THE SPECTRAL THEOREM
FOR LOCALLY NORMAL OPERATORS

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Abstract. We prove the spectral theorem for locally normal operators in terms of a locally spectral measure. In order to do this, we first obtain some characterisations of local projections and we single out and investigate the concept of a locally spectral measure.

Keywords: locally Hilbert space, locally C^*-algebra, locally normal operator, local projection, locally spectral measure.

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1. INTRODUCTION

This article is a continuation of our investigations from [4] and [5] on operator theory on locally Hilbert spaces. This research is motivated, partially, by the theory of certain locally convex ∗-algebras that was initiated by G. Allan [1], C. Apostol [2], A. Inoue in [6], and K. Schmüdgen [10], and continued by N.C. Phillips [9]. The interest for this research got bigger when combined with the theory of Hilbert modules over locally convex ∗-algebras that grew up from the works of A. Mallios [8] and D.V. Voiculescu [14]. This motivated our interest in obtaining an operator model for locally Hilbert C^*-modules in [5] and then in studying locally Hilbert spaces from the topological point of view in [4].

In this article we obtain the spectral theorem for locally normal operators in terms of a spectral measure. In order to do this, we first continue our investigations on locally Hilbert spaces, then we investigate the geometry of local projections on locally Hilbert spaces and obtain a model for locally spectral measures.

Some of the basic tools in this enterprise are the concepts of inductive and projective limits and that of coherent transformations of linear maps, between appropriate inductive or projective limits, that have been carefully reviewed in [4] and we do not repeat them here. Subsection 2.1 starts by recalling a few basic things about locally Hilbert spaces and their topological and geometrical properties from [4] and then

in Subsection 2.2 we review the basic definitions and facts on locally bounded operators. This concept originates in A. Inoue's [6] proof of the generalisation of Gelfand-Naimark Theorem for locally C^*-algebras and was formalised in this way in [4]. One of the most useful facts that we get here are Proposition 2.12 which shows when and how we can assemble a locally bounded operator from its many pieces and Proposition 2.14 that provides a block-matrix characterisation of locally bounded operators in terms of any fixed component. Then we briefly review the locally C^*-algebra $\mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ and two of its weak topologies, the weak operatorial and the strong operatorial topologies.

In Section 3 we consider the basic concepts, like spectrum and resolvent sets, associated to locally bounded operators and then move to a careful investigation on local projections and their geometry. Subsection 3.3 is dedicated to a first encounter of locally normal operators and their basic properties. The main results of this article are contained in Section 4. Our first objective is to get the appropriate concept of a locally spectral measure and then to show that it can be lifted to a well-behaved spectral measure, which is performed in two steps: firstly, in Lemma 4.2 we extend the locally spectral measure to the σ-algebra $\widetilde{\Omega}$ that is generated by the locally σ-algebra Ω and then, in Proposition 4.3 we extend its codomain to $\mathcal{B}(\widetilde{\mathcal{H}})$, where $\widetilde{\mathcal{H}}$ denotes the Hilbert space completion of the inductive limit $\mathcal{H}$. We reach the goal of this article in Subsection 4.3 where we prove in Theorem 4.7 that any locally normal operator has a locally spectral measure, uniquely determined by usual additional properties.

The results we obtained in this article raise the question on obtaining a functional model for locally normal operators. In order to do this, we first have to obtain a functional model for locally Hilbert spaces. All these will be the contents of a forthcoming article.

The locally bounded operators that we consider in this article are, when viewed from the perspective of operator theory on Hilbert spaces, examples of closable and densely defined operators that share a common core, and hence have useful algebraic properties. As a conclusion, from this point of view, what we do here is a special type of spectral theory for unbounded normal operators. We intend to clarify these aspects and apply to concrete operators in future research.

2. LINEAR OPERATORS ON LOCALLY HILBERT SPACES

2.1. LOCALLY HILBERT SPACES

In this subsection, we assume the notation and the facts on inductive limits as in Subsection 2.2 in [4]. A *locally Hilbert space* is an *inductive limit*

$$\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda = \bigcup_{\lambda \in \Lambda} \mathcal{H}_\lambda,$$

of a *strictly inductive system of Hilbert spaces* $\{\mathcal{H}_\lambda\}_{\lambda \in \Lambda}$, that is,

(lhs1) $(\Lambda, \leq)$ is a directed poset,
(lhs2) $\{\mathcal{H}_\lambda, \langle \cdot, \cdot \rangle_{\mathcal{H}_\lambda}\}_{\lambda \in \Lambda}$ is a net of Hilbert spaces,

(lhs3) for each $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ we have $\mathcal{H}_\lambda \subseteq \mathcal{H}_\mu$,

(lhs4) for each $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$ the inclusion map $J_{\mu,\lambda} \colon \mathcal{H}_\lambda \to \mathcal{H}_\mu$ is isometric, that is,

$$\langle x, y \rangle_{\mathcal{H}_\lambda} = \langle x, y \rangle_{\mathcal{H}_\mu}, \text{ for all } x, y \in \mathcal{H}_\lambda. \tag{2.1}$$

For each $\lambda \in \Lambda$, letting $J_\lambda \colon \mathcal{H}_\lambda \to \mathcal{H}$ be the inclusion of $\mathcal{H}_\lambda$ in $\bigcup_{\lambda \in \Lambda} \mathcal{H}_\lambda$, the *inductive limit topology* on $\mathcal{H}$ is the strongest that makes the linear maps J_λ continuous for all $\lambda \in \Lambda$.

On $\mathcal{H}$ a canonical inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ can be defined as follows:

$$\langle h, k \rangle_{\mathcal{H}} = \langle h, k \rangle_{\mathcal{H}_\lambda}, \quad h, k \in \mathcal{H}, \tag{2.2}$$

where $\lambda \in \Lambda$ is any index for which $h, k \in \mathcal{H}_\lambda$.

Remark 2.1. With notation as before, it follows that the definition of the inner product as in (2.2) is correct and, for each $\lambda \in \Lambda$, the inclusion map

$$J_\lambda \colon (\mathcal{H}_\lambda, \langle \cdot, \cdot \rangle_{\mathcal{H}_\lambda}) \to (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$$

is isometric. This implies that, letting $\| \cdot \|_{\mathcal{H}}$ denote the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ on $\mathcal{H}$, the *norm topology* on $\mathcal{H}$ is weaker than the inductive limit topology of $\mathcal{H}$. Since the norm topology is Hausdorff, it follows that the inductive limit topology on $\mathcal{H}$ is Hausdorff as well. In the following we let $\widetilde{\mathcal{H}}$ denote the completion of the inner product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ to a Hilbert space.

In addition, on the locally Hilbert space $\mathcal{H}$ we consider the *weak topology* as well, that is, the locally convex topology induced by the family of seminorms $\mathcal{H} \ni h \to |\langle h, k \rangle|$, indexed by $k \in \mathcal{H}$.

Remark 2.2. Clearly, the weak topology on any locally Hilbert space is Hausdorff separated as well. On the other hand, there is a weak topology on the Hilbert space $\widetilde{\mathcal{H}}$, determined by all linear functionals $\widetilde{\mathcal{H}} \ni h \mapsto \langle h, k \rangle$, for $k \in \widetilde{\mathcal{H}}$, and this induces a topology on $\mathcal{H}$, determined by all linear functionals $\mathcal{H} \ni h \mapsto \langle h, k \rangle$, for $k \in \widetilde{\mathcal{H}}$, different than the weak topology on $\mathcal{H}$; in general, the weak topology of $\mathcal{H}$ is weaker than the topology induced by the weak topology of $\widetilde{\mathcal{H}}$ on $\mathcal{H}$.

For an arbitrary nonempty subset $\mathcal{S}$ of a locally Hilbert space $\mathcal{H}$ we denote, as usually, the *orthogonal companion* of $\mathcal{S}$ by $\mathcal{S}^\perp = \{k \in \mathcal{H} \mid \langle h, k \rangle = 0 \text{ for all } h \in \mathcal{H}\}$.

Remark 2.3. Clearly, if $\mathcal{S}$ is a subset of the locally Hilbert space $\mathcal{H}$, it follows that $\mathcal{S}^\perp$ is always a weakly closed subspace of $\mathcal{H}$. In addition, the weak topology provides a characterisation of those linear manifolds $\mathcal{L}$ in $\mathcal{H}$ such that $\mathcal{L} = \mathcal{L}^{\perp\perp}$. More precisely, if $\mathcal{L}$ is a subspace of $\mathcal{H}$ and we denote by $\overline{\mathcal{L}}^{\mathrm{w}}$ its weak closure, then $\mathcal{L}^\perp$ is weakly closed and $\mathcal{L}^\perp = \overline{\mathcal{L}}^{\mathrm{w}\perp}$, cf. Lemma 2.2 in [4]. In particular, a subspace $\mathcal{L}$ of the inner product space $\mathcal{H}$ is weakly closed if and only if $\mathcal{L} = \mathcal{L}^{\perp\perp}$, cf. Proposition 2.3 in [4].

For two linear subspaces $\mathcal{S}$ and $\mathcal{L}$ of $\mathcal{H}$, that are mutually orthogonal, denoted $\mathcal{S} \perp \mathcal{L}$, we denote by $\mathcal{S} \oplus \mathcal{L}$ their algebraic sum. Also, a linear operator $T \colon \mathcal{H} \to \mathcal{H}$

is called *projection* if $T^2 = T$ and *Hermitian* if $\langle Th, k \rangle = \langle h, Tk \rangle$ for all $h, k \in \mathcal{H}$. It is easy to see that any Hermitian projection T is *positive* in the sense $\langle Th, h \rangle \geq 0$ for all $h \in \mathcal{H}$ and that T is a Hermitian projection if and only if $I - T$ is the same.

For an arbitrary subspace $\mathcal{S}$ of the locally Hilbert space $\mathcal{H}$, the *weak topology of* $\mathcal{S}$ is defined as the locally convex topology generated by the family of seminorms $\mathcal{S} \ni h \mapsto |\langle h, k \rangle|$, indexed on $k \in \mathcal{S}$.

Remark 2.4. With notation as before, it is clear that the weak topology of $\mathcal{S}$ is weaker than the topology on $\mathcal{S}$ induced by the weak topology of $\mathcal{H}$.

On the other hand, as a consequence of the fact that the weak topology of $\mathcal{S}$ is induced by the family of linear functionals $\{f_k\}_{k \in \mathcal{S}}$, which is a linear space, where $f_k(h) = \langle h, k \rangle$, for $h \in \mathcal{S}$, as a consequence of a general fact from duality theory, e.g. see Theorem 1.3.1 in [7], it follows that a linear functional $\varphi \colon \mathcal{S} \to \mathbb{C}$ is continuous with respect to the weak topology of $\mathcal{S}$ if and only if there exists a unique $k_\varphi \in \mathcal{S}$ such that $\varphi(h) = \langle h, k_\varphi \rangle$ for all $h \in \mathcal{S}$, cf. Proposition 2.1 in [4].

The next proposition provides equivalent characterisations for orthocomplementarity of subspaces of a locally Hilbert space and it is actually more general, in the context of inner product spaces, cf. Proposition 2.4 in [4]. However, we state it as it is, since this is the only case when we use it.

Proposition 2.5. *Let $\mathcal{S}$ be a linear subspace of $\mathcal{H}$. The following assertions are equivalent:*

(i) *the weak topology of $\mathcal{S}$ coincides with the topology induced on $\mathcal{S}$ by the weak topology of $\mathcal{H}$, in particular $\mathcal{S}$ is weakly closed in $\mathcal{H}$,*
(ii) *for each $h \in \mathcal{H}$ the functional $\mathcal{S} \ni y \mapsto \langle y, h \rangle$ is continuous with respect to the weak topology of $\mathcal{S}$,*
(iii) $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$,
(iv) *there exists a Hermitian projection $P \colon \mathcal{H} \to \mathcal{H}$ such that $\mathrm{Ran}(P) = \mathcal{S}$.*

For a given locally Hilbert space $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$, it is important to understand the geometry of the components $\mathcal{H}_\lambda$ and their orthogonal complements $\mathcal{H}_\lambda^\perp$ within $\mathcal{H}$. For the proof of the next lemma we refer to Lemma 3.1 in [4].

Lemma 2.6. *For each $\lambda \in \Lambda$ we have $\mathcal{H} = \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda^\perp$, in particular there exists a unique Hermitian projection $P_\lambda \colon \mathcal{H} \to \mathcal{H}$ such that $\mathrm{Ran}(P_\lambda) = \mathcal{H}_\lambda$.*

With respect to the decomposition provided by Lemma 2.6, the underlying locally Hilbert space structure of $\mathcal{H}_\lambda^\perp$ can be explicitly described. For the proof of the next proposition we refer to Proposition 3.2 in [4].

Proposition 2.7. *Let $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$ and, for a fixed but arbitrary $\lambda \in \Lambda$, let us denote by $\Lambda_\lambda = \{\mu \in \Lambda \mid \lambda \leq \mu\}$ the branch of Λ defined by λ. Then, with respect to the induced order relation $\leq$, Λ_λ is a directed poset, $\{\mathcal{H}_\mu \ominus \mathcal{H}_\lambda \mid \mu \in \Lambda_\lambda\}$ is a strictly inductive*

system of Hilbert spaces and, modulo a canonical identification of $\varinjlim\limits_{\mu \in \Lambda_\lambda} (\mathcal{H}_\mu \ominus \mathcal{H}_\lambda)$ *with a subspace of* $\mathcal{H}$, *we have*

$$\mathcal{H}_\lambda^\perp = \varinjlim_{\mu \in \Lambda_\lambda} (\mathcal{H}_\mu \ominus \mathcal{H}_\lambda). \tag{2.3}$$

2.2. LOCALLY BOUNDED OPERATORS

With notation as in Subsection 2.1, let $\mathcal{H} = \varinjlim_{\lambda \in A} \mathcal{H}_\lambda$ and $\mathcal{K} = \varinjlim_{\lambda \in A} \mathcal{K}_\lambda$ be two locally Hilbert spaces generated by strictly inductive systems of Hilbert spaces $(\{\mathcal{H}_\lambda\}_{\lambda \in \Lambda}, \{J_{\nu,\lambda}^{\mathcal{H}}\}_{\lambda \leq \nu})$ and, respectively, $(\{\mathcal{K}_\lambda\}_{\lambda \in \Lambda}, \{J_{\nu,\lambda}^{\mathcal{K}}\}_{\lambda \leq \nu})$, indexed on the same directed poset Λ. For each $\lambda, \nu \in \Lambda$ with $\lambda \leq \nu$, consider the linear map $\pi_{\lambda,\nu} \colon \mathcal{B}(\mathcal{H}_\nu, \mathcal{K}_\nu) \to \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ defined by

$$\pi_{\lambda,\nu}(T) = J_{\nu,\lambda}^{\mathcal{K}}{}^* T J_{\nu,\lambda}^{\mathcal{H}}, \quad T \in \mathcal{B}(\mathcal{H}_\nu, \mathcal{K}_\nu). \tag{2.4}$$

For each $\lambda \in \Lambda$, let $J_\lambda^{\mathcal{H}}$ denote the embedding of $\mathcal{H}_\lambda$ in $\mathcal{H}$ and, similarly $J_\lambda^{\mathcal{K}}$ denote the embedding of $\mathcal{K}_\lambda$ in $\mathcal{K}$. By Lemma 2.6, for each $\lambda \in \Lambda$ let $P_\lambda^{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H}_\lambda$ be the canonical Hermitian projection on $\mathcal{H}_\lambda$, and similarly $P_\lambda^{\mathcal{K}} \colon \mathcal{K} \to \mathcal{K}_\lambda$. By inspection, it follows that the axioms (ps1)–(ps4) from the definition of projective systems, see Subsection 2.1 in [4], are fulfilled by the system $(\{\mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)\}_{\lambda \in \Lambda}, \{\pi_{\lambda,\nu}\}_{\lambda \leq \nu})$. Then, proceeding as in the construction of the projective limit described as in (2.2)–(2.5) in Subsection 2.1 of [4], it follows that the projective limit $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ is canonically embedded in $\mathcal{L}(\mathcal{H}, \mathcal{K})$, the vector space of all linear operators $T \colon \mathcal{H} \to \mathcal{K}$ in the following way: an operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ belongs to $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ if and only if, for each $\lambda \in \Lambda$ the operator

$$T_\lambda := P_\lambda^{\mathcal{K}} T J_\lambda^{\mathcal{H}} \colon \mathcal{H}_\lambda \to \mathcal{K}_\lambda \tag{2.5}$$

is bounded and then $T = \varprojlim_{\lambda \in \Lambda} T_\lambda$ in the sense made precise in the formulae (2.3) and (2.4) in [4]. We summarise these considerations in the following

Proposition 2.8. *With notation as before,* $(\{\mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)\}_{\lambda \in \Lambda}, \{\pi_{\lambda,\nu}\}_{\lambda \leq \nu})$ *is a projective system of Banach spaces and its projective limit* $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ *is canonically embedded in* $\mathcal{L}(\mathcal{H}, \mathcal{K})$.

Remarks 2.9. (a) With notation as before, there is a natural adjoint operation defined for operators $T \in \varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$, more precisely, considering the net $(T_\lambda)_{\lambda \in \Lambda}$, with $T_\lambda \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ defined as in (2.5), then it is easy to see that the net $(T_\lambda^*)_{\lambda \in \Lambda}$ yields a unique operator denoted by $T^* \colon \mathcal{K} \to \mathcal{H}$, such that,

$$T_\lambda^* = P_\lambda^{\mathcal{H}} T^* J_\lambda^{\mathcal{K}}, \quad \lambda \in \Lambda,$$

hence, we have $T^* = \varprojlim_{\lambda \in \Lambda} T_\lambda^*$, that is, $T^* \in \varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{K}_\lambda, \mathcal{H}_\lambda)$. Thus, we have an involution

$$\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda) \ni T \mapsto T^* \in \varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{K}_\lambda, \mathcal{H}_\lambda).$$

In addition, it is easy to see that the operators T and its adjoint T^* satisfy the usual duality with respect to the inner products on $\mathcal{H}$ and $\mathcal{K}$,

$$\langle Th, k \rangle_{\mathcal{K}} = \langle h, T^* k \rangle_{\mathcal{H}}, \quad h \in \mathcal{H}, \ k \in \mathcal{K}, \tag{2.6}$$

and that, $T^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ is the unique operator that satisfies (2.6).

(b) Let us observe that, the canonical injections $J_\lambda^{\mathcal{H}}$ and the canonical projections $P_\lambda^{\mathcal{H}}$ belong to $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda)$ and that, with respect to the involution defined at item (a) we have

$$(J_\lambda^{\mathcal{H}})^* = P_\lambda^{\mathcal{H}}, \quad (P_\lambda^{\mathcal{H}})^* = J_\lambda^{\mathcal{H}}, \quad \lambda \in \Lambda.$$

This can be obtained in many different ways: one simple way is to use (2.6).

One of the deficiencies of the locally convex space $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ is that it has poor properties with respect to composition of operators. In this respect, a much smaller locally convex projective limit space is considered.

A linear map $T \colon \mathcal{H} \to \mathcal{K}$ is called a *locally bounded operator* if it is a continuous double coherent linear map, in the sense defined in Subsection 2.3 in [4], more precisely,

(lbo1) there exists a net of operators $\{T_\lambda\}_{\lambda \in \Lambda}$, with $T_\lambda \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ such that $T J_\lambda^{\mathcal{H}} = J_\lambda^{\mathcal{K}} T_\lambda$ for all $\lambda \in \Lambda$,

(lbo2) the net of operators $\{T_\lambda^*\}_{\lambda \in \Lambda}$ is coherent as well, that is, $T_\nu^* J_{\nu,\lambda}^{\mathcal{K}} = J_{\nu,\lambda}^{\mathcal{H}} T_\lambda^*$, for all $\lambda, \nu \in \Lambda$ such that $\lambda \leq \nu$.

We denote by $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ the collection of all locally bounded operators $T \colon \mathcal{H} \to \mathcal{K}$.

Remarks 2.10. (a) It is easy to see that $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ is a vector space. Actually, there is a canonical embedding

$$\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K}) \subseteq \varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda). \tag{2.7}$$

(b) We can make even more explicit the embedding as in (2.7): the correspondence between $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ and the net of operators $\{T_\lambda\}_{\lambda \in \Lambda}$ as in (lbo1) and (lbo2) is unique. Given $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$, for arbitrary $\lambda \in \Lambda$ we have $T_\lambda h = Th$, for all $h \in \mathcal{H}_\lambda$, with the observation that $Th \in \mathcal{K}_\lambda$. Conversely, if $\{T_\lambda\}_{\lambda \in \Lambda}$ is a net of operators $T_\lambda \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ satisfying (lbo2), then letting $Th = T_\lambda h$ for arbitrary $h \in \mathcal{H}$, where $\lambda \in \Lambda$ is such that $h \in \mathcal{H}_\lambda$, it follows that T is a locally bounded operator: this definition is correct by (lbo2). In accordance with Subsection 2.3 in [4], we will use the notation

$$T = \varprojlim_{\lambda \in \Lambda} T_\lambda.$$

We first record an equivalent characterisation of locally bounded operators within the class of all linear operators between two locally Hilbert spaces. The proof is straightforward and we omit it.

Proposition 2.11. *Let $T\colon \mathcal{H} \to \mathcal{K}$ be a linear operator. Then T is locally bounded if and only if:*

(i) *for all $\lambda \in \Lambda$ we have $T\mathcal{H}_\lambda \subseteq \mathcal{K}_\lambda$ and, letting $T_\lambda := P_\lambda^{\mathcal{K}} T|\mathcal{H}_\lambda \colon \mathcal{H}_\lambda \to \mathcal{K}_\lambda$, where $P_\lambda^{\mathcal{K}}$ is the Hermitian projection of $\mathcal{K}$ onto $\mathcal{K}_\lambda$ as in Lemma 2.6, T_λ is bounded,*
(ii) *for all $\lambda, \nu \in \Lambda$ with $\lambda \leq \nu$, we have $T_\nu \mathcal{H}_\lambda \subseteq \mathcal{K}_\lambda$ and $T_\nu^* \mathcal{K}_\lambda \subseteq \mathcal{H}_\lambda$.*

On the other hand, in order to perform operator theory with locally bounded operators, we will need to assemble a net of bounded operators acting between component spaces into a locally bounded operator acting between the corresponding locally Hilbert spaces. The following result tells us which additional properties this net of bounded operators must have in order to produce a locally bounded operator.

Proposition 2.12. *Let $(T_\lambda)_{\lambda \in \Lambda}$ be a net with $T_\lambda \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ for all $\lambda \in \Lambda$. The following assertions are equivalent:*

(1) *for every $\lambda, \nu \in \Lambda$ such that $\lambda \leq \nu$ we have:*

$$T_\nu | \mathcal{H}_\lambda = J_{\nu,\lambda}^{\mathcal{K}} T_\lambda, \ and \ T_\nu P_{\lambda,\nu}^{\mathcal{H}} = P_{\lambda,\nu}^{\mathcal{K}} T_\nu, \tag{2.8}$$

where $P_{\lambda,\nu}^{\mathcal{H}}$ is the orthogonal projection of $\mathcal{H}_\nu$ onto its subspace $\mathcal{H}_\lambda$,
(2) *for every $\lambda, \nu \in \Lambda$ such that $\lambda \leq \nu$, with respect to the decompositions*

$$\mathcal{H}_\nu = \mathcal{H}_\lambda \oplus (\mathcal{H}_\nu \ominus \mathcal{H}_\lambda), \quad \mathcal{K}_\nu = \mathcal{K}_\lambda \oplus (\mathcal{K}_\nu \ominus \mathcal{K}_\lambda),$$

the operator T_ν has the following block matrix representation

$$T_\nu = \begin{bmatrix} T_\lambda & 0 \\ 0 & T_{\lambda,\nu} \end{bmatrix}, \tag{2.9}$$

for some bounded linear operator $T_{\lambda,\nu} \colon \mathcal{H}_\nu \ominus \mathcal{H}_\lambda \to \mathcal{K}_\nu \ominus \mathcal{K}_\lambda$,
(3) *there exists an operator $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ such that $T|\mathcal{H}_\lambda = J_\lambda^{\mathcal{K}} T_\lambda$ for all $\lambda \in \Lambda$.*

In addition, if any of these assertions holds (*hence all of them hold*)*, the operator $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ as in* (3) *is uniquely determined by $\{T_\lambda\}_{\lambda \in \Lambda}$.*

Proof. (1)$\Rightarrow$(2). Let $\lambda, \nu \in \mathcal{H}$ with $\lambda \leq \nu$ and such that both conditions in (2.8) hold. Since $\mathcal{H}_\lambda \subseteq \mathcal{H}_\nu$, the first condition in (2.8) means that $\mathcal{H}_\lambda$ is invariant under T_ν while the latter means that $\mathcal{H}_\lambda$ is invariant under T_ν^* as well. Therefore, the representation (2.9) holds.

(2)$\Rightarrow$(3). Assume now that, for every $\lambda, \nu \in \Lambda$ with $\lambda \leq \nu$, the representation (2.9) holds. We define an operator $T\colon \mathcal{H} \to \mathcal{K}$ in the following way: for any $h \in \mathcal{H}$, there exists $\lambda \in \Lambda$ such that $h \in \mathcal{H}_\lambda$ and let $Th = J_\lambda^{\mathcal{K}} T_\lambda h$. We have to show that this definition is correct, that is, it does not depend on λ. To see this, let $\nu \in \Lambda$ be such that $h \in \mathcal{H}_\nu$ as well. Since Λ is directed, there exists $\eta \in \Lambda$ such that $\lambda, \nu \leq \eta$. By assumption, the representation (2.9) holds and, with respect to the decompositions

$$\mathcal{H}_\eta = \mathcal{H}_\eta \oplus (\mathcal{H}_\eta \ominus \mathcal{H}_\eta), \quad \mathcal{K}_\eta = \mathcal{K}_\eta \oplus (\mathcal{K}_\eta \ominus \mathcal{K}_\eta),$$

the operator T_η has the following block matrix representation

$$T_\eta = \begin{bmatrix} T_\nu & 0 \\ 0 & T_{\nu,\eta} \end{bmatrix}, \tag{2.10}$$

for some bounded linear operator $T_{\nu,\eta} \colon \mathcal{H}_\eta \ominus \mathcal{H}_\nu \to \mathcal{K}_\eta \ominus \mathcal{K}_\nu$. Since $h \in \mathcal{H}_\lambda \cap \mathcal{H}_\nu \subseteq \mathcal{H}_\eta$, from (2.9) and (2.10) it follows that $T_\lambda h = T_\eta h = T_\nu h$, hence the definition of the operator T is correct. After a moment of thought we see, e.g. by means of Proposition 2.11, that $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$. Clearly, the operator T is uniquely determined by the net $(T_\lambda)_{\lambda \in \Lambda}$.

(3)$\Rightarrow$(1). Let us assume that there exists an operator $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ such that $T|\mathcal{H}_\lambda = J_\lambda^\mathcal{K} T_\lambda$ for all $\lambda \in \Lambda$. This implies that the operators T_λ are exactly those induced by T as in Proposition 2.11 and, since T is assumed to be locally bounded, a straightforward argument shows that both properties as in (2.8) hold. $\qquad\square$

As a consequence of the previous proposition, we can introduce the adjoint operation on $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$. Let $T = \varprojlim_{\lambda \in \Lambda} T_\lambda \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ and hence the net $(T_\lambda)_{\lambda \in \Lambda}$ satisfies the conditions (2.8). Then, consider the net of bounded operators $(T_\lambda^*)_{\lambda \in \Lambda}$, $T_\lambda^* \in \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$ for all $\lambda \in \Lambda$ and observe that, for all $\lambda, \nu \in \Lambda$ with $\nu \geq \lambda$, we have

$$T_\nu^* | \mathcal{K}_\lambda = J_{\nu,\lambda}^\mathcal{H} T_\lambda^*, \text{ and } T_\nu^* P_{\lambda,\nu}^\mathcal{K} = P_{\lambda,\nu}^\mathcal{H} T_\nu^*, \tag{2.11}$$

hence, by Proposition 2.12 there exists a unique operator $T^* \in \mathcal{B}_{\mathrm{loc}}(\mathcal{K}, \mathcal{H})$ such that

$$T^* = \varprojlim_{\lambda \in \Lambda} T_\lambda^*. \tag{2.12}$$

Remarks 2.13. (a) It is easy to see that, with respect to the embedding $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ into $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$, the adjoint operation on $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$, see Remarks 2.9, is just a particular case of the adjoint operation on $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$, in particular the adjoint operation is conjugate linear, involutive, and (2.6) holds.

(b) Given three locally Hilbert spaces $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$, $\mathcal{K} = \varinjlim_{\lambda \in \Lambda} \mathcal{K}_\lambda$, and $\mathcal{G} = \varinjlim_{\lambda \in \Lambda} \mathcal{G}_\lambda$, indexed on the same poset Λ, let observe that the composition of locally compact operators yields locally compact operators, more precisely, whenever $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ and $S \in \mathcal{B}_{\mathrm{loc}}(\mathcal{K}, \mathcal{G})$ it follows that $ST \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{G})$ and usual algebraic properties as associativity and distributivity with respect to addition and multiplication with scalars hold. Moreover, for each $\lambda \in \Lambda$ we have

$$(ST)_\lambda = S_\lambda T_\lambda$$

and hence

$$ST = \varprojlim_{\lambda \in \Lambda} S_\lambda T_\lambda. \tag{2.13}$$

In addition, composition of locally bounded operators behaves as usually with respect to the adjoint operation, that is,

$$(ST)^* = T^* S^*, \quad T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K}), \ S \in \mathcal{B}_{\mathrm{loc}}(\mathcal{K}, \mathcal{G}).$$

(c) With respect to the pre-Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a locally bounded operator $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ is not, in general, a bounded operator. However, when considering $\mathcal{H}$ and $\mathcal{K}$ as dense linear subspaces of its Hilbert space completions $\widetilde{\mathcal{H}}$ and, respectively, $\widetilde{\mathcal{K}}$, it follows that both T and its adjoint T^* are densely defined hence, they are closable and, letting $\widetilde{T}$ denote the closure of T then the closure of T^* is exactly $\widetilde{T}^*$, the closure of T^*. So, when dealing with locally bounded operators we deal with a collection of closed and densely defined operators that have a common core for them, and a common core of all their adjoint operators.

In the following we point out an operator theoretic characterisation of locally bounded operators. Let $T \colon \mathcal{H} \to \mathcal{K}$ be a linear operator and, for arbitrary $\lambda \in \Lambda$, by Lemma 2.6, with respect to the decompositions,

$$\mathcal{H} = \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda^\perp, \quad \mathcal{K} = \mathcal{K}_\lambda \oplus \mathcal{K}_\lambda^\perp,$$

T has the following block matrix representation

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{2.14}$$

Proposition 2.14. *A linear operator $T \colon \mathcal{H} \to \mathcal{K}$ is locally bounded if and only if, for every $\lambda \in \Lambda$, its matrix representation (2.14) is diagonal, i.e. $T_{12} = 0$ and $T_{21} = 0$, and T_{11} is bounded. In addition, if $T = \varprojlim_{\nu \in \Lambda} T_\nu \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ then, for every $\lambda \in \Lambda$, with respect to the block matrix representation (2.14), we have $T_{11} = T_\lambda$ and, with respect to the locally Hilbert spaces $\mathcal{H}_\lambda^\perp$ and $\mathcal{K}_\lambda^\perp$ as in (2.3), $T_{22} \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}_\lambda^\perp, \mathcal{K}_\lambda^\perp)$.*

Proof. Assume that $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ and let $T = \varprojlim_{\nu \in \Lambda} T_\nu$. Then, for every $\lambda, \nu \in \Lambda$ with $\lambda \leq \nu$ we have the block matrix representation (2.10) for some bounded linear operator $T_{\lambda,\nu} \colon \mathcal{H}_\nu \ominus \mathcal{H}_\lambda \to \mathcal{K}_\nu \ominus \mathcal{K}_\lambda$. Then we observe that, for any fixed $\lambda \in \Lambda$, $\{T_{\lambda,\nu}\}_{\nu \in \Lambda_\lambda}$ is a projective system as in (2.8), with respect to the partially ordered set $\Lambda_\lambda = \{\nu \in \Lambda \mid \nu \geq \lambda\}$ and hence, by Proposition 2.12 there exists uniquely an operator $T^\lambda \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}_\lambda^\perp, \mathcal{K}_\lambda^\perp)$ such that

$$T = \begin{bmatrix} T_\lambda & 0 \\ 0 & T^\lambda \end{bmatrix}.$$

Conversely, let us assume that $T \colon \mathcal{H} \to \mathcal{K}$ has the property that for every $\lambda \in \Lambda$ the matrix representation (2.14) is diagonal and T_{11} is bounded. Then, letting $T_\lambda := T_{11}$, it follows that $\{T_\lambda\}_{\lambda \in \Lambda}$ satisfies the condition (1) as in Proposition 2.12 and hence $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$. $\qquad\square$

Remarks 2.15. (a) As a consequence of coherence, see Subsection 2.3 in [4], any locally bounded operator $T \colon \mathcal{H} \to \mathcal{K}$ is continuous with respect to the inductive limit topologies of $\mathcal{H}$ and $\mathcal{K}$.

(b) In general, a locally bounded operator $T \colon \mathcal{H} \to \mathcal{K}$ may not be continuous with respect to the norm topologies of $\mathcal{H}$ and $\mathcal{K}$. An arbitrary linear operator $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ is continuous with respect to the norm topologies of $\mathcal{H}$ and $\mathcal{K}$ if and only if, with respect to the notation as in (lbo1) and (lbo2), $\sup_{\lambda \in \Lambda} \|T_\lambda\|_{\mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)} < \infty$. In this case, the operator T uniquely extends to an operator $\widetilde{T} \in \mathcal{B}(\widetilde{\mathcal{H}}, \widetilde{\mathcal{K}})$, where $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{K}}$ are the Hilbert space completions of $\mathcal{H}$ and, respectively, $\mathcal{K}$, and $\|\widetilde{T}\| = \sup_{\lambda \in \Lambda} \|T_\lambda\|_{\mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)}$.

In general, we do not have equality in (2.7) so, it is of interest to have criteria to distinguish the operators in $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ within $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$. The following proposition is a direct consequence of the definition and Proposition 2.11, hence we omit the proof.

Proposition 2.16. *With respect to the embedding in (2.7), for an arbitrary element* $T = \varprojlim_{\lambda \in \Lambda} T_\lambda \in \varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$, *the following assertions are equivalent:*

(i) $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$,

(ii) *the axiom (lbo2) holds, that is,* $T_\nu^* J_{\nu,\lambda}^{\mathcal{K}} = J_{\nu,\lambda}^{\mathcal{K}} T_\lambda^*$, *for all* $\lambda, \nu \in \Lambda$ *such that* $\lambda \leq \nu$,

(iii) *for all* $\lambda, \nu \in \Lambda$ *with* $\lambda \leq \nu$, *we have* $T_\nu \mathcal{H}_\lambda \subseteq \mathcal{K}_\lambda$ *and* $T_\nu^* \mathcal{K}_\lambda \subseteq \mathcal{H}_\lambda$.

Remarks 2.17. (a) As a consequence of (2.7), $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ has a natural locally convex topology induced by the projective limit locally convex topology of $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$, more precisely, generated by the seminorms $\{q_\lambda\}_{\lambda \in \Lambda}$ defined by

$$q_\nu(T) = \|T_\nu\|_{\mathcal{B}(\mathcal{H}_\nu, \mathcal{K}_\nu)}, \quad T = \{T_\lambda\}_{\lambda \in \Lambda} \in \varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda).$$

(b) With respect to the embedding (2.7), $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ is closed in $\varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda)$, hence complete.

(c) The locally convex space $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})$ can be organised as a projective limit of locally convex spaces, in view of (2.7), more precisely, letting

$$\pi_\nu \colon \varprojlim_{\lambda \in \Lambda} \mathcal{B}(\mathcal{H}_\lambda, \mathcal{K}_\lambda) \to \mathcal{B}(\mathcal{H}_\nu, \mathcal{K}_\nu),$$

for $\nu \in \Lambda$, be the canonical projection, then

$$\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K}) = \varprojlim_{\lambda \in \Lambda} \pi_\lambda(\mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{K})).$$

2.3. THE LOCALLY C^*-ALGEBRA $\mathcal{B}_{\mathrm{loc}}(\mathcal{H})$

If $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$ is a locally Hilbert space then $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}) := \mathcal{B}_{\mathrm{loc}}(\mathcal{H}, \mathcal{H})$ has a natural product and a natural involution $*$, with respect to which it is a $*$-algebra, see Remark 2.13. For each $\mu \in \Lambda$, consider the C^*-algebra $\mathcal{B}(\mathcal{H}_\mu)$ of all bounded linear operators in $\mathcal{H}_\mu$ and $\pi_\mu \colon \mathcal{B}_{\mathrm{loc}}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_\mu)$ be the canonical map:

$$\pi_\mu(T) = T_\mu, \quad T = \varprojlim_{\lambda \in \Lambda} T_\lambda \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}).$$

Let $\mathcal{B}_{\mathrm{loc}}(\mathcal{H}_\mu)$ denote the range of π_μ and note that it is a C^*-subalgebra of $\mathcal{B}(\mathcal{H}_\mu)$. It follows that $\pi_\mu \colon \mathcal{B}_{\mathrm{loc}}(\mathcal{H}) \to \mathcal{B}_{\mathrm{loc}}(\mathcal{H}_\mu)$ is a $*$-morphism of $*$-algebras and, for each $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$, there is a unique $*$-epimorphism of C^*-algebras $\pi_{\lambda,\mu} \colon \mathcal{B}_{\mathrm{loc}}(\mathcal{H}_\mu) \to \mathcal{B}_{\mathrm{loc}}(\mathcal{H}_\lambda)$, such that $\pi_\lambda = \pi_{\lambda,\mu}\pi_\mu$. More precisely, compare with (2.4) and the notation as in Subsection 2.2, $\pi_{\lambda,\mu}$ is the compression of $\mathcal{H}_\mu$ to $\mathcal{H}_\lambda$,

$$\pi_{\lambda,\mu}(S) = J_{\mu,\lambda}^* S J_{\mu,\lambda}, \quad S \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}_\mu).$$

Then $(\{\mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda)\}_{\lambda \in \Lambda}, \{\pi_{\lambda,\mu}\}_{\lambda,\mu \in \Lambda, \ \lambda \le \mu})$ is a projective system of C^*-algebras, in the sense that,

$$\pi_{\lambda,\eta} = \pi_{\lambda,\mu} \circ \pi_{\mu,\eta}, \quad \lambda, \mu, \eta \in \Lambda, \ \lambda \le \mu \le \eta,$$

and, in addition,

$$\pi_\mu(S) P_{\lambda,\mu} = P_{\lambda,\mu} \pi_\mu(S), \quad \lambda, \mu \in \Lambda, \ \lambda \le \mu, \ S \in \mathcal{B}_{\text{loc}}(\mathcal{H}_\mu),$$

such that

$$\mathcal{B}_{\text{loc}}(\mathcal{H}) = \varprojlim_{\lambda \in \Lambda} \mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda),$$

where, the projective limit is considered in the category of locally convex $*$-algebras. In particular, $\mathcal{B}_{\text{loc}}(\mathcal{H})$ is a locally C^*-algebra.

For each $\lambda \in \Lambda$, letting $p_\lambda \colon \mathcal{B}_{\text{loc}}(\mathcal{H}) \to \mathbb{R}$ be defined by

$$p_\lambda(T) = \|T_\lambda\|_{\mathcal{B}(\mathcal{H}_\lambda)}, \quad T = \varprojlim_{\nu \in \Lambda} T_\nu \in \mathcal{B}_{\text{loc}}(\mathcal{H}), \tag{2.15}$$

then p_λ is a C^*-seminorm on $\mathcal{B}_{\text{loc}}(\mathcal{H})$. Then $\mathcal{B}_{\text{loc}}(\mathcal{H})$ becomes a unital locally C^*-algebra with the topology induced by $\{p_\lambda\}_{\lambda \in \Lambda}$.

The C^*-algebra $b(\mathcal{B}_{\text{loc}}(\mathcal{H}))$ is made up of all locally bounded operators $T = \varprojlim_{\lambda \in \Lambda} T_\lambda$ such that $\{T_\lambda\}_{\lambda \in \Lambda}$ is uniformly bounded, in the sense that $\sup_{\lambda \in \Lambda} \|T_\lambda\| < \infty$, equivalently, those locally bounded operators $T \colon \mathcal{H} \to \mathcal{H}$ that are bounded with respect to the canonical norm $\| \cdot \|_{\mathcal{H}}$ on the pre-Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$. In particular $b(\mathcal{A})$ is a C^*-subalgebra of $\mathcal{B}(\widetilde{\mathcal{H}})$, where $\widetilde{\mathcal{H}}$ denotes the completion of $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ to a Hilbert space.

As a locally convex space, $\mathcal{B}_{\text{loc}}(\mathcal{H})$ has its projective limit topology given by the family of seminorms $\{p_\lambda\}_{\lambda \in \Lambda}$ defined at (2.15). In this article, we use two other operator topologies. Briefly, the *weak operator topology* on $\mathcal{B}_{\text{loc}}(\mathcal{H})$ is the locally convex topology associated to the family of seminorms

$$\mathcal{B}_{\text{loc}}(\mathcal{H}) \ni T \mapsto \langle Th, k \rangle_{\mathcal{H}}, \quad h, k \in \mathcal{H}.$$

On the other hand, for each $\lambda \in \Lambda$ there is the weak operator topology $\tau_{\lambda,\text{wo}}$ on $\mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda)$ and $\{(\mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda), \tau_{\lambda,\text{wo}})\}_{\lambda \in \Lambda}$ is a projective system of locally convex spaces. The weak operator topology on $\mathcal{B}_{\text{loc}}(\mathcal{H})$ coincides with the projective limit topology of the projective system of locally convex spaces $\{(\mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda), \tau_{\lambda,\text{wo}})\}_{\lambda \in \Lambda}$.

The *strong operator topology* on $\mathcal{B}_{\text{loc}}(\mathcal{H})$ is the locally convex topology associated to the family of seminorms

$$\mathcal{B}_{\text{loc}}(\mathcal{H}) \ni T \mapsto \|Th\|_{\mathcal{H}}, \quad h \in \mathcal{H}.$$

On the other hand, for each $\lambda \in \Lambda$ there is the strong operator topology $\tau_{\lambda,\text{so}}$ on $\mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda)$ and $\{(\mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda), \tau_{\lambda,\text{so}})\}_{\lambda \in \Lambda}$ is a projective system of locally convex spaces.

The strong operator topology on $\mathcal{B}_{\text{loc}}(\mathcal{H})$ coincides with the projective limit topology of the projective system of locally convex spaces $\{(\mathcal{B}_{\text{loc}}(\mathcal{H}_\lambda), \tau_{\lambda,\text{so}})\}_{\lambda \in \Lambda}$.

3. CLASSES OF LOCALLY BOUNDED OPERATORS

3.1. RESOLVENT AND SPECTRUM

With notation as in the previous section, if $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$, its *resolvent set* is

$$\rho(T) = \{z \in \mathbb{C} \mid zI - T \text{ is invertible in } \mathcal{B}_{\mathrm{loc}}(\mathcal{H})\},$$

and its *spectrum* is

$$\sigma(T) = \mathbb{C} \setminus \rho(T) = \{z \in \mathbb{C} \mid zI - T \text{ is not invertible in } \mathcal{B}_{\mathrm{loc}}(\mathcal{H})\}.$$

Lemma 3.1. *Let $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ and assume that there exists $S \colon \mathcal{H} \to \mathcal{H}$ a linear operator such that $TS = ST = I_{\mathcal{H}}$. Then $S \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$.*

Proof. Let $\lambda \in \Lambda$ be arbitrary. Then, by Proposition 2.14, with respect to the decomposition

$$\mathcal{H} = \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda^\perp, \tag{3.1}$$

T has the following block matrix representation

$$T = \begin{bmatrix} T_\lambda & 0 \\ 0 & T_{22} \end{bmatrix},$$

and $T_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)$. Assume that $S \colon \mathcal{H} \to \mathcal{H}$ is a linear operator such that $ST = TS = I_{\mathcal{H}}$, and consider its matrix representation with respect to the decomposition (3.1)

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \tag{3.2}$$

Then,

$$T_\lambda S_{11} = S_{11} T_\lambda = I_{\mathcal{H}_\lambda}, \quad T_{22} S_{22} = S_{22} T_{22} = I_{\mathcal{H}_\lambda^\perp},$$

hence $S_{11} = T_\lambda^{-1}$ and, by the Closed Graph Theorem it is bounded, while $S_{22} = T_{22}^{-1} \colon \mathcal{H}_\lambda^\perp \to \mathcal{H}_\lambda^\perp$. On the other hand, since $T_\lambda S_{12} = 0$ and T_λ is invertible it follows that $S_{12} = 0$. Similarly, since $T_{22} S_{21} = 0$ and T_{22} is invertible it follows that $S_{21} = 0$. Thus, the matrix representation (3.2) of S is diagonal and S_{11} is bounded. By Proposition 2.14, it follows that $S \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ and $S_{11} = S_\lambda$ for all $\lambda \in \Lambda$.

The converse implication is clear. $\qquad\square$

Proposition 3.2. *Let $T = \varprojlim_{\lambda \in \Lambda} T_\lambda$ be a locally bounded operator in $\mathcal{B}_{\mathrm{loc}}(\mathcal{H})$. Then*

$$\rho(T) = \{z \in \mathbb{C} \mid zI - T \text{ is invertible in } \mathcal{L}(\mathcal{H})\}, \tag{3.3}$$

and

$$\rho(T) = \bigcap_{\lambda \in \Lambda} \rho(T_\lambda), \quad \sigma(T) = \bigcup_{\lambda \in \Lambda} \sigma(T_\lambda). \tag{3.4}$$

Proof. Indeed, the equality of (3.3) is a consequence of Lemma 3.1.

In order to prove the first equality in (3.4), we observe that, for every $z \in \rho(T)$, by Proposition 2.16, it follows that for every $\lambda \in \Lambda$ the operator $zI_\lambda - T_\lambda$ is invertible in $\mathcal{B}(\mathcal{H}_\lambda)$, more precisely, $(zI_\lambda - T_\lambda)^{-1} = P_\lambda^{\mathcal{H}}(zI - T)^{-1}J_\lambda^{\mathcal{H}}$. Conversely, if $z \in \rho(T_\lambda)$ for all $\lambda \in \Lambda$, then we observe that the net $((zI - T_\lambda)^{-1})_{\lambda \in \Lambda}$ satisfies the conditions in assertion (1) of Proposition 2.12 and hence, there exists uniquely an operator $S \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ such that $S|\mathcal{H}_\lambda = J^{\mathcal{H}}(zI_\lambda - T_\lambda)^{-1}$, for all $\lambda \in \Lambda$, and then $S(zI - T) = (zI - T)S = I$, hence $S = (zI - T)^{-1}$ and $z \in \rho(T)$. These prove the first equality in (3.4).

The latter equality in (3.4) is clearly a consequence of the first one. $\square$

Remark 3.3. As a consequence of the previous proposition, we see that the spectrum of a locally bounded operator is always non-empty but it may be neither closed nor bounded.

On the other hand, if Λ is countable, then the resolvent of any operator $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ is a G_δ-set in $\mathbb{C}$, while its spectrum is always an F_σ-subset of $\mathbb{C}$.

3.2. LOCAL PROJECTIONS

A linear operator $E \colon \mathcal{H} \to \mathcal{H}$ is called a *local projection* if $E \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ and $E^2 = E = E^*$. Clearly, E is a local projection if and only if $I - E$ is a local projection. As a consequence of Proposition 2.12 and Remarks 2.13, we have the following characterisation of local projections.

Lemma 3.4. *Given $E \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$, the following assertions are equivalent:*

(i) *E is a local projection,*
(ii) *$E = \varprojlim_{\lambda \in \Lambda} E_\lambda$ with $E_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)$ projections (that is, $E_\lambda^2 = E_\lambda^* = E_\lambda$) for all $\lambda \in \Lambda$ and, for all $\lambda, \nu \in \Lambda$ with $\lambda \leq \nu$ we have*

$$E_\nu|\mathcal{H}_\lambda = J_{\nu,\lambda}E_\lambda \ and \ E_\nu P_{\lambda,\nu} = P_{\lambda,\nu}E_\nu,$$

where $P_{\lambda,\nu} = J_{\nu,\lambda}^*$ denotes the projection of $\mathcal{H}_\nu$ onto $\mathcal{H}_\lambda$.

The following proposition provides equivalent characterisations for the ranges of local projections.

Proposition 3.5. *Let $\mathcal{L}$ be a subspace of the locally Hilbert space $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$. The following assertions are equivalent:*

(i) *there exists a strictly inductive system of Hilbert spaces $\{\mathcal{L}_\lambda\}_{\lambda \in \Lambda}$ such that, for each $\lambda \in \Lambda$, $\mathcal{L}_\lambda$ is isometrically embedded in $\mathcal{H}_\lambda$ and $\mathcal{L} = \varinjlim_{\lambda \in \Lambda} \mathcal{L}_\lambda$,*
(ii) *$\mathcal{H}_\lambda = (\mathcal{L} \cap \mathcal{H}_\lambda) \oplus (\mathcal{L}^\perp \cap \mathcal{H}_\lambda)$ for all $\lambda \in \Lambda$,*
(iii) *there exists a projection $E \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ such that $\mathrm{Ran}(E) = \mathcal{L}$.*

Proof. (i)$\Rightarrow$(ii). By assumption, for each $\lambda \in \Lambda$ we have the decomposition

$$\mathcal{H}_\lambda = \mathcal{L}_\lambda \oplus (\mathcal{H}_\lambda \ominus \mathcal{L}_\lambda). \tag{3.5}$$

Then we consider the net of subspaces $(\mathcal{H}_\lambda \ominus \mathcal{L}_\lambda)_{\lambda \in \Lambda}$ and observe that it is a strict inductive limit of Hilbert spaces and that $\mathcal{L}^\perp = \varinjlim_{\lambda \in \Lambda} (\mathcal{H}_\lambda \ominus \mathcal{L}_\lambda)$, hence

$$\mathcal{H}_\lambda \cap \mathcal{L}^\perp = \mathcal{H}_\lambda \ominus \mathcal{L}_\lambda, \quad \lambda \in \Lambda. \tag{3.6}$$

From (3.5) and (3.6) the desired conclusion follows.

(ii)$\Rightarrow$(iii). For arbitrary $\lambda \in \Lambda$, since $\mathcal{H}_\lambda = (\mathcal{H}_\lambda \cap \mathcal{L}) \oplus (\mathcal{L}^\perp \cap \mathcal{H}_\lambda)$ it follows that there exists a projection $E_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)$ such that $\mathrm{Ran}(E_\lambda) = \mathcal{L} \cap \mathcal{H}_\lambda$ and $\mathrm{Ran}(I_{\mathcal{H}_\lambda} - E_\lambda) = \mathcal{L}^\perp \cap \mathcal{H}_\lambda$. We observe that $\{E_\lambda\}_{\lambda \in \Lambda}$ satisfies the properties (a) and (b) from Proposition 2.12 and hence the operator $E = \varprojlim_\lambda E_\lambda$ is a local projection with $\mathrm{Ran}(E) = \mathcal{L}$.

(ii)$\Rightarrow$(i). If $E = \varprojlim_{\lambda \in \Lambda} E_\lambda$ is a local projection with $\mathrm{Ran}(E) = \mathcal{L}$ then letting $\mathcal{L}_\lambda = \mathrm{Ran}(E_\lambda)$, for all $\lambda \in \Lambda$, it follows that $\{\mathcal{L}_\lambda\}_{\lambda \in \Lambda}$ is a strictly inductive system of Hilbert spaces $\{\mathcal{L}_\lambda\}_{\lambda \in \Lambda}$ such that, for each $\lambda \in \Lambda$, $\mathcal{L}_\lambda$ is isometrically embedded in $\mathcal{H}_\lambda$ and $\mathcal{L} = \varinjlim_{\lambda \in \Lambda} \mathcal{L}_\lambda$. $\qquad \square$

The next proposition shows that the geometry of local projections is close to that of projections in Hilbert spaces.

Proposition 3.6. *Let $\{E_j\}_{j \in \mathcal{J}}$ be a family of local projections on the locally Hilbert space $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$.*

(a) *There exists a unique local projection, denoted $\bigwedge_{j \in \mathcal{J}} E_j$, on $\mathcal{H}$, subject to the following conditions:*
  (i) $\bigwedge_{j \in \mathcal{J}} E_j \leq E_k$ *for all $k \in \mathcal{J}$,*
  (ii) *for any local projection F such that $F \leq E_j$ for all $j \in \mathcal{J}$, it follows $F \leq \bigwedge_{j \in \mathcal{J}} E_j$.*
(b) *There exists a unique local projection, denoted $\bigvee_{j \in \mathcal{J}} E_j$, subject to the following conditions:*
  (i) $E_k \leq \bigvee_{j \in \mathcal{J}} E_j$ *for all $k \in \mathcal{J}$,*
  (ii) *for any local projection F such that $E_j \leq F$ for all $j \in \mathcal{J}$, it follows $\bigvee_{j \in \mathcal{J}} E_j \leq F$.*

Proof. (a) For each $k \in \mathcal{J}$, according to Proposition 3.5, let $\{E_{k,\lambda}\}_{\lambda \in \Lambda}$ be the net of projections such that $E_k = \varprojlim_{\lambda \in \Lambda} E_{k,\lambda}$. For each $\lambda \in \Lambda$ we consider $\bigwedge_{k \in \mathcal{J}} E_{k,\lambda}$ the projection in $\mathcal{H}_\lambda$. By definition, $\bigwedge_{k \in \mathcal{J}} E_{k,\lambda}$ is the maximal projection in $\mathcal{H}_\lambda$ that is dominated by $E_{j,\lambda}$ for all $j \in \mathcal{J}$ and such that, for any projection $F_\lambda \leq E_{j,\lambda}$ for all $j \in \mathcal{J}$ it follows that $F_\lambda \leq \bigwedge_{j \in \mathcal{J}} E_{k,\lambda}$. Then, by Proposition 2.12 it follows that, the net of operators $\{\bigwedge_{k \in \mathcal{J}} E_{k,\lambda}\}_{\lambda \in \Lambda}$ can be assembled to obtain the locally bounded operator $\bigwedge_{k \in \mathcal{J}} E_k$ with the desired properties.

(b) Similar to (a). $\qquad \square$

Recall that for each $\lambda \in \Lambda$ a Hermitian projection $P_\lambda \in \mathcal{L}(\mathcal{H})$ is uniquely defined such that $\mathrm{Ran}(P_\lambda) = \mathcal{H}_\lambda$, cf. Lemma 2.6 and the definition thereafter. In the following proposition we characterise the local projections within the larger class of Hermitian projections.

Proposition 3.7. *Let E be a Hermitian projection in $\mathcal{L}(\mathcal{H})$ and $\mathcal{S} = \mathrm{Ran}(E)$. The following assertions are equivalent:*

(i) *E is a local projection,*
(ii) $EP_\lambda = P_\lambda E$ *for all* $\lambda \in \Lambda$,
(iii) $\mathcal{H}_\lambda = (\mathcal{H}_\lambda \cap \mathcal{S}) \oplus (\mathcal{H}_\lambda \cap \mathcal{S}^\perp)$ *for all* $\lambda \in \Lambda$.

Proof. (i)$\Rightarrow$(ii). This is a consequence of Proposition 2.11.

(ii)$\Rightarrow$(i). Assuming that $EP_\lambda = P_\lambda E$ for all $\lambda \in \Lambda$ it follows that $\mathcal{H}_\lambda$ is invariant under E for each $\lambda \in \Lambda$. Since E is Hermitian it follows that $\mathcal{H}_\lambda$ is invariant under $E^* = E$ as well, hence again by Proposition 2.11 it follows that E is a locally bounded operator.

(i)$\Rightarrow$(iii). If E is a local projection then, in view of (2.13) and (2.12), $E = \varprojlim_{\lambda \in \Lambda} E_\lambda$ with $E_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)$ being an orthogonal projection for each $\lambda \in \Lambda$. $\square$

3.3. LOCALLY NORMAL OPERATORS

A linear operator $T \colon \mathcal{H} \to \mathcal{H}$ is called *locally normal* if $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ and $TT^* = T^*T$, respectively, *locally selfadjoint* if $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ and $T = T^*$. In a similar way, $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ is *locally positive*, also denoted by $T \geq 0$, if $\langle Th, h \rangle \geq 0$. As usually, this opens the possibility of introducing an order relation between locally selfadjoint operators: $A \geq B$ if $A - B \geq 0$.

Remark 3.8. Let $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$, $T = \varprojlim_{\lambda \in \Lambda} T_\lambda$. Then T is locally normal if and only if T_λ is normal for all $\lambda \in \Lambda$. Similarly, T is locally selfadjoint (locally positive) if and only if T_λ is selfadjoint (positive) for all $\lambda \in \Lambda$. In particular, taking into account of Proposition 3.2, it follows that, if T is a locally normal operator then it is locally selfadjoint (locally positive) if and only if $\sigma(T) \subseteq \mathbb{R}$ ($\sigma(T) \subseteq \mathbb{R}_+$).

The following result is a counter-part of the celebrated Fuglede-Putnam Theorem for locally normal operators.

Theorem 3.9. *Let* $N \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ *and* $M \in \mathcal{B}_{\mathrm{loc}}(\mathcal{K})$ *be locally normal operators, with the locally Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$ *modelled on the same poset* Λ, *and let* $B \in \mathcal{B}_{\mathrm{loc}}(\mathcal{K}, \mathcal{H})$ *be such that* $NB = BM$. *Then* $N^*B = BM^*$.

Proof. With notation as in Subsection 2.2, we repeatedly use Proposition 2.12 and Remark 2.13. From the assumption $NB = BM$ it follows that $N_\lambda B_\lambda = B_\lambda M_\lambda$ for all $\lambda \in \Lambda$. Since both N_λ and M_λ are normal operators in $\mathcal{H}_\lambda$ and, respectively, $\mathcal{K}_\lambda$, from the Fuglede-Putnam Theorem, e.g. see Theorem 12.5 in [3], it follows that $N_\lambda^* B = B M_\lambda^*$ for all $\lambda \in \Lambda$, hence $N^*B = BM^*$. $\square$

4. MAIN RESULTS

Recall that, letting (X, Σ) be a measurable space and $\mathcal{H}$ a Hilbert space, a spectral measure with respect to the triple $(X, \Sigma, \mathcal{H})$ is a map $E \colon \Sigma \to \mathcal{B}(\mathcal{H})$ subject to the following conditions:

(sm1) $E(A) = E(A)^*$ for all $A \in \Sigma$,
(sm2) $E(\emptyset) = 0$ and $E(X) = I$,

(sm3) $E(A_1 \cap A_2) = E(A_1)E(A_2)$ for all $A_1, A_2 \in \Sigma$,
(sm4) for any sequence $(A_n)_n$ of mutually disjoint sets in Σ we have

$$E\bigg(\bigcup_{n=1}^{\infty} A_n \bigg) = \sum_{n=1}^{\infty} E(A_n).$$

As a consequence of (sm1) and (sm3) it follows that $E(A)$ is a projection for all $A \in \Sigma$. Also, the convergence of the series in (lsm4) is with respect to the strong operator topology on $\mathcal{B}(\mathcal{H})$. We first single out the concept of a locally spectral measure, as a generalisation of the concept of spectral measure.

4.1. STRICTLY INDUCTIVE SYSTEMS OF MEASURABLE SPACES

Given a poset $(\Lambda; \leq)$, a net $\{(X_\lambda, \Omega_\lambda)\}_{\lambda \in \Lambda}$ is called a *strictly inductive system of measure spaces* if the following conditions hold:

(sim1) for each $\lambda \in \Lambda$, $(X_\lambda, \Omega_\lambda)$ is a measurable space,
(sim2) for each $\lambda, \nu \in \Lambda$ with $\lambda \leq \nu$ we have $X_\lambda \subseteq X_\nu$ and

$$\Omega_\lambda = \{A \cap X_\lambda \mid A \in \Omega_\nu\} \subseteq \Omega_\nu.$$

Given a strictly inductive system of measurable spaces $\{(X_\lambda, \Omega_\lambda)\}_{\lambda \in \Lambda}$, we denote

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda, \quad \Omega = \bigcup_{\lambda \in \Lambda} \Omega_\lambda. \tag{4.1}$$

The pair (X, Ω) is called the *inductive limit* of the strictly inductive system of measurable spaces, and we use the notation

$$(X, \Omega) = \varinjlim_{\lambda \in \Lambda} (X_\lambda, \Omega_\lambda).$$

In general, the inductive limit (X, Ω) of a strictly inductive system of measurable spaces is not a measurable space, since Ω is not a σ-algebra. Actually, Ω is a *ring* of subsets in X, that is, for any $A, B \in \Omega$ it follows $A \setminus B, A \cap B, A \cup B \in \Omega$, but, in general, not a σ-ring, that is, it may not be closed under countable unions or countable intersections. Moreover, Ω is a *locally σ-ring* in the sense that, if $(A_n)_n$ is a sequence of subsets from Ω such that there exists $\lambda \in \Lambda$ with the property that $A_n \in \Omega_\lambda$ for all $n \in \mathbb{N}$, it follows that $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$ are in Ω.

In the following we show that Ω has a canonical extension to a σ-algebra. Let

$$\widetilde{\Omega} := \{A \subseteq X \mid A \cap X_\lambda \in \Omega_\lambda \text{ for all } \lambda \in \Lambda\}. \tag{4.2}$$

Proposition 4.1. $\widetilde{\Omega}$ *is a σ-algebra and* $\Omega \subseteq \widetilde{\Omega}$.

Proof. We first show that $\Omega \subseteq \widetilde{\Omega}$. Let $A \in \Omega$. Then there exists $\lambda_0 \in \Lambda$ such that $A \in \Omega_{\lambda_0}$. For arbitrary $\lambda \in \Lambda$ there exists $\nu \in \Lambda$ such that $\lambda_0, \lambda \leq \nu$ hence $A \in \Omega_\nu$ and, by (sim2), it follows that $A \cap X_\lambda \in \Omega_\lambda$.

We prove that $\widetilde{\Omega}$ is a σ-algebra. First observe that $X \in \widetilde{\Omega}$, by definition. Then, let $A, B \in \widetilde{\Omega}$ be arbitrary. Then, for any $\lambda \in \Lambda$ we have

$$(A \setminus B) \cap X_\lambda = (A \cap X_\lambda) \setminus (B \cap X_\lambda) \in \Omega_\lambda,$$

hence $A \setminus B \in \widetilde{\Omega}$. Let now $(A_n)_{n \in \mathbb{N}}$ be a sequence of subsets in $\widetilde{\Omega}$. Then, for any $n \in \mathbb{N}$ and any $\lambda \in \Lambda$ we have $A_n \cap X_\lambda \in \Omega_\lambda$ and hence

$$\bigcup_{n=1}^{\infty} A_n \cap X_\lambda = \bigcup_{n=1}^{\infty} (A_n \cap X_\lambda) \in \Omega_\lambda.$$

Thus, we proved that $\widetilde{\Omega}$ is a σ-algebra. $\qquad\square$

4.2. LOCALLY SPECTRAL MEASURES

Let $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$ be a locally Hilbert space and consider a strictly inductive system of measurable spaces $\{(X_\lambda, \Omega_\lambda)\}_{\lambda \in \Lambda}$, see Subsection 4.1. We consider a *projective system of spectral measures* $\{E_\lambda\}_{\lambda \in \Lambda}$ with respect to $(X_\lambda, \Omega_\lambda, \mathcal{H}_\lambda)_{\lambda \in \Lambda}$, that is,

(psm1) for each $\lambda \in \Lambda$, E_λ is a spectral measure with respect to $(X_\lambda, \Omega_\lambda, \mathcal{H}_\lambda)$,
(psm2) for any $\lambda, \nu \in \Lambda$ with $\lambda \leq \nu$ we have

$$E_\nu(A)|\mathcal{H}_\lambda = J_{\nu,\lambda} E_\lambda(A), \quad A \in \Omega_\lambda,$$

and

$$E_\nu(A) P_{\lambda,\nu} = P_{\lambda,\nu} E_\nu(A), \quad A \in \Omega_\nu.$$

In the following we consider the inductive limit (X, Ω) of the strictly inductive system of measurable spaces $\{(X_\lambda, \Omega_\lambda)\}_{\lambda \in \Lambda}$, see (4.1), as well as its extension to a measurable space $(X, \widetilde{\Omega})$ as in Proposition 4.1.

Lemma 4.2. *There exists a unique mapping* $E \colon \widetilde{\Omega} \to \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ *such that*

$$E(A)|\mathcal{H}_\lambda = J_\lambda^{\mathcal{H}} E_\lambda(A \cap X_\lambda), \quad A \in \widetilde{\Omega}, \ \lambda \in \Lambda. \tag{4.3}$$

In addition, the mapping E has the following properties:

(i) $E(A) = E(A)^*$ *for all* $A \in \widetilde{\Omega}$,
(ii) $E(\emptyset) = 0$ *and* $E(X) = I$,
(iii) $E(A_1 \cap A_2) = E(A_1) E(A_2)$ *for all* $A_1, A_2 \in \widetilde{\Omega}$,
(iv) *for any sequence $(A_n)_n$ of mutually disjoint sets in $\widetilde{\Omega}$ we have*

$$E\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} E(A_n).$$

Proof. For each $\lambda \in \Lambda$ we define

$$E_\lambda(A) := E_\lambda(A \cap X_\lambda), \quad A \in \widetilde{\Omega},$$

which makes perfectly sense since $A \cap X_\lambda \in \Omega_\lambda$. We observe that, for each $A \in \widetilde{\Omega}$, the net $(E_\lambda(A))_{\lambda \in \Lambda}$ satisfies the conditions (2.8), which follows by the assumption (psm2). Then, by Proposition 2.12, it follows that there exists $E(A) \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ such that (4.3) holds.

In order to prove that E has the properties (i)–(iii), we use its definition, (2.12), and Remark 2.13.(b). Consequently, for each $A \in \widetilde{\Omega}$ the operator $E(A)$ is a local projection. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of mutually disjoint sets in $\widetilde{\Omega}$. Then, $(E(A_n))_{n \in \mathbb{N}}$ is a sequence of mutually orthogonal local projections,

$$E\left(\bigcup_{k=1}^{\infty} A_k \right) \geq E\left(\bigcup_{k=1}^{n} A_k \right) = \sum_{k=1}^{n} E(A_k), \quad n \in \mathbb{N},$$

and hence the sequence $(\sum_{k=1}^{n} E(A_k))_{n \geq 1}$ of local projections is increasing and bounded above by $E(\bigcup_{k=1}^{\infty} A_k)$ hence

$$\sum_{k=1}^{\infty} E(A_k) \leq E\left(\bigcup_{k=1}^{\infty} A_k \right),$$

where the series converges strongly operatorial. Now, for arbitrary $h \in \mathcal{H}$ there exists $\lambda \in \Lambda$ such that $h \in \mathcal{H}_\lambda$ and hence

$$E\left(\bigcup_{k=1}^{\infty} A_k \right) h = E_\lambda \left(\bigcup_{k=1}^{\infty} A_k \right) h$$

$$= E_\lambda \left(\bigcup_{k=1}^{\infty} A_k \cap X_\lambda \right) h = \sum_{k=1}^{\infty} E_\lambda(A_k \cap X_\lambda) = \sum_{k=1}^{\infty} E_\lambda(A_k) h.$$

This proves the property (iv). $\qquad\qquad\square$

We call the mapping $E \colon \widetilde{\Omega} \to \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ obtained in Lemma 4.2 a *locally spectral measure*. On the other hand, when restricted to the locally σ-ring $\Omega \subseteq \widetilde{\Omega}$, the mapping $E|\Omega$ has the following properties:

(i) $E(A) = E(A)^*$ for all $A \in \Omega$,
(ii) $E(\emptyset) = 0$,
(iii) $E(A_1 \cap A_2) = E(A_1)E(A_2)$ for all $A_1, A_2 \in \Omega$,
(iv) for any sequence $(A_n)_n$ of mutually disjoint sets in Ω such that $\bigcup_{n=1}^{\infty} A_n \in \Omega$, we have

$$E\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} E(A_n).$$

We call $(E, X, \Omega, \mathcal{H})$ the *projective limit* of the projective system of spectral measures $\{(E_\lambda, X_\lambda, \Omega_\lambda, \mathcal{H}_\lambda)\}_{\lambda \in \Lambda}$ and use the notation

$$(E, X, \Omega, \mathcal{H}) = \varprojlim_{\lambda \in \Lambda} (E_\lambda, X_\lambda, \Omega_\lambda, \mathcal{H}_\lambda).$$

A locally spectral measure E, as we defined it, cannot be considered as a spectral measure, due to the fact that its codomain is $\mathcal{B}_{\mathrm{loc}}(\mathcal{H})$. However, in the following we show that it can be lifted to a spectral measure. More precisely, taking into account that any local projection is actually a bounded operator on the pre-Hilbert space $\mathcal{H}$, the range of E is actually contained in its bounded part $b\mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ which is canonically embedded in $\mathcal{B}(\widetilde{\mathcal{H}})$. Therefore, for each $A \in \widetilde{\Omega}$, since $E(A) \leq I$, it follows that $E(A)$ is bounded hence it has a unique extension to an operator $\widetilde{E}(A) \in \mathcal{B}(\widetilde{\mathcal{H}})$.

Proposition 4.3. *The mapping $\widetilde{E} \colon \widetilde{\Omega} \to \mathcal{B}(\widetilde{\mathcal{H}})$ is a spectral measure.*

Proof. Indeed, the properties (sm1)–(sm3) of $\widetilde{E}$ from the definition of a spectral measure follow from the properties (i)–(iii) of E as in Lemma 4.2. Let $(A_n)_n$ be a sequence of mutually disjoint sets in $\widetilde{\Omega}$ and let $h \in \widetilde{\mathcal{H}}$ be arbitrary, hence there exists a sequence $(h_k)_{k \geq 1}$ of vectors in $\mathcal{H}$ such that $\|h - h_k\| \to 0$ as $k \to \infty$. Then, in view of the property (iv) as in Lemma 4.2 we have

$$\widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right) h_k = \sum_{j=1}^{\infty} \widetilde{E}(A_j) h_k, \quad k \in \mathbb{N}. \tag{4.4}$$

For arbitrary $n, k \in \mathbb{N}$ we have

$$\left\| \widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right) h - \sum_{j=1}^{n} \widetilde{E}(A_j) h \right\| \leq \left\| \widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right)(h - h_k) \right\|$$

$$+ \left\| \widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right) h_k - \sum_{j=1}^{n} \widetilde{E}(A_j) h_k \right\|$$

$$+ \left\| \sum_{j=1}^{n} \widetilde{E}(A_j)(h_k - h) \right\|$$

$$\leq 2\|h - h_k\| + \left\| \widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right) h_k - \sum_{j=1}^{n} \widetilde{E}(A_j) h_k \right\|.$$

Therefore, for any $\epsilon > 0$ we first choose $k \in \mathbb{N}$ sufficiently large such that $\|h - h_k\| < \epsilon/4$ and then, in view of (4.4) we choose $n \in \mathbb{N}$ sufficiently large such that

$$\left\| \widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right) h_k - \sum_{j=1}^{n} \widetilde{E}(A_j) h_k \right\| < \frac{\epsilon}{2},$$

and conclude that

$$\left\| \widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right) h - \sum_{j=1}^{n} \widetilde{E}(A_j) h \right\| < \epsilon.$$

This is sufficient to conclude that

$$\widetilde{E}\left(\bigcup_{j=1}^{\infty} A_j \right) h = \sum_{j=1}^{\infty} \widetilde{E}(A_j) h.$$

Since $h \in \widetilde{\mathcal{H}}$ is arbitrary, it follows that $\widetilde{E}$ has the property (sm4) as well and hence is a spectral measure. $\qquad\square$

In the following we show how we can produce locally normal operators by integration of locally bounded $\widetilde{\Omega}$-measurable functions with respect to a locally spectral measure. We first clarify the meaning of $\widetilde{\Omega}$-measurability.

Lemma 4.4. *Let $\varphi \colon X \to \mathbb{C}$. The following assertions are equivalent:*

  (i) *φ is $\widetilde{\Omega}$-measurable,*
  (ii) *for each $\lambda \in \Lambda$ the function $\varphi|X_\lambda$ is Ω_λ-measurable.*

Proof. (i)$\Rightarrow$(ii). Assuming that φ is $\widetilde{\Omega}$-measurable, for any $\lambda \in \Lambda$ and any Borel subset B in $\mathbb{C}$ we have $(\varphi|X_\lambda)^{-1}(B) = X_\lambda \cap \varphi^{-1}(B) \in \Omega_\lambda$, by the definition of $\widetilde{\Omega}$, see (4.2).

(ii)$\Rightarrow$(i). Assume that for each $\lambda \in \Lambda$ the function $\varphi|X_\lambda$ is Ω_λ measurable. Then, for any Borel subset B in $\mathbb{C}$ and any $\lambda \in \Lambda$ we have $\varphi^{-1}(B) \cap X_\lambda = (\varphi|X_\lambda)^{-1}(B)$ hence, again by (4.2) it follows that $\varphi^{-1}(B) \in \widetilde{\Omega}$. $\qquad\square$

In the following we denote

$$B_{\mathrm{loc}}(X, \widetilde{\Omega}) := \{\varphi \colon X \to \mathbb{C} \mid \varphi|X_\lambda \text{ is bounded and } \Omega_\lambda\text{-measurable for all } \lambda \in \Lambda\}.$$

It is easy to see that $B_{\mathrm{loc}}(X, \widetilde{\Omega})$ is a $*$-algebra of complex functions, with usual algebraic operations and involution. Letting, for each $\lambda \in \Lambda$,

$$p_\lambda(\varphi) = \sup_{x \in X_\lambda} |\varphi(x)|, \quad \varphi \in B_{\mathrm{loc}}(X, \widetilde{\Omega}),$$

we obtain a family of C^*-seminorms $\{p_\lambda\}_{\lambda \in \Lambda}$ with respect to which $B_{\mathrm{loc}}(X, \widetilde{\Omega})$ becomes a locally C^*-algebra. More precisely, letting $B(X_\lambda, \Omega_\lambda)$ denote the C^*-algebra of all bounded and Ω_λ-measurable functions $f \colon X_\lambda \to \mathbb{C}$, we have

$$B_{\mathrm{loc}}(X, \widetilde{\Omega}) = \varprojlim_{\lambda \in \Lambda} B(X_\lambda, \Omega_\lambda).$$

Let now $\varphi \in B_{\mathrm{loc}}(X, \widetilde{\Omega})$ be fixed. By Proposition 9.4 in [3], for each $\lambda \in \Lambda$ there exists a unique normal operator $N_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)$ such that

$$N_\lambda(\varphi) = \int_{X_\lambda} \varphi(x)\mathrm{d}E_\lambda(x),$$

in the following sense: for each $\epsilon > 0$ and $\{A_1, \dots, A_n\}$ an Ω_λ-partition of X_λ such that, $\sup\{|\varphi(x) - \varphi(y)| \mid x, y \in A_k\} < \epsilon$ for all $k = 1, \dots, n$, and for any choice of points $x_k \in A_k$, for all $k = 1, \dots, n$, we have

$$\left\| N_\lambda(\varphi) - \sum_{k=1}^n \varphi(x_k)E_\lambda(A_k) \right\| < \epsilon, \tag{4.5}$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{B}(\mathcal{H}_\lambda)$.

Proposition 4.5. *For each $\varphi \in B_{\mathrm{loc}}(X, \widetilde{\Omega})$ there exists a unique locally normal operator $N(\varphi) \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ such that $N(\varphi) = \varprojlim_{\lambda \in \Lambda} N_\lambda(\varphi)$. In addition, the mapping $B_{\mathrm{loc}}(X, \widetilde{\Omega}) \ni \varphi \mapsto N(\varphi) \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ is a coherent $*$-representation.*

Proof. We first observe that the net of normal bounded operators $(N_\lambda(\varphi))_\lambda$ satisfies the condition (2.8) due to the condition (psm2) in the definition of the projective system of spectral measures $\{E_\lambda\}_{\lambda \in \Lambda}$ and (4.5). Then, by Proposition 2.12 there exists uniquely $N(\varphi) \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ such that $N(\varphi)|\mathcal{H}_\lambda = N_\lambda(\varphi)J_\lambda^{\mathcal{H}}$ for all $\lambda \in \Lambda$, that is, $N(\varphi) = \varprojlim_{\lambda \in \Lambda} N_\lambda(\varphi)$.

Since, by Proposition 9.6 in [3], for each $\lambda \in \Lambda$ the mapping $B(X_\lambda, \Omega_\lambda) \ni f \mapsto \mathcal{B}(\mathcal{H}_\lambda)$ is a $*$-representation, it follows that the projective limit of these $*$-representations, which is exactly the mapping $B_{\mathrm{loc}}(X, \widetilde{\Omega}) \ni \varphi \mapsto \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$, is a coherent $*$-representation, see [5, p. 651]. $\square$

In view of Proposition 4.5, for any $\varphi \in B_{\mathrm{loc}}(X, \widetilde{\Omega})$ we denote

$$\int_{\sigma(N)} \varphi(x)\mathrm{d}E(x) := N(\varphi) = \varprojlim_{\lambda \in \Lambda} N_\lambda(\varphi) = \varprojlim_{\lambda \in \Lambda} \int_{\sigma(N_\lambda)} \varphi(x)\mathrm{d}E_\lambda(x). \qquad (4.6)$$

4.3. THE SPECTRAL THEOREM

Let $N \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ be a locally normal operator, for some locally Hilbert space $\mathcal{H} = \varinjlim_{\lambda \in \Lambda} \mathcal{H}_\lambda$. By definition, see Subsection 2.2, there exists uniquely the net $(N_\lambda)_{\lambda \in \Lambda}$ with $N = \varprojlim_{\lambda \in \Lambda} N_\lambda$, in the sense of the conditions (lbo1) and (lbo2). By Proposition 3.2 we have $\sigma(N) = \bigcup_{\lambda \in \Lambda} \sigma(N_\lambda)$. For each $\lambda \in \Lambda$ the spectrum $\sigma(N_\lambda)$ is a compact nonempty subset of $\mathbb{C}$ and $(\sigma(N_\lambda), \mathcal{B}_\lambda)$ is a measurable space, where $\mathcal{B}_\lambda$ denotes the σ-algebra of all Borel subsets of $\sigma(N_\lambda)$. Then, $\{(\sigma(N_\lambda), \mathcal{B}_\lambda)\}_{\lambda \in \Lambda}$ is a strictly inductive system of measurable spaces, as in Subsection 4.1, and letting $(\sigma(N), \mathcal{B})$ be its inductive limit in the sense of (4.1), it is easy to see that, with respect to (4.2), $\widetilde{\mathcal{B}}$ is the σ-algebra of all Borel subsets of $\sigma(N)$.

Further on, by the Spectral Theorem for normal operators in Hilbert spaces, e.g. see Theorem 10.2 in [3], for each $\lambda \in \Lambda$, let E_λ denote the spectral measure of N_λ with respect to the triple $(\sigma(N_\lambda), \mathcal{B}_\lambda, \mathcal{H}_\lambda)$. In particular,

$$N_\lambda = \int_{\sigma(N_\lambda)} z\mathrm{d}E_\lambda(z), \quad I_\lambda = \int_{\sigma(N_\lambda)} \mathrm{d}E_\lambda(z), \qquad (4.7)$$

where I_λ denotes the identity operator on $\mathcal{H}_\lambda$.

Lemma 4.6. $\{(E_\lambda, \sigma(N_\lambda), \mathcal{B}_\lambda, \mathcal{H}_\lambda)\}_{\lambda \in \Lambda}$ *is a projective system of spectral measures.*

Proof. We have only to show that the axiom (psm2) holds. Let $\lambda, \nu \in \Lambda$ be such that $\lambda \leq \nu$. Then, by Proposition 2.12, with respect to the decomposition

$$\mathcal{H}_\nu = \mathcal{H}_\lambda \oplus (\mathcal{H}_\nu \ominus \mathcal{H}_\lambda),$$

we have the block-operator matrix representation

$$N_\nu = \begin{bmatrix} N_\lambda & 0 \\ 0 & N_{\lambda,\nu} \end{bmatrix}, \tag{4.8}$$

for some normal operator $N_{\lambda,\nu} \in \mathcal{B}(\mathcal{H}_\nu \ominus \mathcal{H}_\lambda)$. From here, it follows that $\sigma(N_\lambda), \sigma(N_{\lambda,\nu}) \subseteq \sigma(N_\nu)$ and by functional calculus with bounded Borel functions, e.g. see Theorem 10.3 in [3], it follows that for any $f \in B(\sigma(N_\nu))$, by (4.8), we have

$$f(N_\nu) = \begin{bmatrix} f(N_\lambda) & 0 \\ 0 & f(N_{\lambda,\nu}) \end{bmatrix}. \tag{4.9}$$

Then, if A is a Borel subset of $\sigma(N_\lambda)$, we consider $\chi_A \in B(\sigma(N_\lambda)) \subseteq B(\sigma(N_\nu))$ and hence, by (4.9) we get

$$E_\nu(A)|\mathcal{H}_\lambda = \chi_A(N_\nu)|\mathcal{H}_\lambda = J_{\nu,\lambda}\chi_A(N_\lambda) = J_{\nu,\lambda}E_\lambda(A).$$

On the other hand, if now A is a Borel subset of $\sigma(N_\nu)$, we consider $\chi_A \in B(\sigma(N_\nu))$ hence, again by (4.9), we get

$$E_\nu(A)P_{\lambda,\nu} = \chi_A(N_\nu)P_{\lambda,\nu} = \begin{bmatrix} \chi_A(N_\lambda) & 0 \\ 0 & 0 \end{bmatrix} = P_{\lambda,\nu}\chi_A(N_\nu) = P_{\lambda,\nu}E_\nu(A).$$

We have shown that the axiom (psm2) holds as well. $\qquad\square$

We are now in a position to prove the Spectral Theorem for locally normal operators in terms of locally spectral measures.

Theorem 4.7. *For any locally normal operator $N \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ there exists a unique locally spectral measure E, with respect to the Borel measurable space $(\sigma(N), \widetilde{\mathcal{B}})$ and the locally Hilbert space $\mathcal{H}$, with the following properties:*

(i) *$N = \int_{\sigma(N)} z\,\mathrm{d}E(z)$ and $I = \int_{\sigma(N)} \mathrm{d}E(z)$, in the sense of (4.6),*
(ii) *for any nonempty relatively open subset G of $\sigma(N)$ we have $E(G) \neq 0$,*
(iii) *if $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ then $TN = NT$ if and only if $TE(A) = E(A)T$ for all Borel subset of $\sigma(A)$.*

Proof. As a consequence of Lemma 4.6 and Lemma 4.2, let $E\colon \mathcal{B}(\sigma(N)) \to \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ be the locally spectral measure of the projective system of spectral measures $\{(E_\lambda, \sigma(N_\lambda), \mathcal{B}_\lambda, \mathcal{H}_\lambda)\}_{\lambda \in \Lambda}$. Property (i) follows by (4.7) and (4.6).

Let G be a nonempty relatively open subset of $\sigma(N)$. By Proposition 3.2, there exists $\lambda \in \Lambda$ such that $G \cap \sigma(N_\lambda)$ is a nonempty relatively open subset of $\sigma(N_\lambda)$ and hence $E_\lambda(G \cap \sigma(N_\lambda)) \neq 0$. In view of Lemma 4.2 it follows that $E(G) \neq 0$.

Let $T \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ be such that $TN = NT$. Then, by Theorem 3.9 we have $TN^* = N^*T$ as well and hence T commutes with the unital locally C^*-algebra generated by N. Letting $T = \varprojlim_{\lambda \in \Lambda} T_\lambda$ it follows that T_λ commutes with the unital C^*-algebra generated by N_λ for all $\lambda \in \Lambda$ and hence, e.g. by Theorem 10.2 in [3], it follows that T_λ commutes with $E_\lambda(A)$ for all $A \in \sigma(N_\lambda)$. In view of the definition of the locally spectral measure E, see Lemma 4.2, it follows that T commutes with $E(A)$ for any Borel subset A of $\sigma(N)$. The converse implication follows similarly. $\qquad\square$

As a consequence of Theorem 4.7 we can obtain the functional calculus with locally bounded Borel functions.

Theorem 4.8. *Let $N \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ be a locally normal operator and E its locally spectral measure. Then the mapping*

$$B_{\mathrm{loc}}(\sigma(N)) \ni \varphi \mapsto \varphi(N) = \int\limits_{\sigma(N)} \varphi(z) \mathrm{d}E(z) \in \mathcal{B}_{\mathrm{loc}}(\mathcal{H}) \qquad (4.10)$$

is a coherent $*$-representation of the locally C^*-algebra $B_{\mathrm{loc}}(\sigma(N))$, and it is the unique coherent $*$-morphism of locally C^*-algebras such that:

(i) $\zeta(N) = N$ and $1(N) = I$, where $\zeta(z) = z$ and $1(z) = 1$ for all $z \in \mathbb{C}$,
(ii) *for any net $(\varphi_j)_{j \in \mathcal{J}}$ such that $\varphi_j \to 0$ pointwise and*

$$\sup_{j \in \mathcal{J}} \sup_{x \in \sigma(N_\lambda)} |\varphi_j(x)| < +\infty$$

for each $\lambda \in \Lambda$, it follows that $\varphi_j(N) \to 0$ strongly operatorial.

Proof. By Theorem 4.7, the existence of the locally spectral measure $E = \varprojlim_{\lambda \in \Lambda} E_\lambda$ for the locally normal operator $N = \varprojlim_{\lambda \in \Lambda} N_\lambda$ is guaranteed. Then, for any $\varphi = \varprojlim_{\lambda \in \Lambda} \varphi_\lambda \in B_{\mathrm{loc}}(\sigma(N))$, by Proposition 4.5 we have the locally bounded operator

$$\int\limits_{\sigma(N)} \varphi(z) \mathrm{d}E(z) = \varprojlim_{\lambda \in \Lambda} \int\limits_{\sigma(N_\lambda)} \varphi_\lambda(z) \mathrm{d}E_\lambda(z), \qquad (4.11)$$

and the mapping (4.10) is a coherent $*$-representation of the locally C^*-algebra $B_{\mathrm{loc}}(\sigma(N))$.

Letting ζ for φ in (4.11) it follows that

$$\int\limits_{\sigma(N)} z \mathrm{d}E(z) = \varprojlim_{\lambda \in \Lambda} \int\limits_{\sigma(N_\lambda)} z \mathrm{d}E_\lambda(z) = \varprojlim_{\lambda \in \Lambda} N_\lambda = N,$$

and similarly for the function 1.

Let $(\varphi_j)_{j \in \mathcal{J}}$ be a net of functions in $B_{\mathrm{loc}}(\sigma(N))$ such that

$$\sup_{j \in \mathcal{J}} \sup_{x \in \sigma(N_\lambda)} |\varphi_j(x)| < +\infty$$

for every $\lambda \in \Lambda$ and $\varphi_j \to 0$ pointwise. Then, e.g. see Theorem 2.20 in [12], for each $\lambda \in \Lambda$ it follows that $\varphi_j(N_\lambda) \to 0$ strongly operatorial which implies that $\varphi(N) \to 0$ strongly operatorial.

For the uniqueness part, let $\pi \colon B_{\mathrm{loc}}(\sigma(N)) \to \mathcal{B}_{\mathrm{loc}}(\mathcal{H})$ be a coherent $*$-representation having the properties (i) and (ii). From (i) it follows that $\pi(\varphi) = \varphi(N)$ for any complex polynomial in the variables z and $\overline{z}$. Since π is coherent it is automatically continuous on the locally C^*-algebra $C(\sigma(N))$ and then, in view of

Stone-Weierstrass Theorem, it follows that $\pi(\varphi) = \varphi(N)$ for any $\varphi \in C(\sigma(N))$. In view of Baire's Theorem, any function $\varphi \in B_{\mathrm{loc}}(\sigma(N))$ can be approximated by a net $(\varphi_j)_{j \in \mathcal{J}}$ of functions in $C(\sigma(N))$ such that

$$\sup_{j \in \mathcal{J}} \sup_{x \in \sigma(N_\lambda)} |\varphi_j(x)| < +\infty$$

for every $\lambda \in \Lambda$, hence $\pi(\varphi) = \varphi(N)$ for all $\varphi \in B_{\mathrm{loc}}(\sigma(N))$. $\square$

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