{n_P+1}v_{n_P+3}P^+v_m$ is a longest path with $n_P' \geq n_P + 1$. Thus $v_{n_P+3}$ is adjacent neither to $v_{n_P}$, nor to $v_{n_P+1}$, and so $v_{n_P}v_{n_P+1}v_{n_P+2}v_{n_P+3}$ is an induced path $P_4$ in $G$. Since $G \in \mathcal{F}(P_4, k)$ and $d_G(v_{n_P}) < k/2$, it follows that $d_G(v_{n_P+2}) \geq k/2$. \(\square\)
Claim 3.3. \( d_G(v_2) < k/2 \) and \( v_2v_{n_p+3} \notin E(G) \), \( v_2v_{n_p+1} \in E(G) \).

Proof. Clearly, if \( d_G(v_2) \geq k/2 \), then \( v_2P^+v_{n_p+1}v_{n_p+2}P^+v_m \) is a longest path with \( d_G(v_2) \geq k/2 \), and if \( v_2v_{n_p+3} \in E(G) \), then, by Claim 3.2, \( v_{n_p+2}v_1v_{n_p+1}P^-v_2v_{n_p+3}P^+v_m \) is such path.

Now suppose that \( v_2v_{n_p+1} \notin E(G) \). Since \( d_G(v_2), d_G(v_{n_p}) < k/2 \), the set \( \{v_2; v_1, v_{n_p+1}, v_{n_p}\} \) can not induce \( P_4 \) and the set \( \{v_{n_p+2}; v_2, v_{n_p+1}, v_{n_p+3}\} \) can not induce a claw. It follows from Claim 3.2 that \( v_2v_{n_p} \in E(G) \) and \( v_2v_{n_p+2} \notin E(G) \). But now \( v_2v_{n_p}v_{n_p+1}v_{n_p+2} \) is an induced \( P_4 \) in \( G \), a contradiction.

\( \square \)

Claim 3.4. There are no edges between the vertices \( v_1, v_{n_p}, v_{n_p+1} \) and the vertices from the set \( \{v_{n_p+3}, \ldots, v_m\} \).

Proof. From the definition of \( n_P \) it follows that to prove that \( v_1 \) is not adjacent to any of the vertices from the set \( \{v_{n_p+3}, \ldots, v_m\} \) it suffices to show that it is not adjacent to \( v_{n_p+3} \). This is clearly true, since otherwise \( v_{n_p+2}P^-v_1v_{n_p+3}P^+v_m \) would be a longest path with \( d_G(v_{n_p+2}) \geq k/2 \), by Claim 3.2.

Recall that \( v_{n_p+3} \notin E(G) \) and \( v_{n_p+1}v_{n_p+3} \notin E(G) \), by Claim 3.2, and \( v_2v_{n_p+1} \in E(G) \), by Claim 3.3. Suppose that \( v_{n_p} \) is adjacent to \( v_{n_p+j} \) for some \( 3 < j \leq m - n_p \). Then the path \( P' = v_{n_p}P^-v_2v_{n_p+1}v_1v_{n_p+2}P^+v_m \) is a longest path in \( G \) with \( n_{P'} \geq n_p + 3 \), contradicting the choice of \( P \).

From the observations made so far it follows that if \( v_{n_p+1} \) is adjacent to some vertex \( v_{n_p+j} \) with \( 3 < j \leq m - n_p \), then \( \{v_{n_p+1}; v_1, v_{n_p}, v_{n_p+j}\} \) induces a claw. Since \( d_G(v_1), d_G(v_{n_p}) < k/2 \), this contradicts \( G \) being a graph from the family \( \mathcal{F}(K_{1,3}, k) \).

\( \square \)

Next claim provides a characterization of properties of the vertices that lie on \( P \) between \( v_1 \) and \( v_{n_p} \).

Claim 3.5. For \( i \in \{2, \ldots, n_p\} \) the following holds:

(i) \( d_G(v_i) < k/2 \),
(ii) \( v_iv_{n_p+3} \notin E(G) \),
(iii) \( v_iv_{n_p+1} \in E(G) \),
(iv) either \( v_i \) is adjacent to both \( v_1 \) and \( v_{n_p+2} \) or else it is not adjacent to any of them,
(v) \( v_i \) is not adjacent to any of the vertices from the set \( \{v_{n_p+3}, \ldots, v_m\} \).

Proof. The proof is by induction on \( i \). For \( i = 2 \) the statements (i), (ii) and (iii) are true by Claim 3.3. To show that the condition (iv) holds, we first observe that \( v_2 \) is adjacent to \( v_1 \). Suppose \( v_2v_{n_p+2} \notin E(G) \). Then under the assumptions of the case and depending on the existence of the edge \( v_2v_{n_p} \), either \( v_{n_p}v_2v_1v_{n_p+2} \) is an induced path or the set \( \{v_{n_p+1}; v_2, v_{n_p}, v_{n_p+2}\} \) induces a claw. Since the degrees of \( v_1, v_2 \) and \( v_{n_p} \) are strictly less than \( k/2 \), this contradicts \( G \) being a member of the family \( \mathcal{F}(K_{1,3}, P_4, k) \).

For the proof of (v) suppose that \( v_2 \) is adjacent to some vertex \( v \in \{v_{n_p+3}, \ldots, v_m\} \). The path \( v_2v_{n_p+1}v_{n_p} \) can not be an induced one, since \( d_G(v_2), d_G(v_{n_p}) < k/2 \). Thus
it follows from Claim 3.4 that \(v_2\) is adjacent to \(v_{n_P}\). But now \(\{v_2; v, v_{n_P}, v_1\}\) induces a claw with \(d_G(v_2), d_G(v_{n_P}) < k/2\), a contradiction.

Now assume that for some \(i < n_P\) the conditions (i)-(v) hold for the vertices \(v_2, \ldots, v_i\). It will be shown that they hold also for \(v_{i+1}\).

First observe that \(d_G(v_{i+1}) < k/2\), since otherwise, by the condition (iii) for \(v_i\), the path \(v_{i+1}P^tv_{n_{n_P}+1}v_iP^tv_{n_{n_P}+2}P^tv_m\) is a longest path in \(G\) with its first vertex having degree at least \(k/2\). The validity of the condition (ii) is also straightforward: if \(v_{i+1}v_{n_{n_P}+3} \in E(G)\), then a longest path with its first vertex having degree not less than \(k/2\) is the path \(v_{n_{n_P}+2}v_1P^tv_{n_{n_P}+1}P^tv_{i+1}v_{n_{n_P}+3}P^tv_m\), by Claim 3.2.

Now suppose that the condition (iii) is not true, i.e., that \(v_{i+1}v_{n_{n_P}+1}\) is not an edge in \(G\). It follows that \(v_{i+1}\) is not adjacent to \(v_{n_{n_P}+2}\), since otherwise, by (ii) for \(v_{i+1}\) and by Claim 3.4, the set \(\{v_{n_{n_P}+2}; v_{i+1}, v_{n_{n_P}+1}, v_{n_{n_P}+3}\}\) induces a claw with \(d_G(v_{i+1}), d_G(v_{n_{n_P}+1}) < k/2\).

If \(v_iv_{n_P}\) is not an edge in \(G\), then by (iii) for \(v_i\), the vertex \(v_{i+1}\) is adjacent to \(v_{n_P}\) to avoid induced path \(v_{i+1}v_i; v_{n_{n_P}+1}v_{n_{n_P}}\) with \(d_G(v_i), d_G(v_{n_{n_P}}) < k/2\). But now \(v_{i+1}v_{n_{n_P}}v_{n_{n_P}+1}v_{n_{n_P}+2}\) is an induced \(P_4\) with none of the vertices \(v_{i+1}\) and \(v_{n_{n_P}+1}\) having degree not less than \(k/2\), a contradiction. Thus assume \(v_iv_{n_{n_P}} \in E(G)\).

Note that \(v_i\) can not be adjacent to \(v_{n_{n_P}+2}\). If this is not the case, then, depending on the existence of the edge \(v_{i+1}v_{n_{n_P}}\), either \(\{v_i; v_{n_P}, v_{n_{n_P}+2}, v_{i+1}\}\) is an induced claw in \(G\) or else \(v_{i+1}v_{n_{n_P}}v_{n_{n_P}+1}v_{n_{n_P}+2}\) is an induced \(P_4\) that does not satisfy the Fan’s condition.

From the fact that \(v_iv_{n_{n_P}+2}\) is not an edge in \(G\) and from the condition (iv) for \(v_i\) it follows that \(v_iv_{n_{n_P}+1} \notin E(G)\). This implies that \(v_{i+1}\) is adjacent to \(v_i\), since otherwise the path \(v_{i+1}v_i; v_{n_{n_P}+1}v_{n_{n_P}}\) is an induced \(P_4\) with \(d_G(v_i), d_G(v_{n_{n_P}}) < k/2\). But now \(v_{i+1}v_i; v_{n_{n_P}+1}v_{n_{n_P}+2}\) is an induced \(P_4\) with \(d_G(v_i), d_G(v_{n_{n_P}}) < k/2\), a contradiction. Thus the condition (iii) holds for \(v_{i+1}\).

To show that the condition (iv) is satisfied by \(v_{i+1}\), first suppose that \(v_{i+1}v_{n_{n_P}} \notin E(G)\). Then \(v_{i+1}\) is adjacent to both \(v_i\) and \(v_{n_{n_P}+2}\) to avoid induced claws \(\{v_{n_{n_P}+1}; v_{i+1}, v_{n_{n_P}}, v_i\}\) and \(\{v_{n_{n_P}+1}; v_{i+1}, v_{n_{n_P}}, v_{n_{n_P}+2}\}\) with both \(v_{i+1}\) and \(v_{n_{n_P}}\) having degrees less than \(k/2\).

Now suppose that \(v_{i+1}\) is adjacent to \(v_{n_{n_P}}\). If \(v_{i+1}\) is a neighbour of \(v_{n_{n_P}+1}\), then the same is true for \(v_{n_{n_P}+2}\), since otherwise \(v_{n_{n_P}}v_{i+1}v_{n_{n_P}+1}v_{n_{n_P}+2}\) is an induced \(P_4\) with \(d_G(v_{n_{n_P}}), d_G(v_{n_{n_P}+1}) < k/2\). Similarly, \(v_{i+1}v_{n_{n_P}+2} \in E(G)\) implies that \(v_{i+1}\) is adjacent to \(v_{n_{n_P}}\), to avoid induced path \(v_{n_{n_P}}v_{i+1}v_{n_{n_P}+2}v_1\). This proves (iv).

Finally, suppose that \(v_{i+1}\) is adjacent to some vertex \(v \in \{v_{n_{n_P}+3}, \ldots, v_m\}\). By Claim 3.4 we can assume that \(i + 1 < n_P\). If \(v_{i+1}v_i \notin E(G)\), then \(v_1v_{n_{n_P}+1}v_{i+1}v\) is an induced path \(P_4\), by Claim 3.4. Since the degrees of both \(v_1\) and \(v_{i+1}\) are less than \(k/2\), this contradicts \(G\) belonging to the family \(\mathcal{F}(P_4, k)\). Now suppose that \(v_{i+1}\) is adjacent to \(v_i\). Then \(v_{i+1}v_{n_{n_P}} \notin E(G)\) to avoid induced claw \(\{v_{i+1}; v_i, v_{n_{n_P}}, v\}\). But now \(v_{n_{n_P}}v_{n_{n_P}+1}v_{i+1}v\) is an induced path \(P_4\), by Claim 3.4. This final contradiction shows that the property (v) holds for \(v_{i+1}\). By mathematical induction the claim is true.

\[\Box\]

Claim 3.6. For every \(i \in \{1, \ldots, n_P + 1\}\) the neighbourhood \(N_G(v_i)\) of the vertex \(v_i\) is a subset of the set \(\{v_1, v_2, \ldots, v_{n_{n_P}+2}\}\).
Proof. Note that by Claims 3.4 and 3.5 the vertex $v_i$, with $1 \leq i \leq n_P + 1$, has no neighbours in the set $\{v_{n_P+3}, \ldots, v_m\}$. Thus to prove the claim it suffices to show that $v_i$ is not adjacent to any $v \in V(G) \setminus V(P)$. Clearly, if one of the vertices $v_1$, $v_2$ and $v_{n_P + 1}$ was adjacent to some vertex $v \not\in V(P)$, this would create a path in $G$ longer than $P$, i.e., one of the paths $vv_1P^+v_m$, $v_2P^+v_{n_P+1}v_{n_P+2}P^+v_m$ or $v_{n_P+1}P^-v_1v_{n_P+2}P^+v_m$. Hence, the claim is true for $i \in \{1, 2, n_P + 1\}$.

For a proof by induction assume that the claim holds for the values from the set $\{1, 2, \ldots, i\}$, where $2 \leq i \leq n_P - 1$. It will be shown that this implies the validity of the claim for $i + 1$.

Suppose that there is a vertex $v \in V(G) \setminus V(P)$ adjacent to $v_{i+1}$. Then $v$ is not adjacent to any of $v_i$ and $v_{i+2}$, since such an edge would create a path in $G$ longer than $P$. Recall that $d_G(v_i), d_G(v_{i+2}) < k/2$, by Claim 3.5, and so $\{v_{i+1}; v, v_{i+2}\}$ cannot induce a claw in $G$. Thus $v_i, v_{i+2} \in E(G)$. We observe that if $v_{i+1}$ is not adjacent to some vertex $v_k$ with $1 \leq k \leq i - 1$, then choosing $k$ of largest possible value gives an induced path $v_kv_{k+1}v_{i+1}v_{i+2}$, by the induction hypothesis. This contradicts $G$ being a member of the family $\mathcal{F}(4, k)$, by Claim 3.5. Thus $v_{i+1}$ is adjacent to every vertex preceding it on the path $P$, in particular $v_1v_{i+1} \in E(G)$. But now $vv_1v_1P^+v_{i+1}P^+v_m$ is a path longer than $P$, a contradiction.

Now it follows from Claim 3.6 that $G - v_{n_P + 2}$ is not connected. This contradicts $G$ being 2-connected and completes the proof of this case.

Case 2. $v_1v_{n_P + 2} \not\in E(G)$, $v_{n_P}v_{n_P + 2} \in E(G)$.

We begin the proof of this case with a counterpart of Claim 3.4.

Claim 3.7. There are no edges between the vertices $v_1, v_{n_P}, v_{n_P + 1}$ and the vertices from the set $\{v_{n_P+3}, \ldots, v_m\}$.

Proof. The validity of the claim for $v_1$ follows immediately from the definition of $n_P$ and the assumptions of this case. For $v_{n_P + 1}$ we first observe that $v_{n_P + 1}v_{n_P + 3} \not\in E(G)$, since otherwise the path $P' = v_{n_P + 2}v_{n_P}P^{-}v_1v_{n_P+1}v_{n_P+3}P^+v_m$ is a longest path in $G$ with $n_{P'} \geq n_P + 1$, contradicting the choice of $P$. With this observation it is easy to see that if $v_{n_P + 1}v \in E(G)$ for some vertex $v \in \{v_{n_P + 4}, \ldots, v_m\}$, then the path $P' = v_{n_P + 1}v_1P^+v_{n_P}v_{n_P + 2}P^+v_m$ is a longest path with $n_{P'} \geq n_P + 3$. Finally, if $v_{n_P}$ has a neighbour in the set $\{v_{n_P+3}, \ldots, v_m\}$, say $v$, then $v_1v_{n_P + 1}v_{n_P}v$ is an induced $P_4$ in $G$ with $d_G(v_{n_P}), d_G(v_1) < k/2$. A contradiction.

Next we establish some properties of the vertices that precede $v_{n_P}$ on $P$.

Claim 3.8. For $i \in \{1, 2, \ldots, n_P - 2\}$ the following holds:

(i) $d_G(v_{n_P-i}) < k/2$,
(ii) $v_{n_P-i}$ is adjacent to at least one of the vertices $v_1$ and $v_{n_P+1}$,
(iii) $v_{n_P-i}v_{n_P+2} \in E(G)$ or else $v_{n_P-i}$ is adjacent to $v_1$,
(iv) $v_{n_P-i}$ is not adjacent to any of the vertices from the set $\{v_{n_P+3}, \ldots, v_m\}$. 

Proof. We use induction on \(i\). For \(i = 1\) it is clear that \(d_G(v_{n_P-1}) < k/2\), since the path \(v_{n_P-1}P - v_1v_{n_P+1}v_{n_P}v_{n_P+2}P+ v_m\) is a longest path in \(G\) beginning with \(v_{n_P-1}\). Thus (i) holds. Recall that the degrees of both \(v_1\) and \(v_{n_P}\) are less than \(k/2\), and so the path \(v_{n_P-1}v_{n_P}v_{n_P+1}v_1\) cannot be an induced one. This implies (ii).

To show (iii) assume that \(v_{n_P-1}\) is not adjacent to \(v_{n_P+2}\) and suppose \(v_1v_{n_P-1} \notin E(G)\). Then \(v_{n_P-1}\) is adjacent to \(v_{n_P+1}\) to avoid induced path \(v_{n_P-1}v_{n_P}v_{n_P+1}v_1\) with \(d_G(v_{n_P}), d_G(v_1) < k/2\). But now \(\{v_{n_P+1}; v_1, v_{n_P-1}, v_{n_P+2}\}\) induces a claw. By (i), this contradicts \(G\) belonging to the family \(F(K_{1,3}, k)\).

For the proof of (iv) suppose that \(v_{n_P-1}\) has a neighbour, say \(v\), in the set \(\{v_{n_P+3}, \ldots, v_m\}\). Then \(v_{n_P-1}\) is not adjacent to \(v_1\), since otherwise \(\{v_{n_P-1}; v_1, v_{n_P}, v\}\) induces a claw, by Claim 3.7. It follows from (ii) that \(v_{n_P-1}v_{n_P+1} \in E(G)\). But now \(v_1v_{n_P+1}v_{n_P-1}v\) is an induced path \(P_4\) with \(d_G(v_1), d_G(v_{n_P-1}) < k/2\), a contradiction. This proves (iv) for \(i = 1\).

Now assume that the claim holds for the values from the set \(\{1, 2, \ldots, i\}\), where \(1 \leq i \leq n_P - 3\). It will be shown that this implies the validity of the claim for \(i+1\).

By the condition (iii) for \(v_{n_P-i}\) there is a longest path in \(G\) beginning with \(v_{n_P-i-1}\), namely \(v_{n_P-i-1}P - v_1v_{n_P+1}v_{n_P}v_{n_P+2}P+ v_m\) or \(v_{n_P-i-1}P - v_1v_{n_P-i}P+ v_m\). Thus \(d_G(v_{n_P-i-1}) < k/2\), proving (i). For the proof of (ii) suppose that \(v_{n_P-i-1}\) is not adjacent neither to \(v_1\) nor to \(v_{n_P+1}\). This implies that both \(v_1\) and \(v_{n_P+1}\) are neighbours of \(v_{n_P-i}\), since otherwise, by (ii), one of the paths \(v_{n_P-1}v_{n_P-i}v_{n_P-i-1}v_{n_P+1}\) would be an induced \(P_4\) in \(G\). Furthermore, \(v_{n_P}\) is not adjacent to \(v_{n_P-i-1}\), to avoid induced path \(v_{n_P-1}v_{n_P}v_{n_P+1}v_{n_P+1}v_1\). It is also not adjacent to \(v_{n_P-i}\), since otherwise \(\{v_{n_P-i-1}; v_1, v_{n_P-i-1}, v_{n_P}\}\) induces a claw. But now \(v_{n_P-i}v_{n_P-i-1}v_{n_P+1}v_{n_P}\) is an induced path with four vertices. Since the degrees of the vertices of this path are less than \(k/2\), this contradicts \(G\) belonging to the family \(F(P_4, k)\) and proves (ii).

Now assume \(v_{n_P-i-1}v_{n_P+2} \notin E(G)\) and suppose that \(v_{n_P-i-1}\) is not adjacent to \(v_1\). From the condition (ii) for \(v_{n_P-i-1}\) it follows that \(\{v_{n_P+1}; v_1, v_{n_P-i-1}, v_{n_P+2}\}\) induces a claw. Since the degrees of both \(v_1\) and \(v_{n_P-i-1}\) are strictly less than \(k/2\), this is a contradiction. Thus (iii) holds.

Finally, suppose that \(v_{n_P-i-1}\) is adjacent to some vertex \(v \in \{v_{n_P+3}, \ldots, v_m\}\). Claim 3.7 implies that \(n_P-i+1 > 1\). If \(v_{n_P-i-1}v_1 \notin E(G)\), then it follows from the condition (ii) and Claim 3.7 that the path \(v_1v_{n_P-i}v_{n_P-i-1}v\) is an induced \(P_4\) in \(G\) with the degrees of both \(v_1\) and \(v_{n_P-i-1}\) being less than \(k/2\). Thus \(v_{n_P-i-1}v_1 \in E(G)\). This implies that \(v_{n_P-i-1}\) is not adjacent to \(v_{n_P}\), since otherwise \(\{v_{n_P-i-1}; v_1, v_{n_P}, v\}\) induces a claw, by Claim 3.7. Furthermore, in order to avoid induced path \(v_{n_P}v_{n_P+1}v_{n_P-i-1}v\), the vertex \(v_{n_P-i-1}\) cannot be adjacent to \(v_{n_P+1}\). But now we obtain an induced path \(v_{n_P-i-1}v_1v_{n_P+1}v_{n_P}\), a contradiction. By mathematical induction the claim is true.

\[
\text{\textbf{Claim 3.9.}} \quad \text{For every } i \in \{0, 1, \ldots, n_P\} \text{ the neighbourhood } \mathcal{N}_G(v_{n_P-i+1}) \text{ of the vertex } v_{n_P-i+1} \text{ is a subset of the set } \{v_1, \ldots, v_{n_P+2}\}. \]

\textbf{Proof.} Note that by Claims 3.7 and 3.8 the vertex \(v_{n_P-i+1}\), with \(0 \leq i \leq n_P\), has no neighbours in the set \(\{v_{n_P+3}, \ldots, v_m\}\). Thus to prove the claim it suffices to show that \(v_{n_P-i+1}\) is not adjacent to any \(v \in V(G) \setminus V(P)\). Clearly, if one of the vertices \(v_1, v_{n_P}\)
and \(v_{n_p+1}\) was adjacent to some vertex \(v\) lying outside the path \(P\), this would create a path in \(G\) longer than \(P\), i.e., one of the paths \(v_1P^+v_m\), \(v_{n_p}P^-v_1v_{n_p+1}P^+v_m\) or \(v_{n_p+1}v_1P^+v_{n_p}v_{n_p+2}P^+v_m\). Hence, the claim is true for \(i \in \{0, 1, n_p\}\).

For a proof by induction assume that the claim holds for the values from the set \(\{0, 1, \ldots , i\}\), where \(1 \leq i \leq n_p - 2\). It will be shown that this implies the validity of the claim for \(i + 1\).

Suppose that there is a vertex \(v \in V(G) \setminus V(P)\) adjacent to \(v_{n_p-i}\). Then \(v\) is not adjacent to any of \(v_{n_p-i-1}\) and \(v_{n_p-i+1}\), to avoid creating a path in \(G\) longer than \(P\). Recall that \(d_G(v_{n_p-i-1}), d_G(v_{n_p-i+1}) < k/2\), by Claim 3.8 and by the fact that \(d_G(v_1) < k/2\), and thus \(v_{n_p-i-1}, v_{n_p-i+1} \) can not induce a claw in \(G\). Thus \(v_{n_p-i-1}v_{n_p-i+1} \in E(G)\). Next we note that if \(v_{n_p-i}\) is not adjacent to some vertex \(v_k\) with \(n_p - i < k \leq n_p\), then choosing \(k\) of smallest possible value gives an induced path \(v_{n_p-i}v_k\), by the induction hypothesis. This contradicts \(G\) being a member of the family \(\mathcal{F}(P_4, k)\), by Claim 3.8 and by the fact that \(d_G(v_{n_p}) < k/2\). Thus \(v_{n_p-i}\) is adjacent to every vertex from the set \(\{v_{n_p-i+1}, \ldots , v_{n_p+1}\}\). But now the path \(v_{n_p-i}v_{n_p+1}v_1P^+v_{n_p-i-1}v_{n_p-i+1}P^+v_{n_p}v_{n_p+2}P^+v_m\) is a path longer than \(P\), a contradiction. 

Similarly to the previous case of the proof, now it follows from Claim 3.9 that \(G - v_{n_p+2}\) is not connected, a contradiction with the assumption of 2-connectivity of \(G\).

**Case 3.** \(v_1v_{n_p+2} \in E(G)\), \(v_{n_p}v_{n_p+2} \in E(G)\).

Recall that the degrees of the vertices \(v_1\), \(v_{n_p}\), and \(v_{n_p+1}\) are less than \(k/2\). Keeping that in mind, we first establish some basic facts regarding the vertex \(v_{n_p-1}\).

**Claim 3.10.** \(d_G(v_{n_p-1}) < k/2\), \(v_{n_p-1}v_{n_p+1} \notin E(G)\), \(v_{n_p-1}v_1 \in E(G)\).

**Proof.** Note that under the assumptions of this case the path \(v_{n_p-1}P^-v_1v_{n_p+1}v_{n_p}v_{n_p+2}P^+v_m\) is a longest path in \(G\). Thus \(d_G(v_{n_p-1}) < k/2\).

Now suppose that \(v_{n_p-1}\) is adjacent to \(v_{n_p+1}\). Then the path \(P' = v_1P^+v_{n_p-1}v_{n_p+1}v_{n_p}v_{n_p+2}P^+v_m\) is a longest path in \(G\) with \(n_{p'} \geq n_p + 1\), contradicting the choice of \(P\). Hence, \(v_{n_p-1}v_{n_p+1} \notin E(G)\). This implies that \(v_{n_p-1}v_1 \in E(G)\), since otherwise the path \(v_1v_{n_p+1}v_{n_p}v_{n_p-1}\) would be an induced \(P_4\) in \(G\).

**Claim 3.11.** Every neighbour of \(v_1\) in \(G\) is adjacent to at least one of the vertices \(v_{n_p-1}, v_{n_p+1}\).

**Proof.** If this is not the case, then there exists a neighbour \(v\) of \(v_1\) such that \(\{v_1, v_{n_p-1}, v_{n_p+1}, v\}\) induces a claw, by Claim 3.10. A contradiction follows from Claim 3.10.

Now we focus our attention on the edges \(v_{n_p-1}v_{n_p+2}, v_{n_p-1}v_{n_p+3}\) and \(v_{n_p+1}v_{n_p+3}\). We begin with the following observation.

**Claim 3.12.** \(v_{n_p+3}\) is adjacent to exactly one of the vertices \(v_{n_p-1}\) and \(v_{n_p+1}\).
Proof. Suppose the contrary. If the vertex \( v_{n+3} \) is not adjacent to any of the vertices \( v_{n-1}, v_{n+1}, \) then it follows from Claim 3.11 that \( v_1v_{n+3} \notin E(G) \). Now, depending on the existence of the edge \( v_nv_{n+3} \), we obtain induced path \( v_{n+3}v_nv_{n+1}v_1 \) or induced claw \( \{v_{n+2}; v_n, v_1, v_{n+3}\} \), a contradiction.

If both \( v_{n+3}v_{n-1} \) and \( v_{n+3}v_{n+1} \) are edges in \( G \), then the path \( P' = v_{n-1}P^-v_1v_{n+2}v_nv_{n+1}v_{n+3}P^+v_m \) is a longest path in \( G \) with \( n_P' \geq n_P + 2 \), by Claim 3.10. This contradicts the choice of \( P \).

Claim 3.13. \( v_{n-1}v_{n+3} \) is not an edge in \( G \).

Proof. Suppose that the opposite holds. If \( v_{n-1}v_{n+2} \) is not an edge in \( G \), then the path \( P' = v_{n-1}P^-v_1v_{n+1}v_nv_{n+2}v_{n+3}P^+v_m \) is a longest path in \( G \) with \( n_P' \geq n_P + 2 \), contradicting the choice of \( P \). Thus \( v_{n-1}v_{n+2} \in E(G) \).

It follows from Claim 3.12 that \( v_{n+1} \) is not adjacent to \( v_{n+3} \). Since \( d_G(v_{n+1}) < k/2 \) and, by Claim 3.10, \( d_G(v_{n-1}) < k/2 \), the path \( v_{n+1}v_nv_{n-1}v_{n+3} \) can not be an induced one. Thus it follows from Claim 3.10 that \( v_{n+1}v_{n+3} \) is not an edge in \( G \). Now to avoid induced path \( v_1v_{n+1}v_nv_{n+3} \), \( v_1v_{n+3} \in E(G) \). But then the path \( P' = v_1P^+v_{n-1}v_{n+2}v_{n+1}v_nv_{n+3}P^+v_m \) is a longest path in \( G \) with \( n_P' \geq n_P + 2 \), a contradiction.

From Claims 3.12 and 3.13 it follows that the vertex \( v_{n+3} \) is not adjacent to \( v_{n-1} \) and that it is adjacent to \( v_{n+1} \). Next we observe that to avoid induced path \( v_{n-1}v_nv_{n+1}v_{n+3} \) the vertex \( v_{n+3} \) is adjacent to \( v_{n+1} \), by Claim 3.10. It follows that \( v_{n+3} \) is adjacent also to \( v_1 \), since otherwise the path \( v_1v_{n-1}v_{n+1}v_{n+3} \) is an induced one, also by Claim 3.10. But now, depending on the existence of the edge \( v_{n-1}v_{n+2} \), one of the paths \( P' = v_1P^+v_{n-1}v_{n+2}P^-v_nv_{n+3}P^+v_m \) and \( P'' = v_{n-1}P^-v_1v_{n+1}v_{n+2}v_nv_{n+3}P^+v_m \) is a longest path in \( G \). Since \( n_P' \geq n_P + 2 \) and \( n_P'' \geq n_P + 1 \), this contradicts the choice of \( P \). This final contradiction completes the proof of this case and shows that there exists a longest path in \( G \) with end vertex \( v_m \) and with the other end vertex of degree at least \( k/2 \).

In the above argument, each longest path considered has \( v_m \) as one of the end vertices. Thus, since one of the end vertices of \( P \) has degree \( \geq k/2 \), it could have been initially assumed that \( P \) is a longest path with \( d_G(v_m) \geq k/2 \) and with \( n_P \) of largest possible value. The above argument then shows that there exists a longest path \( P \) with both endvertices of degree \( \geq k/2 \). This contradiction with Theorem 3.1 completes the proof of the theorem.

\[Q.E.D.\]

4. FAN’S CONDITION AND PANCYCLICITY

4.1. PRELIMINARIES

Lemma 4.1 (Benhocine and Wojda [3]). If a graph \( G \) of order \( n \geq 4 \) has a cycle \( C \) of length \( n - 1 \), such that the vertex not in \( V(C) \) has degree at least \( n/2 \), then \( G \) is pancyclic.
The next four lemmas provide a description of the cycle structure of hamiltonian graphs with two vertices that lie close (i.e., with distance one or two along the cycle) to each other on some hamiltonian cycle and have large degree sum.

**Lemma 4.2** (Bondy [5]). Let $G$ be a graph of order $n$ with a hamiltonian cycle $C$. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 1$ and $d_G(x) + d_G(y) \geq n + 1$, then $G$ is pancyclic.

**Lemma 4.3** (Schmeichel and Hakimi [20]). Let $G$ be a graph of order $n$ with a hamiltonian cycle $C = v_1v_2\ldots v_nv_1$. If $d_G(v_1) + d_G(v_n) \geq n$, then $G$ is pancyclic unless $G$ is bipartite or else $G$ is missing only $(n - 1)$-cycles.

Furthermore, when $G$ is missing only $(n - 1)$-cycles and $d_G(v_1) = d_G(v_2) = n/2$, then the adjacency structure near $v_1$ and $v_2$ is the following: the path $v_{n-2}v_{n-1}v_nv_1v_2v_3$ is an induced one, and $v_nv_{n-3}, v_nv_{n-4}, v_1v_4, v_1v_5$ are edges in $G$.

**Lemma 4.4** (Ferrara, Jacobson, and Harris [13]). Let $G$ be a graph of order $n$ with a hamiltonian cycle $C$. If there exist two vertices $x, y \in V(G)$ such that $d_C(x, y) = 2$ and $d_G(x) + d_G(y) \geq n + 1$, then $G$ is pancyclic.

**Lemma 4.5** (Han [16]). Let $G$ be a graph of order $n$ with a hamiltonian cycle $C$. If there exist two non-adjacent vertices $x, y \in V(G)$ such that $d_C(x, y) = 2$ and $d_G(x) + d_G(y) \geq n$, then $G$ is pancyclic, unless $G$ is bipartite or else $G$ is missing only the $(n - 1)$-cycle, or the 3-cycle.

4.2. PROOF OF THEOREM 1.9

For the convenience of the reader we restate Theorem 1.9 below.

**Theorem 1.9.** Let $G$ be a 2-connected graph of order $n \geq 3$. If $G \in F(\{K_{1,3}, P_4\}, n)$, then $G$ is pancyclic unless $n = 4r, r \geq 2$ and $G$ is $F_{4r}$, or $n$ is even and $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.

We first prove three auxiliary lemmas that deal with the exceptional non-pancyclic graphs and establish the existence of short cycles in a graph satisfying the assumptions of Theorem 1.9.

**Lemma 4.6.** Let $G$ be a 2-connected, bipartite graph of order $n \geq 3$. If $G \in F(\{K_{1,3}, P_4\}, n)$, then $G = K_{n/2,n/2}$ or else $n \geq 6$ and $G = K_{n/2,n/2} - e$.

**Proof.** First suppose that $G$ is $\{K_{1,3}, P_4\}$-free. Then it follows from Theorem 1.2 that $G$ is a cycle. Since there are no induced paths with four vertices in $G$, $G$ is a cycle $K_{2,2}$.

Now assume that $G$ contains an induced claw or an induced path $P_4$. Let $(X, Y)$ be a bipartition of $V(G)$. It follows from the assumptions that there is a vertex, say $u$, in $G$ with $d_G(u) \geq n/2$. Clearly, if $|V(G)| = 4$, then $G$ is isomorphic to $K_{2,2}$. Thus assume $|V(G)| \geq 6$. Without loss of generality let $X$ be the set of bipartition containing $u$. It follows that $|Y| \geq n/2 \geq 3$. Note that since $G \in F(K_{1,3}, n)$ and $u$ together with any three of its neighbours induce a claw, at most one neighbour of $u$ has degree less than $n/2$. This implies $|X| = |Y| = n/2$. By the symmetry, at most one vertex in $X$ might
have less neighbours than \( n/2 \). Let \( x \in X \) and \( y \in Y \) be those only vertices in \( G \), the degrees of which are not necessarily equal to \( n/2 \). Clearly, every vertex of \( Y \) other than \( y \) is adjacent to \( x \) and every vertex from \( X \) other than \( x \) is adjacent to \( y \). Thus, depending on the existence of the edge \( xy \) in \( G \), \( G \) is isomorphic either to \( K_{n/2,n/2} \) or else to \( K_{n/2,n/2} - e \). □

**Lemma 4.7.** Let \( G \) be a 2-connected, non-bipartite graph of order \( n \). If \( G \in \mathcal{F}(\{K_{1,3}, P_4\}, n) \) and there are no cycles of length \( n-1 \) in \( G \), then \( G \) is isomorphic to \( F_{4r} \), with \( r > 2 \).

**Proof.** Suppose that \( G \) is \( \{K_{1,3}, P_4\}\)-free. Similarly to the previous Lemma, this implies that \( G \) is a cycle \( K_{2,2} \), by Theorem 1.2. This contradicts the assumption of \( G \) not being bipartite. Hence, we can assume that \( G \) contains an induced claw or an induced path \( P_4 \), and so there are at least two heavy vertices in \( G \).

Note that by Theorem 1.8 \( G \) is hamiltonian. It is easy to check that if \( G \) has no more than five vertices, then it is pancyclic. Thus we assume \( |V(G)| \geq 6 \). Let \( C = v_0 \ldots v_{n-1}v_0 \) be a hamiltonian cycle in \( G \). Clearly, under the assumptions of the Lemma there are no edges of the form \( v_iv_i+2 \) in \( G \). In the following any arithmetic involving the subscripts of the vertices of \( C \) is modulo \( n \). We begin the proof with an observation regarding heavy vertices of \( G \).

**Claim 4.8.** If \( v_i \) is a heavy vertex in \( G \), then at least one of the vertices \( v_{i-1} \) and \( v_{i+1} \) is also heavy.

**Proof.** Suppose to the contrary that neither \( v_{i-1} \) nor \( v_{i+1} \) is heavy. Since \( G \in \mathcal{F}(P_4,n) \), this implies that none of the paths \( v_{i-2}v_{i-1}v_iv_{i+1} \) and \( v_{i-1}v_iv_{i+1}v_{i+2} \) can be an induced one. Since there are no cycles of length \( n-1 \) in \( G \), \( v_{i-2}v_i, v_{i-1}v_i, v_i \in E(G) \), implying that \( v_{i-2}v_{i+1} \) and \( v_{i-1}v_{i+2} \) are edges in \( G \). Now consider the path \( P = v_{i+3}C^+v_{i-3} \). Clearly, \( dp(v_i) \geq n/2 - 2 = (|V(P)| + 1)/2 \). If \( v_i \) is adjacent to two consecutive vertices of the path, say \( v_k \) and \( v_{k+1} \), then the cycle \( v_{i+1}C^+v_{i+2}v_{i+3}C^+v_{i+4}v_{i+5} \) is a cycle of length \( n-1 \), a contradiction. This implies that \( |V(P)| \) is odd and that the neighbourhood of \( v_i \) in \( P \) is \( N_P(v_i) = \{v_{i+3}, v_{i+5}, \ldots, v_{i-5}, v_{i-3}\} \). Clearly, if \( v_{i-1}v_{i+3} \in E(G) \), then there is a cycle of length \( n-1 \) in \( G \), namely \( v_{i+1}v_{i+2}v_{i+3}C^+v_{i+4}v_{i+5} \). Thus \( v_{i-1}v_{i+3} \notin E(G) \). But now \( \{v_i; v_{i-1}, v_{i+1}, v_{i+3}\} \) induces a claw in \( G \). Since neither \( v_{i-1} \) nor \( v_{i+1} \) is heavy, this contradicts \( G \) being a member of the family \( \mathcal{F}(K_{1,3}, n) \). □

**Claim 4.9.** If \( v_i \) is a heavy vertex in \( G \), then \( d_G(v_i) = n/2 \).

**Proof.** By Claim 4.8, assume that both \( v_i \) and \( v_{i+1} \) are heavy. If the degree of \( v_i \) is strictly greater than \( n/2 \), then \( d_G(v_i) + d_G(v_{i+1}) \geq n + 1 \) and so \( G \) is pancyclic by Lemma 4.2. This contradicts the assumption of \( G \) missing the \((n-1)\)-cycle. □

**Claim 4.10.** If \( v_i \) and \( v_{i+1} \) are heavy vertices in \( G \), then none of the vertices \( v_{i-2}, v_{i-1}, v_{i+2} \) and \( v_{i+3} \) is heavy and the vertices \( v_{i+4} \) and \( v_{i+5} \) are both heavy. Furthermore, the path \( v_{i-2}v_{i-1}v_{i+1}v_{i+2}v_{i+3} \) is an induced one, and \( v_{i-3}, v_{i-4}, v_{i+1}v_{i+4}, v_{i+1}v_{i+5} \) are edges in \( G \).
Proof. Since $G$ is missing the $(n - 1)$-cycle, it is not bipartite and, by Claim 4.9, the degrees of both $v_i$ and $v_{i+1}$ are equal to $n/2$, it follows from Lemma 4.3 that $v_{i-2}v_{i-1}v_iv_{i+1}v_{i+2}v_{i+3}$ is an induced path $P_6$ in $G$ and that $v_i$ is adjacent to $v_{i-3}$ and $v_{i-4}$, and $v_{i+1}$ is adjacent to $v_{i+4}$ and $v_{i+5}$. Suppose $v_{i-1}$ is heavy. Then applying Lemma 4.3 to the pair $v_{i-1}$, $v_i$ leads to a contradiction with the adjacency structure it provides, since $v_iv_{i+3} \notin E(G)$. Similar contradiction arises if we suppose that $v_{i+2}$ is heavy and apply Lemma 4.3 to the pair $v_{i+1}$, $v_{i+2}$. Thus neither $v_{i-1}$ nor $v_{i+2}$ is heavy. Now suppose that $v_{i-2}$ is heavy. From the previous observation and from Claim 4.8 it follows that $v_{i-3}$ is also heavy. Since $v_{i-2}v_{i+1} \notin E(G)$ this again leads to a contradiction with the structure described by Lemma 4.3, when applied to the pair $v_{i-2}$, $v_{i-3}$. Similar contradiction is obtained if one assumes that $v_{i+3}$ is heavy. Thus the first part of the Claim holds.

Now it will be shown that $v_{i+4}$ is heavy. Suppose to the contrary that $d_G(v_{i+4}) < n/2$. Since the degree of $v_{i+2}$ is also less than $n/2$, the path $v_{i+2}v_{i+3}v_{i+4}v_{i+5}$ can not be an induced one. This implies that $v_{i+2}v_{i+5} \in E(G)$. Similarly, to avoid induced claw \{v_{i+1}; v_i, v_{i+2}, v_{i+4}\}, $v_i$ is adjacent to $v_{i+4}$. But these two edges create in $G$ a cycle of length $n - 1$, namely $v_{i+2}v_{i+5}C^+v_iv_{i+4}v_{i+3}v_{i+2}$, a contradiction. Thus $v_{i+4}$ is heavy. Since $d_G(v_{i+3}) < n/2$, the heaviness of $v_{i+5}$ follows from Claim 4.8.

Since there is a heavy vertex in $G$, we can assume without loss of generality that the vertices $v_0$ and $v_1$ are heavy, by Claim 4.8. It follows from Claim 4.10 that $v_4$ and $v_5$ are also heavy. Applying Claim 4.10 to the pair $v_4$, $v_5$ we obtain the heaviness of the vertices $v_8$ and $v_9$, and so on, i.e., every vertex $v$ of $G$ with $d_C(v_0, v) \in \{4k, 4k+1\}$ for some non-negative integer $k$, is heavy. Similarly, every $v \in V(G)$ with $d_C(v_0, v) \in \{4k + 2, 4k + 3\}$ is not heavy. Thus the number of vertices of $G$ is divisible by four. Let $n = 4r$, with $r \geq 2$. Then the set of heavy vertices of $G$ is \{v_0, v_1, v_4, v_5, \ldots, v_{4r-4}, v_{4r-3}\} and the remaining vertices are not heavy. With the following claim we establish the existence of a perfect matching between the sets of heavy and non-heavy vertices of $G$.

Claim 4.11. Every heavy vertex of $G$ is adjacent to exactly one non-heavy vertex.

Proof. Suppose the contrary. Let $v_i$ be a heavy vertex of $G$ with at least two non-heavy neighbours. From Claims 4.8, 4.9 and 4.10 it follows that at least one of these neighbours, say $v_k$, satisfies $d_C(v_i, v_k) \geq 5$. Claims 4.8 and 4.10 imply that exactly one of the vertices $v_{i-1}$ and $v_{i+1}$ is also not heavy. Thus \{v_i; v_{i-1}, v_{i+1}, v_k\} can not induce a claw, since $G \in \mathcal{F}(K_{1,3}, n)$. Since there are no cycles of length $n - 1$ in $G$, it follows that $v_kv_{i-1}$ or $v_kv_{i+1}$ is an edge in $G$.

Depending on which of the vertices $v_{k-1}$ and $v_{k+1}$ is heavy, either $v_{k-1}v_k$ or else $v_{k-2}v_{k+1}$ is an edge in $G$, by Claim 4.10. Denote this edge $w_1w_2$. This, together with the previous observations, implies that either $v_iC^+w_1w_2C^+v_{i-1}v_kv_i$ or $v_iC^-w_2w_1C^-v_{i+1}v_kv_i$ is a cycle in $G$. Since the length of this cycle is $n - 1$, this contradicts the assumption of $G$ missing the $(n - 1)$-cycle.

Claim 4.11 implies that, since there are $2r$ heavy vertices and $2r$ non-heavy vertices in $G$, in order for the heavy vertices to be indeed heavy, every two of them are adjacent. Thus the heavy vertices induce a clique in $G$ and there is a perfect matching between
this clique and the set of non-heavy vertices, since every heavy vertex has a non-heavy neighbour that lies next to it on the cycle \( C \). Clearly, every non-heavy vertex \( v \) has a unique non-heavy neighbour \( u \) with \( d_C(v, u) = 1 \). To complete the proof it suffices to show that every non-heavy vertex is adjacent to exactly one non-heavy vertex.

Suppose this is not the case. Let \( v_k \) be a non-heavy vertex with \( v_{k+1} \) being also not heavy. Suppose \( v_k \) has a neighbour in a pair of non-heavy vertices \( \{v_m, v_{m+1}\} \). From Claim 4.10 it follows that \( d_C(v_k, v_m) \geq 7 \). Since the heavy vertices of \( G \) induce a clique, either \( v_kv_mv_{m-1}v_{m+2}C^+v_{k-1}v_{m-2}C^-v_k \) or \( v_kv_{m+1}C^+v_{k-1}v_{m-1}C^-v_k \) is a cycle in \( G \). The length of this cycle is \( n - 1 \). This final contradiction completes the proof of Lemma 4.7.

\[ \square \]

**Lemma 4.12.** Let \( G \) be a 2-connected graph of order \( n \geq 3 \) and let \( u \) and \( v \) be heavy vertices in \( G \). If \( G \in \mathcal{F}(\{K_{1,3}, P_4\}, n) \), then

(i) if \( G \) is not bipartite, then \( G \) contains a triangle,

(ii) there is a cycle of length four in \( G \).

**Proof.** For the proof of (i) assume that \( G \) is not bipartite. As the statement is easy to verify for \( n \leq 4 \), we further assume that \( n \geq 5 \). Clearly, if there is an edge in the subgraph of \( G \) induced by the neighbourhood \( N_G(u) \) of \( u \), then there is a triangle in \( G \). Suppose that \( G[N_G(u)] \) is edgeless. Since \( G \in \mathcal{F}(K_{1,3}, n) \), it follows that at most one of the neighbours of \( u \) is not heavy. Observe that \( G \) is hamiltonian by Theorem 1.8.

Let \( C = v_1 \ldots v_nv_1 \) be a hamiltonian cycle in \( G \) with \( v_1 = u \). Since at least one of the vertices \( v_2 \) and \( v_n \) is heavy, Lemma 4.3 implies that there is a triangle in \( G \).

Now it will be shown that (ii) holds. Clearly, if \( u \) and \( v \) have at least two common neighbours, then \( G \) contains \( C_4 \). Thus suppose they have at most one common neighbour. Since both \( u \) and \( v \) are heavy, it follows that \( uv \in E(G) \). If \( u \) and \( v \) have no common neighbours, then \( V(G) = A \cup B \cup \{u, v\} \), where \( N_G(u) = A \cup \{v\} \), \( N_G(v) = B \cup \{u\} \) and \( A \cap B = \emptyset \). Since \( G \) is 2-connected, there is an edge \( ab \) in \( G \) for some \( a \in A \) and \( b \in B \). This edge creates the cycle \( uabvu \) of length four.

Assume that there is exactly one common neighbour of \( u \) and \( v \) in \( G \), say \( w \). Let \( N_G(u) = A \cup \{v, w\} \) and \( N_G(v) = B \cup \{u, w\} \), where \( A \cap B = \emptyset \). Furthermore, assume that \( N_G[w] \cap (A \cup B) = \emptyset \) and that there are no edges between the sets \( A \) and \( B \), since otherwise there is a cycle of length four in \( G \). From the 2-connectivity of \( G \) it follows that there is a path connecting \( A \) and \( B \) that is disjoint with the vertices \( u \) and \( v \). Hence, there is a vertex in \( V(G) \) that does not belong to \( A \cup B \cup \{u, v, w\} \).

This implies that

\[ |A| + |B| + 3 < n. \]

On the other hand, since \( u \) and \( v \) are heavy, both \( A \) and \( B \) contain at least \( n/2 - 2 \) vertices. Thus

\[ |A| + |B| + 4 \geq n. \]

Hence, \( |A| + |B| + 4 = n \), \( |A| = |B| = n/2 - 2 \), and there is exactly one vertex, say \( x \), in the set \( V(G) \setminus (A \cup B \cup \{u, v, w\}) \). In order to create a path between \( A \) and \( B \) with the set of vertices disjoint with both \( u \) and \( v \), the vertex \( x \) is as adjacent to some \( a \in A \) and some \( b \in B \). Clearly, none of the vertices from \( A \cup B \) is heavy. Since
$G \in \mathcal{F}(P_4, n)$, it follows that neither the path $uxxb$ nor the path $vbxu$ can be induced in $G$. Thus $xu, xv \in E(G)$. These edges create a cycle of demanded length, namely $uxvwu$.

Now we are ready to prove Theorem 1.9.

**Proof of Theorem 1.9.** Let $G$ be a graph satisfying the assumptions of the Theorem. Assume that $G$ is not one of $K_{n/2,n/2}, K_{n/2,n/2} - e$ and $F_4$. Lemmas 4.6 and 4.7 imply that $G$ is neither bipartite nor missing the $(n - 1)$-cycle. Furthermore, there is a hamiltonian cycle in $G$, by Theorem 1.8.

Toward a contradiction, suppose that $G$ is not pancyclic. Then it follows from Theorem 1.2 that $G$ is not $\{K_{1,3}, P_4\}$-free and so there are at least two heavy vertices in $G$. The following Claim gathers the pieces of information regarding cycles in $G$ that we have obtained so far.

**Claim 4.13.** $G$ contains cycles of lengths three, four, $n - 1$ and $n$.

**Proof.** The existence of the long cycles is clear. The fact that there are cycles $C_3$ and $C_4$ in $G$ follows from Lemma 4.12.

By Claim 4.13, if $n \leq 6$, then $G$ is pancyclic. So we assume that $n \geq 7$.

**Claim 4.14.** If $x, y \in V(G)$ are heavy in $G$, then for every hamiltonian cycle $C$ in $G$ holds $d(C, (x, y)) \geq 2$. Furthermore, if $d(C(x, y)) = 2$, then $d(C(x)) = d(C(y)) = n/2$ and $xy \in E(G)$.

**Proof.** Clearly, if $d(C, (x, y)) = 1$, then $G$ is pancyclic by Lemma 4.3. If $d(C, (x, y)) = 2$ and the degree of at least one of $x$ and $y$ is strictly greater than $n/2$, then $G$ is pancyclic by Lemma 4.4. Finally, if $d(C, (x, y)) = 2$ and $x$ is not adjacent to $y$, pancyclicity of $G$ follows from Claim 4.13 and Lemma 4.5.

Let $u$ be a vertex in $G$ with $d_G(u) \geq n/2$.

**Case 1.** $G - u$ is not 2-connected.

Under the assumptions of this case there is a vertex in $G$, say $v$, such that $G - \{u, v\}$ is not connected. Since $G$ is hamiltonian, we can set $C = uy_1 ... y_{h-1} vx_{h-1} ... x_1 u$ to be a hamiltonian cycle with $H_1 = \{x_1, ..., x_{h-1}\}$ and $H_2 = \{y_1, ..., y_h\}$ being the components of $G - \{u, v\}$. The following simple observation is crucial for the further reasoning.

**Claim 4.15.** There are no heavy vertices in at least one of the sets $H_1$ and $H_2$.

**Proof.** Suppose this is not the case. Then $h_1 = h_2 = (n - 2)/2$ and there are vertices $x \in H_1, y \in H_2$ such that $N_G(x) = H_1 \cup \{u, v\}$ and $N_G(y) = H_2 \cup \{u, v\}$. Thus $uyvxu$ is a cycle of length four in $G$. To this cycle we can append all vertices from $H_2$, one-by-one, creating cycles $uy_1yvxu, uy_1y_2yvxu, ..., uC^+yy_{h'}vxxu, uC^+yy_{h'+1}y_{h'}vxxu, ... , uC^+vxxu$. The vertices from $H_1$ can be appended to the longest of these cycles in a similar manner. In this way we obtain $[4, n]$-cycles in $G$. Since $G$ contains a triangle, by Claim 4.13, it is pancyclic. A contradiction. \qed
It follows from Claim 4.15 that for the rest of the proof of this case we may assume a lack of heavy vertices in $H_1$. We also assume that $y_1$ is not heavy, since the opposite yields a contradiction with Claim 4.14.

The next three claims describe the neighbourhood $N_G(u)$ of the vertex $u$.

**Claim 4.16.** $N_{H_2}[u] \subset N_G[y_1]$.

*Proof.* Otherwise $u$ is adjacent to some vertex $y \in H_2 \setminus N_G[y_1]$. Then $\{u; y, y_1, x_1\}$ induces a claw in $G$. Since neither $x_1$ nor $y_1$ is heavy, this contradicts $G$ being a member of the family $\mathcal{F}(K_{1,3}, n)$. $\square$

**Claim 4.17.** $N_{H_1}(u) = H_1$ and $N_{H_1}[u]$ induces a clique.

*Proof.* Since the statement is clearly true for $h_1 = 1$, assume that there are at least two vertices in $H_1$. Suppose that there is a vertex $x_i \in H_1$ such that $wx_i \notin E(G)$. Choose minimal $i$ with this property. Then the path $y_1ux_{i-1}x_i$ is induced in $G$. Since there are no heavy vertices in $H_1$ and $y_1$ is not heavy, this is a contradiction with $G$ belonging to the family $\mathcal{F}(P_4, n)$.

Now suppose that there are two nonadjacent neighbours of $u$ in $H_1$, say $x$ and $x'$. Then $\{u; x, x', y_1\}$ induces a claw, with none of its endvertices being heavy. Since $G \in \mathcal{F}(K_{1,3}, n)$, this is a contradiction. $\square$

**Claim 4.18.** $N_{H_2}(u) \neq H_2$.

*Proof.* Suppose the contrary. Then $uy_{h_2}vx_{h_1}u$ is a cycle in $G$, by Claim 4.17. To this cycle we can append the vertices from $H_1$, one-by-one, also by Claim 4.17. To the longest of the cycles obtained the vertices from $H_2$ can be appended in a similar way. With this procedure we obtain $[4, n]$-cycles in $G$. The pancyclicity of $G$ follows from Claim 4.13. $\square$

It follows from Claim 4.18 that there is a vertex $y_k$ in $N_{H_2}(u)$ such that $y_{k+1} \in H_2$ and $u$ is not adjacent to $y_{k+1}$. Choose minimal $k$ satisfying these conditions.

**Claim 4.19.** $y_k$ is heavy. In consequence, $k \geq 2$, both $y_{k-1}$ and $y_{k+1}$ are not heavy, and $y_{k-1}y_{k+1} \notin E(G)$.

*Proof.* Clearly, the path $x_1uy_{k}\{y_{k+1}\}$ is an induced one. Since $G \in \mathcal{F}(P_4, n)$ and $x_1$ is not heavy, it follows that $y_k$ is heavy. Now Claim 4.14 implies that $k \geq 2$ and that neither $y_{k-1}$ nor $y_{k+1}$ is heavy. The fact that $y_{k-1}$ is not adjacent to $y_{k+1}$ follows from Lemma 4.1. $\square$

**Claim 4.20.** There are $[n - h_1, n]$-cycles in $G$.

*Proof.* Claim 4.19 implies that $u$ is adjacent to $y_2$. Thus $C' = uy_2C^+x_{h_1}u$ is a cycle of length $n - h_1$, by Claim 4.17. Since $u$ is adjacent to every vertex of $H_1$, all these vertices can be appended to $C'$, one-by-one, creating cycles of demanded lengths. $\square$
Claim 4.21. \( uv \in E(G) \).

Proof. Suppose the contrary. Then \( y_1v \in E(G) \) to avoid induced path \( y_1ux_{h_1}v \) with neither \( y_1 \) nor \( x_{h_1} \) being heavy. Now it follows from Claims 4.16 and 4.17 that \( d_G(y_1) \geq n/2 - h_1 + 1 \). Set \( G' = G - \{x_1, \ldots, x_{h_1-1}\} \) if \( h_1 > 1 \) or \( G' = G \) otherwise. Since \( y_1y_k \in E(G) \), by Claim 4.16, \( G' \) is hamiltonian, with \( C' = uy_k^{-1}C^+y_1y_kC^+x_{h_1}u \) being its hamiltonian cycle. Note that \( d_{G'}(y_1) + d_{G'}(y_k) \geq n/2 - h_1 + 1 + n/2 = \left|G'\right| \), by Claim 4.19, and that \( uy_1y_2u \) is a triangle in \( G' \). Thus it follows from Lemma 4.3 that \( G' \) is either pancyclic or else missing only \((n-h_1)\)-cycle. In either case Claim 4.20 implies pancyclicity of \( G \).

The next claim provides a full description of the neighbourhood of the vertex \( y_1 \).

Claim 4.22. \( N_G[y_1] = N_{H_2}[u] \).

Proof. Suppose that the Claim is not true. Then it follows from Claims 4.16, 4.17 and 4.21 that \( d_G(y_1) \geq n/2 - h_1 - 1 + 1 = n/2 - h_1 \). By Claim 4.21 the cycle \( uy_{k-1}C^{-}y_1y_kC^{+}vu \) is a hamiltonian cycle in \( G' = G - H_1 \). Since \( d_{G'}(y_1) + d_{G'}(y_k) \geq \left|G'\right| \) and \( uy_1y_2u \) is a triangle in \( G' \), it follows from Lemma 4.3 that \( G' \) is either pancyclic or else missing only \((n-h_1)\)-cycle. By Claim 4.20, the same is true for \( G \). Since \( uy_2C^{+}vu \) is a cycle of length \( n - h_1 - 1 \), \( G \) is pancyclic.

Claim 4.23. \( y_{h_2} \) is adjacent neither to \( u \) nor to \( y_1 \).

Proof. Suppose this is not the case. Then, by Claim 4.22, \( y_{h_2} \) is adjacent to both \( u \) and \( y_1 \). If \( vy_k \notin E(G) \), then set \( G' = G - (H_1 \cup \{v\}) \). Note that the cycle \( uy_{k-1}C^{-}y_1y_kC^{+}y_{h_2}u \) is a hamiltonian cycle in \( G' \) and \( uy_2C^{+}y_{h_2}u \) is a cycle of length \( \left|G'\right| - 1 \). Since \( d_{G'}(y_1) + d_{G'}(y_k) \geq \left|G'\right| \), Lemma 4.3 implies that \( G' \) is pancyclic. Together with Claim 4.20 this implies pancyclicity of \( G \).

Now assume \( vy_k \in E(G) \). If \( vy_{k-1} \in E(G) \), then consider \( G' = G - H_1 \). Again, \( G' \) is a graph with both \( \left|G'\right| \)- and \( (\left|G'\right| - 1) \)-cycles, namely, \( uy_{k-1}C^{-}uy_kC^{+}v \) and \( uy_2C^{+}vu \). Clearly, \( d_{G'}(u) + d_{G'}(y_k) \geq \left|G'\right| \), by Claim 4.19, and \( G' \) is not bipartite. Thus \( G' \) is pancyclic, by Lemma 4.3, and so \( G \) is pancyclic, by Claim 4.20.

Hence, \( v \) is adjacent to \( y_k \) and not adjacent to \( y_{k-1} \). Now to avoid \( \{yk, y_{k-1}, v, y_{k+1}\} \) inducing a claw with neither \( y_{k-1} \) nor \( y_{k+1} \) being heavy, \( v \) is adjacent to \( y_{k+1} \). But then \( y_{k+1}vC^{+}uy_kC^{-}y_1y_{h_2}C^{-}y_{k+1} \) is a hamiltonian cycle in \( G \) with both \( u \) and \( y_k \) being heavy. This contradicts Claim 4.14.

Observe that, by Claims 4.21, 4.22 and 4.23, the path \( y_1uvy_{h_2} \) is an induced one. Since \( y_1 \) is not heavy, it follows that \( v \) is heavy. In consequence, \( y_{h_2} \) is not heavy, by Claim 4.14.

Claim 4.24. \( y_{h_2} \) is adjacent to both \( y_k \) and \( y_{k+1} \).

Proof. We first observe that \( vy_{h_2-1} \in E(G) \). Clearly, otherwise the path \( x_{h_1}vy_{h_2}y_{h_2-1} \) is an induced one. Since neither \( x_{h_1} \) nor \( y_{h_2} \) is heavy, this contradicts \( G \) belonging to the family \( \mathcal{F}(P_4, n) \).

Now suppose that \( y_{h_2} \) is not adjacent to \( y_k \). Set \( G' = G - (H_1 \cup y_{h_2}) \). It follows from Claims 4.19, 4.21, 4.22 and 4.23 that \( d_{G'}(y_1) + d_{G'}(y_k) \geq \left|G'\right| \). Since
\[ y_1 y_k C^+ y_{h_2 - 1} v u y_{k - 1} C^- y_1 \] is a hamiltonian cycle and \[ u y_2 C^+ y_{h_2 - 1} v u \] is a cycle of length \(|G'| - 1| in \ G', \) Lemma 4.3 implies pancyclicity of \(G'\). Thus there are \([3, n - h_1 - 1]\)-cycles in \(G\) and so \(G\) is pancyclic, by Claim 4.20.

Hence, \(y_{h_2} y_k \in E(G)\). Suppose \(y_{h_2} y_{k + 1} \notin E(G)\). It follows from Claims 4.22 and 4.23 and the choice of \(k\) that \({y_k; y_1, y_{h_2}, y_{k + 1}\}) induces a claw. Since none of the endvertices of this claw is heavy, this is a contradiction. Thus \(y_{h_2}\) is adjacent to \(y_{k + 1}\).

**Claim 4.25.** \(v\) is adjacent to every vertex from the set \(\{y_k, y_{k + 1}, \ldots, y_{h_2}\}\).

**Proof.** Suppose that the above statement is not true. Then there exists a vertex \(y_m \in N_{H_2}(v)\) such that \(y_m - 1 \in \{y_k, y_{k + 1}, \ldots, y_{h_2 - 1}\}\) and \(v y_{m - 1} \notin E(G)\). Choose maximal \(m\) satisfying these conditions. Note that, since \(v\) is heavy and \(G - v\) is not 2-connected, we could change \(u\) with \(v\) in the beginning of the proof of this case and repeat the reasoning conducted so far, obtaining in particular that \(N_{H_1}(v) = H_1\), and \(N_{H_2}(v) \neq H_2\). Then \(y_m\) would be an equivalent of \(y_k\) for \(u\), and thus we could show that \(y_m\) is heavy, and so on. Finally, similarly to Claim 4.24, i.e., the existence of the edge \(y_{h_2} y_{k + 1}\), by symmetry we would obtain the existence of the edge \(y_1 y_{m - 1}\). But then the cycle \(u y_k C^- y_1 y_{m - 1} C^- y_{k + 1} y_{h_2} C^- y_m v C^+ u\) is a hamiltonian cycle in \(G\) with \(d_G(u) + d_G(y_k) \geq n\), a contradiction with Claim 4.14.

Now it follows from Claim 4.25 that \(u y_k v u\) is a triangle in \(G\). Since \(\{y_1, \ldots, y_{k - 1}\} \subset N_G(u)\) and \(\{y_{k + 1}, \ldots, y_{h_2}\} \subset N_G(v)\), we can append the vertices from \(H_2\) to this triangle, one-by-one, obtaining cycles of all lengths from three up to \(h_2 + 2 = n - h_1\). Since there are also \([n - h_1, n]\)-cycles in \(G\), by Claim 4.20, this implies that \(G\) is pancyclic. This final contradiction completes the proof of this case.

**Case 2.** \(G - u\) is 2-connected.

Set \(G' = G - u\). Note that \(G'\) is not hamiltonian, by Lemma 4.1, and that for every heavy vertex \(v\) of \(G\) other than \(u\) we have \(d_{G'}(v) \geq n/2 - 1 = (n - 2)/2\). Thus \(G' \in F(\{K_{1, 3}, P_3\}, n - 2)\). It follows from Theorem 1.8 that there is a cycle of length \(n - 2\) in \(G'\), say \(C' = w_0 w_1 \ldots w_{n - 3} w_0\). In the following any arithmetic involving the subscripts of the vertices of \(C'\) is modulo \(n - 2\).

Let \(x\) be the vertex of \(G'\) such that \(x \notin V(C')\). Lemma 4.1 implies that \(d_{G'}(x) \leq (n - 2)/2\). Next we will show that this inequality is in fact strict.

**Claim 4.26.** \(d_{G'}(x) < (n - 2)/2\).

**Proof.** Suppose that the above statement is not true, i.e., that \(d_{G'}(x) = (n - 2)/2\). Since \(G'\) is not hamiltonian, we can assume \(N_{G'}(x) = \{w_0, w_2, \ldots, w_{n - 4}\}\). It is not difficult to see, that if \(u\) is joined by an edge with two consecutive vertices of \(C'\), then \(G\) is pancyclic. Thus

\[ n/2 \leq d_G(u) = d_{G'}(u) + e(u, x) \leq (n - 2)/2 + 1 = n/2, \]

implying that \(ux \in E(G)\) and \(u\) is joined to either each vertex of the set \(\{w_0, w_2, \ldots, w_{n - 4}\}\) or else to each vertex of the set \(\{w_1, w_3, \ldots, w_{n - 3}\}\). If the first
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In case occurs, then $G$ is clearly pancyclic. Thus assume the latter is true. Since $G$ is not bipartite, there is a chord in $C'$ joining two vertices whose indices have the same parity. One can easily check that $G$ is pancyclic.

\begin{claim}
ux \in E(G) and d_G(u) = n/2.
\end{claim}

\begin{proof}
If at least one of the above conditions is not satisfied, then $d_{C'}(u) \geq (n - 1)/2$, implying pancyclicity of $G - x$, by Lemma 4.1, and, in consequence, pancyclicity of $G$. \hfill \Box
\end{proof}

Fix $k$ for which there are no $k$-cycles in $G$. It follows from Claim 4.13 and the existence of $C'$ that $k \in \{5, 6, \ldots, n - 3\}$. Furthermore, for every $i$ from the set $\{0, 1, \ldots, n - 3\}$ we have $e(u, w_i) + e(u, w_{i+k-2}) \leq 1$, since otherwise $uw_iC'w_{i+k-2}u$ is a cycle $C_k$. This implies, together with Claim 4.27, that

$$n - 2 \leq 2d_{C'}(u) = \sum_{i=0}^{n-3}[e(u, w_i) + e(u, w_{i+k-2})] \leq n - 2.$$ 

Thus $d_{C'}(u) = (n - 2)/2$ and the following holds:

$$\forall i \in \{0, 1, \ldots, n - 3\}: e(u, w_i) + e(u, w_{i+k-2}) = 1. \quad (4.1)$$

We also note that in order to avoid the cycle $xw_iC'w_{i+k-3}ux$ of length $k$, for every $i$ with $0 \leq i \leq n - 3$ the following inequality holds:

$$e(x, w_i) + e(u, w_{i+k-3}) \leq 1. \quad (4.2)$$

Now we examine relations between the vertices $u$ and $x$ and the vertices of the cycle $C'$.

\begin{claim}
Let $l$ be an integer satisfying $1 \leq l \leq k - 4$. If $w_i$ is a neighbour of $x$ in $V(C')$, then

(i) $xw_{i-l} \notin E(G)$,
(ii) $uw_{i-l} \in E(G)$,
(iii) $w_{i-l}$ is not heavy in $G$,
(iv) $w_{i-l}w_{i+l+1} \in E(G)$.
\end{claim}

\begin{proof}
The proof is by induction on $l$. Clearly, $xw_{i-1}, xw_{i+1} \notin E(G)$ to avoid hamiltonian cycle in $G$. Since $x$ is adjacent to $w_i$, it follows from (4.2) that $uw_{i+k-3} \notin E(G)$. Thus, by (4.1), $u$ is adjacent to $w_{i-1}$. Note that $uxw_iC'w_{i-1}u$ is a hamiltonian cycle in $G$. Since $u$ is heavy, Claim 4.14 implies that $w_{i-1}$ is not heavy. To prove (iv) observe that if $w_{i-1}w_{i+1}$ is not an edge in $G$, then $\{w_i; w_{i-1}, x, w_{i+1}\}$ induces a claw. Since neither $w_{i-1}$ nor $x$, by Claim 4.26, is heavy, this contradicts $G$ being a member of the family $\mathcal{F}(K_{1,3}, n)$.

Assume that the Claim holds for the values from the set $\{1, 2, \ldots, l\}$ with $l$ satisfying $l < k - 4$. We will show that this implies the validity of the claim for $l + 1$.

Suppose $xw_{i-l-1} \in E(G)$. Then, by the condition (iv) for $l$, there is a hamiltonian cycle in $G'$, namely $xw_{i-l-1}C'-w_{i+1}w_{i-l}C'^+w_i$. This contradiction proves (i).
By the conditions (i) and (ii) the vertex \( w_{i-l} \) is adjacent to \( u \) and not adjacent to \( x \). Thus \( uw_{i-l-1} \in E(G) \) to avoid induced path \( xuw_{i-l}w_{i-l-1} \) with neither \( x \) nor \( w_{i-l} \) being heavy. This proves (ii). Now, since \( uw_{i-l-1} \in E(G) \) and, by (iv), \( w_{i-l} \) is adjacent to \( w_{i+1} \), the cycle \( uw_{i-l-1}C'-w_{i+1}w_{i-l}C'^+w_{i}xu \) is a hamiltonian cycle in \( G \). Since \( u \) is heavy, Claim 4.14 implies that \( w_{i-l-1} \) is not heavy.

For the proof of (iv) suppose that \( w_{i-l-1} \) is not adjacent to \( w_{i+1} \). Note that \( uw_{i+1} \in E(G) \) to avoid induced path \( xuw_{i-l}w_{i-l-1} \) with neither \( x \) nor \( w_{i-l} \) being heavy in \( G \). But this implies that \( \{u; x, w_{i-l-1}, w_{i+1}\} \) induces a claw in \( G \). Since none of the vertices \( x \) and \( w_{i-l-1} \) is heavy, this contradicts \( G \) belonging to the family \( \mathcal{F}(K_{1,3}, n) \).

By mathematical induction the Claim is true.

Since \( G \) is 2-connected, there is a vertex \( w_i \in V(C') \) adjacent to \( x \). From Claim 4.28 it follows that \( uxw_iw_{i+1}w_{i-l-1}C'-w_{i-k+4}u \) is a cycle in \( G \). Since the length of this cycle is \( k \), this contradicts the choice of \( k \). This final contradiction completes the proof.

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