ON ONE CONDITION
OF ABSOLUTELY CONTINUOUS SPECTRUM
FOR SELF-ADJOINT OPERATORS
AND ITS APPLICATIONS

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Communicated by Sergey N. Naboko

Abstract. In this work the method of analyzing of the absolutely continuous spectrum for self-adjoint operators is considered. For the analysis it is used an approximation of a self-adjoint operator A by a sequence of operators A_n with absolutely continuous spectrum on a given interval $[a, b]$ which converges to A in a strong sense on a dense set. The notion of equi-absolute continuity is also used. It was found a sufficient condition of absolute continuity of the operator A spectrum on the finite interval $[a, b]$ and the condition for that the corresponding spectral density belongs to the class $L_p[a, b]$ $(p \geq 1)$. The application of this method to Jacobi matrices is considered. As one of the results we obtain the following assertion: Under some mild assumptions, suppose that there exist a constant $C > 0$ and a positive function $g(x) \in L_p[a, b]$ $(p \geq 1)$ such that for all n sufficiently large and almost all $x \in [a, b]$ the estimate $\frac{1}{g(x)} \leq b_n(P_{n+1}^2(x) + P_n^2(x)) \leq C$ holds, where $P_n(x)$ are 1st type polynomials associated with Jacobi matrix (in the sense of Akhiezer) and b_n is a second diagonal sequence of Jacobi matrix. Then the spectrum of Jacobi matrix operator is purely absolutely continuous on $[a, b]$ and for the corresponding spectral density $f(x)$ we have $f(x) \in L_p[a, b]$.

Keywords: self-adjoint operators, absolutely continuous spectrum, equi-absolute continuity, spectral density, Jacobi matrices.

Mathematics Subject Classification: 47A10, 47A58.

1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Let us consider the following problem. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of linear self-adjoint operators in a separable Hilbert space H with absolutely continuous spectrum on some interval $[a, b]$ and Φ be some dense set in H. Suppose that on Φ there exists in a strong

sense the limit $\lim_{n\to\infty} A_n$ and the closure of this limit operator is a self-adjoint operator A, that is, for any $e \in \Phi$

$$\lim_{n\to\infty} \|(A_n - A)e\| = 0.$$

Suppose also that $[a, b] \subseteq \sigma(A)$. There arises the question: what is the condition which guarantees that the spectrum of a limit operator A is also absolutely continuous on $[a, b]$?

Let $E_\lambda^{(n)}$ and E_λ be the resolutions of the identity of the operators A_n and A, respectively, and let

$$\sigma_n(\lambda; e) = (E_\lambda^{(n)} e, e), \quad e \in H,$$
$$\sigma(\lambda; e) = (E_\lambda e, e), \quad e \in H.$$

By assumption all functions $\sigma_n(\lambda; e)$ are absolutely continuous on $[a, b]$ so that

$$\sigma_n(\lambda; e) = \sigma_n(a; e) + \int_a^\lambda f_n(t; e)\, dt, \quad e \in H,\ \lambda \in [a, b],$$

where $f_n(t; e)$ is a sequence of non-negative summable on $[a, b]$ functions.

We may suppose that the convergence of the sequence A_n to the operator A in a strong sense on the set Φ should leads to the convergence of the sequence $\sigma_n(\lambda; e)$ to the function $\sigma(\lambda; e)$. This turns out to be true.

Lemma 1.1. *If λ is not an eigenvalue of the operator A, then for all $e \in H$*

$$\lim_{n\to\infty} \sigma_n(\lambda; e) = \sigma(\lambda; e). \tag{1.1}$$

Proof. It is known [2] that the strong convergence of the sequence A_n to A on a dense subset in H leads to the strong convergence of the sequence $E_\lambda^{(n)}$ to E_λ for those λ which are not the eigenvalues of A. That is, if λ is not an eigenvalue of the operator A then for all $e \in H$

$$\lim_{n\to\infty} \|(E_\lambda^{(n)} - E_\lambda)e\| = 0.$$

Further, we have

$$|\sigma_n(\lambda; e) - \sigma(\lambda; e)| = |((E_\lambda^{(n)} - E_\lambda)e, e)| \le \|(E_\lambda^{(n)} - E_\lambda)e\|,$$

whence the statement of the lemma readily follows. $\qquad\square$

Note that as we suppose that the space H is separable, the set of eigenvalues of the operator A is no more than denumerable and hence the equality (1.1) is fulfilled for almost all λ.

For further consideration we need the notion of equi-absolute continuity [15, 16]. The sequence $g_n(x)$ of absolutely continuous on the interval $[a, b]$ functions

$$g_n(x) = g_n(a) + \int_a^x f_n(t)\, dt$$

is called equi-absolutely continuous on this interval if for any $\epsilon > 0$ there exists $\delta > 0$ (which is independent of n) such that for any measurable subset $e \subset [a, b]$ the inequality

$$\left| \int_e f_n(t) \, dt \right| < \epsilon$$

is valid, provided $m(e) < \delta$ ($m(\cdot)$ is the Lebesgue measure). The sequence of summable functions $f_n(t)$ is said to be have equi-absolutely continuous integrals on $[a, b]$.

Let us give now the answer on the question stated above.

Theorem 1.2. *In order that the spectrum of the operator A on the interval $[a, b]$ ($[a, b] \subseteq \sigma(A)$) be absolutely continuous it is sufficient that for any vector e from generating subspace of the operator A the sequence of functions $\{\sigma_n(\lambda; e)\}$ be equi-absolutely continuous on $[a, b]$, or that is the same, the sequence of functions $\{f_n(t; e)\}$ has equi-absolutely continuous integrals on $[a, b]$. Under these conditions the sequence $\{\sigma_n(\lambda; e)\}$ uniformly converges to $\sigma(\lambda; e)$ on $[a, b]$.*

Proof. Fix some vector e from generating subspace of A. For the brevity we will omit this vector in the arguments of all functions.

Consider the sequence of functions $\sigma_n(\lambda)$ on the interval $[a, b]$

$$\sigma_n(\lambda) = \sigma_n(a) + \int_a^\lambda f_n(t) \, dt, \quad \lambda \in [a, b].$$

Here $f_n(t) \geq 0$ on $[a, b]$ but the case $f_n(t) \equiv 0$ is also possible if for example the corresponding vector e belongs to the discrete subspace of the operator A_n. In this case the function $\sigma_n(\lambda)$ is constant on $[a, b]$.

From Lemma 1.1 we conclude that for almost all $\lambda \in [a, b]$ $\lim_{n \to \infty} \sigma_n(\lambda) = \sigma(\lambda)$. By assumption the sequence $\sigma_n(\lambda)$ is equi-absolutely continuous on $[a, b]$. Besides, by virtue of its definition, it is uniformly bounded ($0 \leq \sigma_n(\lambda) \leq \|e\|^2$, $\lambda \in \mathbb{R}$). Hence (see [16]), from the sequence $\sigma_n(\lambda)$ one can extract a partial subsequence $\sigma_{n_k}(\lambda)$ which uniformly converges on $[a, b]$. Moreover, the limit function $\sigma_1(\lambda)$ is absolutely continues on $[a, b]$. So, we obtain

$$\lim_{n \to \infty} \sigma_n(\lambda) = \sigma(\lambda) \quad \text{for almost all } \lambda \text{ from } [a, b],$$
$$\lim_{k \to \infty} \sigma_{n_k}(\lambda) = \sigma_1(\lambda) \quad \text{for all } \lambda \text{ from } [a, b]. \tag{1.2}$$

Comparing this two limit equalities we can note that $\sigma_1(\lambda) = \sigma(\lambda)$ for almost all $\lambda \in [a, b]$. Let us show that the first equality (1.2) is fulfilled everywhere in $[a, b]$. Let λ_0 be a point in which the equality (1.2) perhaps is not valid. Consider the sequence $\sigma_n(\lambda_0)$. We have

$$|\sigma_n(\lambda_0) - \sigma_m(\lambda_0)| \leq |\sigma_n(\lambda_0) - \sigma_n(\lambda')| + |\sigma_n(\lambda') - \sigma_m(\lambda')| + |\sigma_m(\lambda') - \sigma_m(\lambda_0)|.$$

Here λ' is an arbitrary point of the interval $[a, b]$. Since the set of the possible points λ_0 has measure zero, in any arbitrarily small neighbourhood of the point λ_0 there

exists an infinite number of points in which the sequence $\sigma_n(\lambda)$ converges. Hence one can choose the point λ' such that on the one hand the sequence $\sigma_n(\lambda')$ converges and on the other hand the inequality

$$|\sigma_n(\lambda_0) - \sigma_n(\lambda')| < \epsilon/3$$

holds for any $\epsilon > 0$ and any n.

The validity of this inequality for any n is possible by virtue of equi-absolute continuity of the sequence $\sigma_n(\lambda)$.

Fixing such λ' and ϵ, one can choose a number N such that the inequality

$$|\sigma_n(\lambda') - \sigma_m(\lambda')| < \epsilon/3$$

is valid, provided $n, m > N$. Then for the same n and m

$$|\sigma_n(\lambda_0) - \sigma_m(\lambda_0)| < \epsilon.$$

Since ϵ is arbitrarily small, the sequence $\sigma_n(\lambda_0)$ converges by Cauchy's criterion.

Thus the sequence $\sigma_n(\lambda)$ converges everywhere in $[a, b]$ and therefore $\sigma(\lambda) = \sigma_1(\lambda)$ also everywhere in $[a, b]$. Hence the function $\sigma(\lambda)$ is absolutely continuous on this interval. Note that the sequence $\sigma_n(\lambda)$ converges to $\sigma(\lambda)$ uniformly on $[a, b]$. It can be proved by contradiction. Actually suppose it is not true. Then there exist a positive ϵ, subsequence n_k and corresponding to it points $x_k \in [a, b]$ such that

$$|\sigma_{n_k}(\lambda_k) - \sigma(\lambda_k)| \geq \epsilon(> 0), \quad k = 0, 1, 2, \ldots \tag{1.3}$$

The sequence $\sigma_{n_k}(\lambda)$ contains uniformly convergent subsequence $\sigma_{n_{k_m}}(\lambda)$ in $[a\,b]$ and the limit function is $\sigma(\lambda)$. Hence

$$|\sigma_{n_{k_m}}(\lambda) - \sigma(\lambda)| < \epsilon,$$

provided $\lambda \in [a, b]$ and $n_{k_m} > N$. In particular,

$$|\sigma_{n_{k_m}}(\lambda_{k_m}) - \sigma(\lambda_{k_m})| < \epsilon$$

for m sufficiently large. But this inequality contradicts to (1.3), and the uniform convergence is proved.

Thus we proved that for any vector e from generating subspace of A the functions $\sigma(\lambda) \equiv \sigma(\lambda; e)$ are absolutely continuous on $[a, b]$. Since $[a, b] \subseteq \sigma(A)$, it follows already that the spectrum of the operator A is absolutely continuous on this interval. The theorem is proved.   □

Corollary 1.3. *If for any vector e from generating subspace of A the sequence of functions $\{\sigma_n(\lambda; e)\}$ is equi-absolutely continuous on any finite interval of a real axis, then the spectrum of the operator A is purely absolutely continuous.*

In [8] it was used in fact a simple sufficient condition of equi-absolute continuity when $f_n(t) \leq g(t)$ where $g(t) \in L_\infty[a, b]$. One can easily generalize it on the case of any L_p, $p \geq 1$.

Lemma 1.4. *Suppose there exists a positive function $g(t) \in L_p[a, b]$ $(p \geq 1)$ such that $f_n(t) \leq g(t)$ for all $n \in \mathbb{N}$ and almost all $t \in [a, b]$. Then the sequence of functions $f_n(t)$ has equi-absolutely continuous integrals on $[a, b]$.*

Proof. Let $e \subset [a, b]$ be a measurable set and $m(e) < \delta$. By Gölder's inequality

$$\int_e f_n(t)\, dt \leq \left(\int_e f_n^p(t)\, dt \right)^{1/p} \left(\int_e dt \right)^{1/q} \leq \|g\|_p\, \delta^{1/q},$$

whence equi-absolute continuity readily follows. $\qquad \square$

Of course, if under conditions of Lemma 1.4 we have pointwise convergence or convergence almost everywhere $f_n(t)$ to a function $f(t)$ then Lebesgue's dominated convergence theorem gives at once $f(t) \in L_p$. But such convergence does not take place generally even under conditions of the Theorem 1.2 as it follows for instance from the following example.

Let $f_n(t) = 1 + \cos(nt)$. We have uniformly in $x \in [0; 1]$

$$\int_0^x (1 + \cos(nt))\, dt \to \int_0^x dt$$

as $n \to \infty$. But the limit $\lim_{n \to \infty} f_n(t)$ exists nowhere on $[0; 1]$ except $t = 0$.

Nevertheless one can state the following almost obvious assertion.

Theorem 1.5. *Assume that there exists a summable on $[a, b]$ non-negative function $f(t)$ such that for all $x \in [a, b]$*

$$\lim_{n \to \infty} \int_a^x f_n(t)\, dt = \int_a^x f(t)\, dt.$$

Assume also that the sequence $f_n(t)$ is dominated by a positive function $g(t) \in L_p[a, b]$ $(p \geq 1)$, that is, $f_n(t) \leq g(t)$ for all $n \in \mathbb{N}$ and almost all $t \in [a, b]$. Then $f(t) \in L_p[a, b]$.

Proof. By reverse Fatou's lemma we have for any $[\alpha, \beta] \subseteq [a, b]$

$$\int_\alpha^\beta f(t)\, dt = \lim_{n \to \infty} \int_\alpha^\beta f_n(t)\, dt \leq \int_\alpha^\beta \limsup_{n \to \infty} f_n(t)\, dt \leq \int_\alpha^\beta g(t)\, dt.$$

From this it follows for $h > 0$ and $x \in [a, b]$ $(f(t) = g(t) \equiv 0$ for $t > b)$

$$\frac{1}{h} \int_x^{x+h} f(t)\, dt \leq \frac{1}{h} \int_x^{x+h} g(t)\, dt.$$

Passing to the limit $h \to 0$ in this inequality, we find almost everywhere on $[a, b]$

$$f(x) \leq g(x),$$

and hence $f(t) \in L_p[a, b]$. $\qquad \square$

Till now we supposed that $[a, b] \subseteq \sigma(A)$ but it is easy to give a sufficient condition for that in terms of functions $f_n(x)$. Actually by the same way as it was made in Theorem 1.5 one can prove that if $f_n(t) \geq C > 0$ for all $n \in \mathbb{N}$ and almost all $t \in [a, b]$, then $f(t) \geq C > 0$ for almost all $t \in [a, b]$ and hence $\sigma(A)$ is not empty on $[a, b]$.

Taking this into account and combining the results of Lemmas 1.1, 1.4 and Theorems 1.2, 1.5, we obtain the following result which we will formulate for the brevity for the operators A with simple spectrum.

Theorem 1.6. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of linear self-adjoint operators in a separable Hilbert space H with absolutely continuous spectrum on some interval $[a, b]$ and Φ be some dense set in H. Suppose that on Φ there exists in a strong sense the limit $\lim_{n \to \infty} A_n$ and the closure of this limit operator is self-adjoint operator A with simple spectrum and generating vector e_0. Let $E_{\lambda}^{(n)}$ and E_{λ} be the resolutions of the identity of the operators A_n and A respectively, and let*

$$\sigma_n(\lambda) = (E_{\lambda}^{(n)} e_0, e_0) = \sigma_n(a) + \int_a^{\lambda} f_n(t)\, dt, \quad \lambda \in [a, b],$$

$$\sigma(\lambda) = (E_{\lambda} e_0, e_0), \quad \lambda \in [a, b].$$

If there exist a constant $C > 0$ and a positive function $g(t) \in L_p[a, b]$ $(p \geq 1)$ such that $C \leq f_n(t) \leq g(t)$ for all $n \in \mathbb{N}$ and almost all $t \in [a, b]$, then the spectrum of the operator A on $[a, b]$ is purely absolutely continuous,

$$\sigma(\lambda) = \sigma(a) + \int_a^{\lambda} f(t)\, dt, \quad \lambda \in [a, b]$$

and $f(t) \in L_p[a, b]$. Under these conditions the sequence $\sigma_n(\lambda)$ uniformly converges to $\sigma(\lambda)$ on $[a, b]$.

Remark 1.7. If the functions $f_n(t)$ are continuous and converge uniformly on $[a, b]$ to a positive function $f(t)$, then the spectrum of A on $[a, b]$ is of course also purely absolutely continuous and $f(t) \in C[a, b]$.

2. APPLICATION TO JACOBI MATRICES

A Jacobi matrix is the tridiagonal matrix of the form

$$\begin{pmatrix} a_0 & b_0 & 0 & 0 & \dots \\ b_0 & a_1 & b_1 & 0 & \dots \\ 0 & b_1 & a_2 & b_2 & \dots \\ 0 & 0 & b_2 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{2.1}$$

where all the a_n are real and all the b_n positive.

An operator in a separable Hilbert space H can be associated with this matrix as follows [1]. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis in H. Let us define basically the operator A on the basis vectors e_n, according to the matrix representation (2.1):

$$Ae_n = b_{n-1}e_{n-1} + a_ne_n + b_ne_{n+1}, \quad n = 1, 2, 3, \ldots,$$
$$Ae_0 = a_0e_0 + b_0e_1.$$

Further, the operator A is defined by linearity on all finite vectors, the set of which is dense in H. (Finite vector has a finite number of non-zero components in the basis $\{e_n\}_{n=0}^{\infty}$.) Due to symmetricity of a Jacobi matrix, for any two finite vectors f and g we have

$$(Af, g) = (f, Ag).$$

Therefore, the operator A is symmetric and permits a closure. This minimal closed operator is a subject of the present consideration and it is naturally to save for it the notation A.

The operator A can have deficiency indices $(1, 1)$ or $(0, 0)$. In the second case having the main interest in the present paper, operator A is self-adjoint. A simple sufficient condition for self-adjointness of A is Carleman's condition

$$\sum_{n=0}^{\infty} \frac{1}{b_n} = +\infty. \tag{2.2}$$

In the self-adjoint case the spectrum of A is simple and e_0 is a generating element. The information about the spectrum of a self-adjoint operator A is contained in function

$$R(\lambda) = ((A - \lambda E)^{-1}e_0, e_0) = \int_{-\infty}^{\infty} \frac{d\sigma(t)}{t - \lambda}, \quad \sigma(t) = (E_te_0, e_0), \tag{2.3}$$

defined at $\lambda \notin \sigma(A)$. (E_t is the resolution of the identity of the operator A.) The distribution function $\sigma(t)$ contains complete information about the spectrum. It has the representation

$$\sigma(\lambda) = \sigma_d(\lambda) + \sigma_{ac}(\lambda) + \sigma_{sc}(\lambda),$$

where $\sigma_d(\lambda)$ is a saltus function defining eigenvalues, $\sigma_{ac}(\lambda)$ and $\sigma_{sc}(\lambda)$ are the absolutely continuous and the singularly continuous parts, respectively.

For the function $R(\lambda)$ there exists an afficient approximation algorithm based on the continued fractions. Namely, the following representation is valid

$$R(\lambda) = \cfrac{1}{a_0 - \lambda - \cfrac{(b_0)^2}{a_1 - \lambda - \cfrac{(b_1)^2}{a_2 - \lambda - \cfrac{(b_2)^2}{a_3 - \lambda - \ldots}}}}. \tag{2.4}$$

The possibility of such representation and also the conditions and character of convergence are defined by the following theorem [1].

Theorem 2.1. *If the operator A is self-adjoint then the continued fraction in* (2.4) *uniformly converges in any closed bounded domain of λ without common points with real axis to the analytic function defined by formula* (2.3).

The continued fraction convergence here means the existence of the finite limit of the n-th approximant

$$\cfrac{1}{a_0 - \lambda - \cfrac{b_0^2}{a_1 - \lambda - \cfrac{b_1^2}{a_2 - \lambda - \ldots - \cfrac{b_{n-2}^2}{a_{n-1} - \lambda}}}} = -\frac{Q_n(\lambda)}{P_n(\lambda)},$$

where $P_n(\lambda)$ and $Q_n(\lambda)$ are 1st and 2nd type polynomials respectively. These polynomials form a pair of linearly independent solutions of a second order finite difference equation

$$b_{n-1}\, y_{n-1} + a_n\, y_n + b_n\, y_{n+1} = \lambda\, y_n, \quad n = 1, 2, 3, \ldots, \tag{2.5}$$

with initial conditions

$$P_0(\lambda) = 1, \quad P_1(\lambda) = \frac{\lambda - a_0}{b_0}, \quad Q_0(\lambda) = 0, \quad Q_1(\lambda) = \frac{1}{b_0}. \tag{2.6}$$

The following equality is also valid

$$P_{n-1}(\lambda)\, Q_n(\lambda) - P_n(\lambda)\, Q_{n-1}(\lambda) = \frac{1}{b_{n-1}}, \quad n = 1, 2, 3, \ldots \tag{2.7}$$

Thus

$$R(\lambda) = -\lim_{n \to \infty} \frac{Q_n(\lambda)}{P_n(\lambda)}, \quad \operatorname{Im} \lambda \neq 0.$$

Exactly an infinite part of the continued fraction (the behavior of a_n and b_n at infinity) defines essentially the spectrum of A. Particularly, an absolutely continuous part of spectrum is determined by the behavior of a_n and b_n at infinity too. Therefore we need to allocate an infinite part of the continued fraction starting from arbitrary place.

Lemma 2.2. *Denote by $K_n(\lambda)$ the infinite part ("tail") of continued fraction* (2.4) *starting with n-th element of sequences a_n and b_n*

$$K_n(\lambda) = \cfrac{1}{a_n - \lambda - \cfrac{(b_n)^2}{a_{n+1} - \lambda - \cfrac{(b_{n+1})^2}{a_{n+2} - \lambda - \cfrac{(b_{n+2})^2}{a_{n+3} - \lambda - \ldots}}}}.$$

Then

$$R(\lambda) = -\frac{Q_n(\lambda) + Q_{n-1}(\lambda)\, b_{n-1} K_n(\lambda)}{P_n(\lambda) + P_{n-1}(\lambda)\, b_{n-1} K_n(\lambda)}, \quad n = 1, 2, 3, \ldots$$

Proof. The proof is obtained by induction. The validity of this formula for $n = 1$ follows from initial conditions (2.6) for polynomials P_n and Q_n. Assuming the validity of formula

$$R(\lambda) = -\frac{Q_{n-1} + Q_{n-2}\, b_{n-2} K_{n-1}}{P_{n-1} + P_{n-2}\, b_{n-2} K_{n-1}},$$

let us prove the same one for the value of index greater on unit. Expressing K_{n-1} through K_n, we obtain

$$R(\lambda) = -\frac{Q_{n-1} + Q_{n-2}\, b_{n-2}\, \dfrac{1}{a_{n-1} - \lambda - (b_{n-1})^2\, K_n}}{P_{n-1} + P_{n-2}\, b_{n-2}\, \dfrac{1}{a_{n-1} - \lambda - (b_{n-1})^2\, K_n}}$$

$$= -\frac{(a_{n-1} - \lambda)Q_{n-1} + Q_{n-2}\, b_{n-2} - Q_{n-1}\,(b_{n-1})^2\, K_{n-1}}{(a_{n-1} - \lambda)P_{n-1} + P_{n-2}\, b_{n-2} - P_{n-1}\,(b_{n-1})^2\, K_{n-1}}.$$

Using (2.5), we find

$$R(\lambda) = -\frac{-Q_n\, b_{n-1} - Q_{n-1}\,(b_{n-1})^2\, K_{n-1}}{-P_n\, b_{n-1} - P_{n-1}\,(b_{n-1})^2\, K_{n-1}} = -\frac{Q_n + Q_{n-1}\, b_{n-1} K_n}{P_n + P_{n-1}\, b_{n-1} K_n}.$$

Thus the formula is valid for all natural n. $\qquad\square$

To construct an approximating sequence of operators A_n one can choose the infinite part $K_n(\lambda)$ such that the spectrum of A_n is absolutely continuous on some interval $[a, b]$ and, on the other hand, $K_n(\lambda)$ is rather simple and convenient for given A.

Choose the approximating sequence A_n as follows:

$$A_n = \begin{pmatrix} a_0 & b_0 & \dots & 0 & 0 & 0 & 0 & \dots \\ b_0 & a_1 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1} & b_{n-1} & 0 & 0 & \dots \\ 0 & 0 & \dots & b_{n-1} & a_n & b_n & 0 & \dots \\ 0 & 0 & \dots & 0 & b_n & a_n & b_n & \dots \\ 0 & 0 & \dots & 0 & 0 & b_n & a_n & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad n = 0, 1, \dots$$

The matrix A_n differs from A by that the elements of the diagonals are not changed starting with n-th number. If we denote the elements of the diagonal sequences of A_n by $a_k^{(n)}$ and $b_k^{(n)}$, then

$$a_k^{(n)} = \begin{cases} a_k, & k < n, \\ a_n, & k \geq n, \end{cases} \qquad b_k^{(n)} = \begin{cases} b_k, & k < n, \\ b_n, & k \geq n. \end{cases}$$

Such matrices A_n were used for approximation in works [3, 4, 8, 19]. Note that they were used also in work [6], where the absolutely continuous spectrum of Jacobi matrices was also considered, but in connection with commutation relations rather than with approximation.

It is clear that the matrices A_n describe a sequence of bounded self-adjoint operators (Carleman's condition (2.2) is obviously fulfilled) and it is natural to save the same notation for it.

The sequence A_n strongly converges to the operator A on the set of finite vectors since for any finite vector the difference $A - A_n$ vanishes, provided n is sufficiently large. To use Theorem 1.6 we should state the conditions under which the spectrum of A_n is absolutely continuous. Note that the spectrum of the operators A_n and A is simple and all information about the spectrum contains in the functions $\sigma_n(\lambda) = (E_\lambda^{(n)} e_0, e_0)$ and $\sigma(\lambda) = (E_\lambda e_0, e_0)$ (generating vector e_0 is the same for A and A_n). Below we will state the absolute continuity conditions of the spectrum of A_n. We will give also an explicit expression for the functions $f_n(t) = \sigma_n'(t)$. It is possible to do with the help of continued fraction approach.

Denote by $R_n(\lambda)$ the function $R(\lambda)$ for the operator A_n defined by (2.3). Applying Lemma 2.2 and taking into account the definition of the operators A_n, we can present $R_n(\lambda)$ in the form

$$R_n(\lambda) = -\frac{Q_n(\lambda) + Q_{n-1}(\lambda)\, b_{n-1} K_n(\lambda)}{P_n(\lambda) + P_{n-1}(\lambda)\, b_{n-1} K_n(\lambda)}, \qquad (2.8)$$

where

$$K_n(\lambda) = \cfrac{1}{a_n - \lambda - \cfrac{(b_n)^2}{a_n - \lambda - \cfrac{(b_n)^2}{a_n - \lambda - \cfrac{(b_n)^2}{a_n - \lambda - \dots}}}}. \qquad (2.9)$$

A continued fraction for $K_n(\lambda)$ describes an infinite part of the matrix of the operators A_n in which the elements of the sequences are constant and equal a_n and b_n. The function $K_n(\lambda)$ can be regarded as a matrix element of the resolvent of the operator represented by a Jacobi matrix J_n with constant elements along the diagonals

$$J_n = \begin{pmatrix} a_n & b_n & 0 & 0 & \dots \\ b_n & a_n & b_n & 0 & \dots \\ 0 & b_n & a_n & b_n & \dots \\ 0 & 0 & b_n & a_n & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This Jacobi matrix is naturally connected with Tchebychev's polynomials, and the function $K_n(\lambda)$ can be found explicitly. Note first that the operator J_n is self-adjoint and by Theorem 2.1 the continued fraction (2.9) converges to the function $K_n(\lambda)$

for any non-real λ. Write next the identity for the k-th and $(k-1)$-th approximants which follows readily from the structure of the continued fraction (2.9)

$$K_n^{(k)}(\lambda) = \frac{1}{a_n - \lambda - (b_n)^2 \, K_n^{(k-1)}(\lambda)}.$$

Letting here $\operatorname{Im} \lambda \neq 0$ and passing to the limit, $k \to \infty$, one obtains the simple equation for $K_n(\lambda)$

$$K_n(\lambda) = \frac{1}{a_n - \lambda - (b_n)^2 \, K_n(\lambda)},$$

which has a solution

$$K_n(\lambda) = \frac{a_n - \lambda + \sqrt{(a_n - \lambda)^2 - 4(b_n)^2}}{2 \, (b_n)^2}.$$

One of the two branches of the square root is chosen from additional condition

$$\operatorname{Im} K_n(\lambda) > 0, \quad \operatorname{Im} \lambda > 0,$$

which is valid for a matrix element of the resolvent of any self-adjoint operator [2]. Using this one obtains for any $x \in \mathbb{R}$

$$\lim_{\epsilon \to +0} K_n(x + i\epsilon) = D_n(x) + i \, B_n(x), \tag{2.10}$$

$$D_n(x) = \begin{cases} \dfrac{a_n - x}{2 \, (b_n)^2}, & |a_n - x| \leq 2b_n, \\[2ex] \dfrac{a_n - x \pm \sqrt{(a_n - x)^2 - 4(b_n)^2}}{2 \, (b_n)^2}, & |a_n - x| \geq 2b_n, \end{cases} \tag{2.11}$$

$$B_n(x) = \begin{cases} \dfrac{\sqrt{4(b_n)^2 - (a_n - x)^2}}{2 \, (b_n)^2}, & |a_n - x| \leq 2b_n, \\[2ex] 0, & |a_n - x| \geq 2b_n. \end{cases} \tag{2.12}$$

In the expression for $D_n(x)$ the sign "$+$" before the square root is taken when $x - a_n > 2b_n$, the sing "$-$" when $a_n - x > 2b_n$.

From (2.10)–(2.12) it follows that the spectrum of J_n is absolutely continuous and concentrated on the interval $\sigma(J_n) = [a_n - 2b_n, a_n + 2b_n]$. The function $K_n(\lambda)$ is the matrix element of the resolvent of self-adjoint operator J_n. Hence for the function $K_n(\lambda)$ one has the integral representation

$$K_n(\lambda) = \int\limits_{-\infty}^{\infty} \frac{f_{J_n}(x) \, dx}{x - \lambda} = \frac{1}{2\pi(b_n)^2} \int\limits_{a_n - 2b_n}^{a_n + 2b_n} \frac{\sqrt{4(b_n)^2 - (a_n - x)^2} \, dx}{x - \lambda},$$

where

$$f_{J_n}(x) = \begin{cases} \dfrac{\sqrt{4(b_n)^2 - (a_n - x)^2}}{2\pi \, (b_n)^2}, & |a_n - x| \leq 2b_n, \\[2ex] 0, & |a_n - x| > 2b_n. \end{cases} \tag{2.13}$$

The function $f_{J_n}(x)$ at $a_n = 0$, $b_n = 1/2$ is the spectral weight for Tchebychev's polynomials [17, 20]. The element a_n defines the center of the spectrum and b_n defines the width of the spectrum. The center and the width of the spectrum change according to the behavior of a_n and b_n.

We prove that under some conditions the spectrum of A_n is the same as the spectrum of J_n.

Theorem 2.3. *Inside the interval $I_n = [a_n - 2b_n, a_n + 2b_n]$ the spectrum of the operator A_n is always absolutely continuous. The spectral weight $f_n(x)$ with accuracy to the values on a set of measure zero is defined by*

$$f_n(x) = \frac{\frac{1}{\pi} B_n(x)}{P_n^2(x) - \frac{b_{n-1}}{b_n} P_{n-1}(x) P_{n+1}(x)} = \frac{f_{J_n}(x)}{P_n^2(x) - \frac{b_{n-1}}{b_n} P_{n-1}(x) P_{n+1}(x)}, \quad (2.14)$$

where $f_{J_n}(x)$ is the spectral weight of J_n. If for any $n \in \mathbb{N}$ the inequalities

$$b_n \geq b_{n-1}, \quad |a_n - a_{n-1}| \leq 2(b_n - b_{n-1})$$

hold, then the spectrum of A_n is purely absolutely continuous, concentrated on the interval I_n and $I_n \subseteq I_{n+1}$.

Proof. We first prove that the absolutely continuous part of the spectrum of A_n coincides with the spectrum of J_n. For this purpose transform the expression (2.8) for $R_n(\lambda)$

$$R_n(\lambda) = -\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)} - \frac{Q_n(\lambda) P_{n-1}(\lambda) - Q_{n-1}(\lambda) P_n(\lambda)}{P_{n-1}(\lambda) (P_n(\lambda) + P_{n-1}(\lambda) b_{n-1} K_n(\lambda))}$$

or, using (2.7),

$$R_n(\lambda) = -\frac{Q_{n-1}(\lambda)}{P_{n-1}(\lambda)} - \frac{1}{b_{n-1} P_{n-1}(\lambda) (P_n(\lambda) + P_{n-1}(\lambda) b_{n-1} K_n(\lambda))}.$$

Let $\lambda = x + i\epsilon$, $x \in \mathbb{R}$. When $\epsilon \to 0$, the imaginary parts of P_n and Q_n tend to zero, and the unique nontrivial imaginary part comes from $K_n(\lambda)$. Using (2.10), (2.12) one obtains

$$\lim_{\epsilon \to 0^+} \operatorname{Im} R_n(x + i\epsilon) = \frac{B_n(x)}{(P_n(x) + b_{n-1} P_{n-1}(x) D_n(x))^2 + (b_{n-1} P_{n-1}(x) B_n(x))^2}. \tag{2.15}$$

It is easy to note that the denominator of the fraction (2.15) is nowhere zero within the interval of the absolutely continuous spectrum of J_n. Indeed, if $|x - a_n| < 2b_n$, then the denominator may be zero if and only if x is a common zero of P_n and P_{n-1}. But this is impossible since P_n and P_{n-1} have no common zeros [1].

Thus from the limit expression (2.15) it follows readily that the absolutely continuous parts of the spectrum of A_n and J_n coincide. Let us prove that under the conditions of the theorem the singular part of the spectrum of A_n is absent. For this purpose, return to the formula (2.8). The numerator and the denominator in

this formula are analytic functions on the set $\mathbb{C} \setminus \sigma(J_n)$. Therefore the point $x \in \mathbb{R}$ belongs to the singular part of the spectrum of A_n for $|x - a_n| \leq 2b_n$ if and only if

$$\lim_{\epsilon \to 0^+} (P_n(x + i\epsilon) + P_{n-1}(x + i\epsilon) \, b_{n-1} \, K_n(x + i\epsilon))$$
$$= P_n(x) + P_{n-1}(x) \, b_{n-1} D_n(x) + i \, P_{n-1}(x) \, b_{n-1} B_n(x) = 0 \tag{2.16}$$

and for $|x - a_n| > 2b_n$ if and only if x is a zero of the denominator:

$$P_n(x) + P_{n-1}(x) \, b_{n-1} K_n(x) = P_n(x) + P_{n-1}(x) \, b_{n-1} D_n(x) = 0. \tag{2.17}$$

But as it was shown before, the equality (2.16) cannot hold for $|x - a_n| < 2b_n$. It remains to consider the case $|x - a_n| \geq 2b_n$. In this case the conditions (2.16) and (2.17) coincide since $B_n(x) = 0$.

Thus, let $|x - a_n| \geq 2b_n$. From the definition (2.11) of $D_n(x)$ it follows that for all $x \in \mathbb{R}$

$$|D_n(x)| \leq \frac{1}{b_n}.$$

Using this estimate and the assumed monotonicity of the sequence b_n one obtains

$$|P_n(x) + P_{n-1}(x) \, b_{n-1} D_n(x)| \geq |P_n(x)| - |P_{n-1}(x)| b_{n-1} |D_n(x)| \geq$$
$$\geq |P_n(x)| - |P_{n-1}(x)| \frac{b_{n-1}}{b_n} \geq |P_n(x)| - |P_{n-1}(x)|. \tag{2.18}$$

On the other hand, from the recurrence relations (2.5) for the polynomials P_n one has

$$|P_n(x)| = \left| \frac{(x - a_{n-1})P_{n-1}(x) - b_{n-2}P_{n-2}(x)}{b_{n-1}} \right| \geq \frac{|x - a_{n-1}|}{b_{n-1}} |P_{n-1}(x)| - \frac{b_{n-2}}{b_{n-1}} |P_{n-2}|.$$

If $|x - a_n| \geq 2b_n$, then by virtue of the theorem condition $|a_n - a_{n-1}| \leq 2(b_n - b_{n-1})$, the inequality $|x - a_{n-1}| \geq 2b_{n-1}$ also holds. Therefore, from the last inequality one obtains

$$|P_n(x)| \geq 2|P_{n-1}(x)| - |P_{n-2}(x)|$$

or

$$|P_n(x)| - |P_{n-1}(x)| \geq |P_{n-1}(x)| - |P_{n-2}(x)|.$$

Continuing this process by induction one obtains

$$|P_n(x)| - |P_{n-1}(x)| \geq |P_1(x)| - |P_0(x)| \geq 2 - 1 = 1.$$

Finally, substituting this inequality into (2.18) one arrives at the following estimate of the denominator in the formula (2.8) in the case $|x - a_n| \geq 2b_n$

$$|P_n(x) + P_{n-1}(x) \, b_{n-1} D_n(x)| \geq 1,$$

whence it follows the absence in this area the points of a singular spectrum of A_n.

It remains to prove the formula (2.14). Using (2.13) and (2.15) one has

$$f_n(x) = \frac{1}{\pi} \lim_{\epsilon \to +0} \operatorname{Im} R_n(x + i\epsilon)$$

$$= \frac{\frac{1}{\pi} B_n(x)}{(P_n(x) + b_{n-1} P_{n-1}(x) D_n(x))^2 + (b_{n-1} P_{n-1}(x) B_n(x))^2}$$

$$= \frac{f_{J_n}(x)}{(P_n(x) + b_{n-1} P_{n-1}(x) D_n(x))^2 + (b_{n-1} P_{n-1}(x) B_n(x))^2}.$$

Let us transform the denominator. When $|x - a_n| \le 2b_n$, using (2.11), (2.12), (2.5), one has

$$(P_n(x) + b_{n-1} P_{n-1}(x) D_n(x))^2 + (b_{n-1} P_{n-1}(x) B_n(x))^2$$

$$= \left(P_n + \frac{a_n - x}{2b_n^2} b_{n-1} P_{n-1} \right)^2 + P_{n-1}^2 b_{n-1}^2 \frac{4b_n^2 - (a_n - x)^2}{4b_n^4}$$

$$= P_n^2 + \frac{a_n - x}{b_n^2} b_{n-1} P_{n-1} P_n + \frac{b_{n-1}^2}{b_n^2} P_{n-1}^2 = P_n^2 + P_{n-1} \frac{b_{n-1}}{b_n^2} ((a_n - x) P_n + b_{n-1} P_{n-1})$$

$$= P_n^2 - \frac{b_{n-1}}{b_n} P_{n-1} P_{n+1}.$$

The theorem is proved. $\square$

Additional condition of the theorem means some subordination of a_n to b_n. Indeed, for $k \in \mathbb{N}$, one has

$$|a_k| - |a_{k-1}| \le |a_k - a_{k-1}| \le 2(b_k - b_{k-1}).$$

Adding these inequalities for $k = 1, 2, \ldots, n$, one obtains

$$|a_n| \le |a_0| + 2(b_n - b_0).$$

Note also that the expression

$$P_n^2(x) - \frac{b_{n-1}}{b_n} P_{n-1}(x) P_{n+1}(x)$$

appears probably first time in work [13]. In [7] it is also called a "Turan determinant" although the classical Turan determinant [22] is

$$P_n^2(x) - P_{n-1}(x) P_{n+1}(x) = \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix}.$$

Definition 2.4. Let us call the system of intervals I_n *centered* on some interval $[a, b]$ if this interval is contained inside the interval I_n for all sufficiently large values of n.

Now, when the absolute continuity conditions for the spectrum of A_n are stated, we can apply Theorem 1.6. Combining the results of Theorems 1.6 and 2.3 and using Definition 2.4, we get the following result.

Theorem 2.5. *Assume that the operator A which is represented by (2.1) is self-adjoint and the system of intervals I_n is centered on some interval $[a, b]$. Let*

$$f_n(x) = \frac{\sqrt{4(b_n)^2 - (a_n - x)^2}}{2\pi\, b_n\, [b_n P_n^2(x) - b_{n-1} P_{n-1}(x) P_{n+1}(x)]}, \quad x \in [a, b].$$

Assume that there exist a constant $C > 0$ and a positive function $g(x) \in L_p[a, b]$ $(p \geq 1)$ such that for all sufficiently large n and almost all $x \in [a, b]$

$$C \leq f_n(x) \leq g(x).$$

Then the spectrum of the operator A is purely absolutely continuous on $[a, b]$. The sequence $\sigma_n(\lambda)$ uniformly on $[a, b]$ converges to $\sigma(\lambda)$,

$$\sigma(\lambda) = \sigma(a) + \int_a^\lambda f(x)\, dx,$$

and $f(x) \in L_p[a, b]$.

Remark 2.6. The functions $f_n(x)$ are continuous on $[a, b]$, if the system I_n is centered on this interval. Hence if the system $f_n(x)$ converges uniformly on $[a, b]$ to a positive function $f(x)$, then the spectrum of A on $[a, b]$ is purely absolutely continuous and $f(x) \in C[a, b]$.

Using the recurrence relations for polynomials P_n, one can present the denominator of functions $f_n(x)$ in the form

$$b_n P_n^2(x) - b_{n-1} P_{n-1}(x) P_{n+1}(x) = b_n(P_{n+1}^2(x) + P_n^2(x)) - (x - a_n) P_{n+1}(x) P_n(x).$$

Hence, if $|x - a_n| \leq 2b_n q$ where $0 < q < 1$, then

$$b_n \left(P_{n+1}^2(x) + P_n^2(x) \right)(1 - q) \leq b_n P_n^2(x) - b_{n-1} P_{n-1}(x) P_{n+1}(x)$$
$$\leq b_n \left(P_{n+1}^2(x) + P_n^2(x) \right)(1 + q).$$

Therefore,

$$\frac{\frac{\sqrt{1-q^2}}{\pi(1+q)}}{b_n(P_{n+1}^2(x) + P_n^2(x))} \leq f_n(x) \leq \frac{\frac{1}{\pi(1-q)}}{b_n(P_{n+1}^2(x) + P_n^2(x))},$$

whence one obtains the following result.

Theorem 2.7. *Suppose that the operator A represented by Jacobi matrix (2.1) is self-adjoint. Suppose that there exist the constants $C > 0$, $0 < q < 1$ and a positive function $g(x) \in L_p[a, b]$ $(p \geq 1)$ such that for all sufficiently large n and almost all $x \in [a, b]$ the following estimates hold:*

1. $|x - a_n| \leq 2b_n q$,
2. $\dfrac{1}{g(x)} \leq b_n(P_{n+1}^2(x) + P_n^2(x)) \leq C.$

Then the spectrum of the operator A is purely absolutely continuous on $[a, b]$. The sequence $\sigma_n(\lambda)$ uniformly on $[a, b]$ converges to $\sigma(\lambda)$,

$$\sigma(\lambda) = \sigma(a) + \int\limits_a^\lambda f(x)\, dx,$$

and $f(x) \in L_p[a, b]$.

Note that the right hand side of the second estimate of this theorem is a consequence of subordination theory as it was shown in [10, 18] (see also [9, 11, 14]) and it provides the presence of absolutely continuous part of the spectrum on $[a, b]$. Here it is obtained by a natural and simple way which is not connected with subordination theory.

The most simple form the second estimate has in the case L_∞. Actually then it takes the form

$$C_1 \le b_n(P_{n+1}^2(x) + P_n^2(x)) \le C_2,$$

where C_1, C_2 are some positive constants.

One can state a simple conditions on the coefficients a_n and b_n of Jacobi matrix in the case $f(x) \in C(-\infty, +\infty)$ (with accuracy up to equivalence). Some kinds of this result were obtained by different methods in works [4,5,9,12]. Here we give independent, new and simple proof of it.

Theorem 2.8. *Suppose that the operator A represented by Jacobi matrix (2.1) is self-adjoint. Let a_n and b_n satisfy the conditions:*

1. $\lim\limits_{n\to\infty} b_n = +\infty$,

2. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = s, \qquad 0 \le |s| < 2$,

3. $\lim\limits_{n\to\infty} \dfrac{b_n}{b_{n+1}} = 1$,

4. $\left\{ \dfrac{b_{n-1}}{b_n} - \dfrac{b_{n-2}}{b_{n-1}} \right\} \in l_1, \quad \left\{ \dfrac{1}{b_n} - \dfrac{1}{b_{n-1}} \right\} \in l_1, \quad \left\{ \dfrac{a_n}{b_n} - \dfrac{a_{n-1}}{b_{n-1}} \right\} \in l_1.$

Then the spectrum of the operator A is purely absolutely continuous $\sigma(A) = \sigma_{ac}(A) = \mathbb{R}$, the corresponding spectral density $f(x) \in C(-\infty, +\infty)$ and the sequence $\sigma_n(x)$ uniformly converges to the function $\sigma(x)$ on any finite interval.

Proof. Fix an arbitrary finite interval $[a, b]$ of a real axis. From conditions (1) and (2) it follows that for all $x \in [a, b]$ and all sufficiently large n the estimation $|x - a_n| \le 2b_n q$ holds for $|s|/2 < q < 1$. Denote

$$\Delta_n(x) = b_n P_n^2(x) - b_{n-1} P_{n-1}(x) P_{n+1}(x).$$

Then

$$\lim\limits_{n\to\infty} f_n(x) = \frac{1}{\pi} \lim\limits_{n\to\infty} \frac{1}{\Delta_n(x)},$$

and due to Remark 2.6 to Theorem 2.5 we have to prove that the sequence $\Delta_n(x)$ uniformly on $[a, b]$ converges to a positive function (recall that $\Delta_n(x)$ is always positive for all sufficiently large n when the system I_n is centered on $[a, b]$). We have

$$\Delta_{n+1} - \Delta_n = \left(1 - \frac{b_n}{b_{n+1}}\right) b_{n+1} P_{n+1}^2 - \left(1 - \frac{b_n}{b_{n+1}}\right) b_n P_n^2$$
$$+ \left(x\left(\frac{1}{b_n} - \frac{1}{b_{n+1}}\right) + \left(\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n}\right)\right) b_n P_n P_{n+1}.$$

Adding consecutively this equalities, one obtains for $n > m + 1$

$$\Delta_n - \Delta_m = \gamma_n b_n P_n^2 - \gamma_m b_m P_m^2 + \sum_{k=m+1}^{n-1} \alpha_k b_k P_k^2 + \sum_{k=m}^{n-1} \beta_k b_k 2 P_k P_{k+1},$$

where

$$\alpha_k = \frac{b_k}{b_{k+1}} - \frac{b_{k-1}}{b_k}, \quad \beta_k = \frac{1}{2}\left(x\left(\frac{1}{b_k} - \frac{1}{b_{k+1}}\right) + \left(\frac{a_{k+1}}{b_{k+1}} - \frac{a_k}{b_k}\right)\right), \quad \gamma_k = 1 - \frac{b_{k-1}}{b_k}.$$

Since for all n sufficiently large and all $x \in [a, b]$

$$b_n P_n^2 (1 - q) \le b_n \left(P_{n+1}^2 + P_n^2\right)(1 - q) \le \Delta_n,$$

we have for m sufficiently large ($n > m + 1$) and all $x \in [a, b]$

$$(1 - q)|\Delta_n - \Delta_m| \le |\gamma_n| \Delta_n + |\gamma_m| \Delta_m + \sum_{k=m}^{n-1}(|\alpha_k| + |\beta_k|)\Delta_k. \qquad (2.19)$$

Suppose that the sequence Δ_n is uniformly bounded in $[a, b]$ (we will prove it below). Then from conditions (3) and (4) of the theorem it follows that for any $\epsilon > 0$ and any $x \in [a, b]$

$$|\Delta_n - \Delta_m| < \epsilon,$$

provided $m > N$ and any $n > m$. In the case $n = m + 1$ this estimation is also fulfilled. It follows that the sequence Δ_n uniformly converges to a nonnegative continuous on $[a, b]$ function $\Delta(x)$.

It remains to prove uniform boundedness of $\Delta_n(x)$ and positivity of $\Delta(x)$. Let us return to the inequality (2.19). One has

$$(1 - \delta_n)\Delta_n \le (1 + \delta_m)\Delta_m + \sum_{k=m}^{n-1} \epsilon_k \Delta_k,$$

where

$$\delta_k = \frac{|\gamma_k|}{1 - q}, \quad \epsilon_k = \frac{|\alpha_k| + |\beta_k|}{1 - q}.$$

Denote

$$B_n = \sum_{k=m}^{n-1} \epsilon_k \Delta_k.$$

We have
$$B_n - B_{n-1} = \epsilon_{n-1}\Delta_{n-1} \le \epsilon_{n-1}\frac{B_{n-1} + (1+\delta_m)\Delta_m}{1 - \delta_{n-1}}$$
and hence
$$B_n \le \left(1 + \frac{\epsilon_{n-1}}{1-\delta_{n-1}}\right)B_{n-1} + \frac{\epsilon_{n-1}}{1-\delta_{n-1}}(1+\delta_m)\Delta_m.$$
From this we consecutively deduce that
$$B_n \le (B_m + (1+\delta_m)\Delta_m)\prod_{k=m}^{n-1}\left(1 + \frac{\epsilon_k}{1-\delta_k}\right)$$
$$\le (B_m + (1+\delta_m)\Delta_m)\prod_{k=m}^{\infty}\left(1 + \frac{\epsilon_k}{1-\delta_k}\right) = C < +\infty.$$

This estimation is uniform in $x \in [a, b]$. Here we used again the uniform convergence of series $\sum \frac{\epsilon_k}{1-\delta_k}$ which follows from conditions (3) and (4) of the theorem. From
$$(1-\delta_n)\Delta_n \le (1+\delta_m)\Delta_m + B_n$$
it follows that Δ_n is also uniformly bounded.

Let us prove at last that the limit function $\Delta(x)$ is strictly positive in $[a, b]$. From (2.19) one has
$$(1+\delta_n)\Delta_n \ge (1-\delta_m)\Delta_m - \sum_{k=m}^{n-1}\epsilon_k\Delta_k$$

(in the same notations as before). Since $\sum\limits_{k=m}^{n-1}\epsilon_k\Delta_k \le \epsilon\Delta_m$, where ϵ is arbitrary small for m sufficiently large, we have
$$(1+\delta_n)\Delta_n \ge (1-\delta_m-\epsilon)\Delta_m = C > 0$$
for m sufficiently large and all $n > m+1$. Passing to the limit $n \to \infty$, we obtain $\Delta \ge C > 0$ (C depends on x of course).

Thus the sequence $f_n(x)$ uniformly converges to a positive continuous function $f(x) = \dfrac{1}{\pi\Delta(x)}$ in $[a, b]$ and Theorem 2.5 gives the pure absolute continuity of the operator A spectrum on $[a, b]$. Since the interval $[a, b]$ is arbitrary the theorem is proved. $\qquad\square$

In the work [21] it was shown that in the case $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = s$, $0 < |s| < 2$ one can extend the class of weights. Namely, the basic conclusions of Theorem 2.8 (except $f(x) \in C(-\infty, +\infty)$) remains valid if the sequences
$$\frac{a_{n-1}a_n}{b_n^2}, \quad \frac{a_{n-1}+a_n}{b_n^2}, \quad \frac{1}{b_n^2}$$
have bounded variation. Using Theorem 2.5 and the main ideas of [21] one can show that in this case also $f(x) \in C(-\infty, +\infty)$.

3. CONCLUSION

Note that the method of analysis of the absolutely continuous spectrum developed here can be applied not to the finite-difference operators only but to any self-adjoint operators in separable Hilbert space provided that one can find a convenient approximative sequence of operators A_n with absolutely continuous spectrum strongly converging to the operator A on a dense set.

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