Opuscula Math. 39, no. 4 (2019), 543–555
https://doi.org/10.7494/OpMath.2019.39.4.543 Opuscula Mathematica

ON UNITARY EQUIVALENCE OF BILATERAL OPERATOR VALUED WEIGHTED SHIFTS

Jakub Kośmider

Communicated by P.A. Cojuhari

Abstract. We establish a characterization of unitary equivalence of two bilateral operator valued weighted shifts with quasi-invertible weights by an operator of diagonal form. We provide an example of unitary equivalence between shifts with weights defined on $\mathbb{C}^2$ which cannot be given by any unitary operator of diagonal form. The paper also contains some remarks regarding unitary operators that can give unitary equivalence of bilateral operator valued weighted shifts.

Keywords: unitary equivalence, bilateral weighted shift, quasi-invertible weights, partial isometry.

Mathematics Subject Classification: 47B37, 47A62.

1. INTRODUCTION AND PRELIMINARIES

Classical weighted shift operators and their properties have already been studied for a long time by many authors (see, e.g., [2, 5, 14, 16]). By classical weighted shifts we understand both unilateral and bilateral weighted shifts defined on ℓ_2 and $\ell_2(\mathbb{Z})$, respectively. There are many papers devoted to various problems related to weighted shifts in more general context in which these operators are operator valued weighted shifts (see [3, 6, 8, 9, 11–13, 15]). In some of them authors prove or use certain results concerning unitary equivalence of the latter class of operators (see [6, 9, 11–13, 15]). Jabłoński, Jung and Stochel introduced in [10] the class of weighted shifts on directed trees, which generalizes classical unilateral and bilateral weighted shifts.

Unitary equivalence of unilateral operator valued weighted shifts with invertible weights was characterized by Lambert in [11, Corollary 3.3]. Orovčanec provided a similar characterization in the case of unilateral operator valued weighted shifts with quasi-invertible weights (see [13, Theorem 1]). Later on, a similar result was proved in [1, Theorem 2.3] with weaker assumptions, namely, for unilateral shifts with weights having dense ranges. Jabłoński proved in [9, Proposition 2.2] that a unilateral operator valued weighted shift with invertible weights is unitarily equivalent to a unilateral operator valued weighted shift with weights $\{T_n\}_{n=0}^{\infty}$ such that the product $T_n \dots T_0$ is a positive operator for all $n \in \mathbb{N}$.

On the other hand, there are some partial results regarding unitary equivalence of bilateral operator valued weighted shifts. Pilidi proved that if some certain sequences of two shifts are unitarily equivalent, then these shifts are unitarily equivalent (see [15, Theorem 4]). The opposite implication is also true under an additional assumption that certain algebras are equal to $\boldsymbol{B}(\mathcal{H})$ (see [15, Theorem 3] for details). In addition to this, he stated (without proof) a characterization of unitary equivalence of such operators when $\dim \mathcal{H} = 2$ (see [15, Theorem 5]). Li, Ji and Sun proved that each bilateral operator valued weighted shift with invertible weights defined on $\mathbb{C}^m$ for $m \geq 2$ is unitarily equivalent to a shift with upper triangular weights (see [12, Theorem 2.1]). Shields provided in [16] a characterization of unitary equivalence in the case of classical bilateral shifts. Guyker proved in [6] a result regarding unitary equivalence of bilateral operator valued weighted shift with a shift having positive weights. The proof required an additional assumption, i.e., the normality and commutativity of certain weights. In the same paper, among several results regarding reducibility of bilateral shifts, Guyker showed that under some additional assumptions a bilateral operator valued weighted shift is unitarily equivalent to a countable direct sum of classical bilateral shifts (see [6, Theorem 3]).

In what follows, we denote by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ the sets of non-negative integers, positive integers, integers, real numbers, non-negative real numbers and complex numbers, respectively. Throughout the paper by $\mathcal{H}$ we denote a nonzero complex Hilbert space. The symbol $\boldsymbol{B}(\mathcal{H})$ stands for the $\mathcal{C}^*$-algebra of all bounded operators defined on $\mathcal{H}$. All operators considered in this paper are assumed to be linear. By $\mathcal{R}(A)$ and $\mathcal{N}(A)$ we understand the range and the kernel of an operator $A \in \boldsymbol{B}(\mathcal{H})$, respectively. As usual, $I \in \boldsymbol{B}(\mathcal{H})$ stands for the identity operator. A unitary equivalence of operators $A, B \in \boldsymbol{B}(\mathcal{H})$ is denoted by $A \cong B$. We also write $A \cong_U B$ to emphasize that the unitary equivalence is given by U, that is, $UA = BU$. For a closed subspace $\mathcal{M}$ of $\mathcal{H}$, by $\mathcal{M}^\perp$ we denote its orthogonal complement. If $\mathcal{M}$ and $\mathcal{N}$ are two closed subspaces of $\mathcal{H}$, which are orthogonal, then we write $\mathcal{M} \perp \mathcal{N}$. We say that an operator $A \in \boldsymbol{B}(\mathcal{H})$ is *quasi-invertible*, if A is injective and $\overline{\mathcal{R}(A)} = \mathcal{H}$. The reader can verify that, if $A \in \boldsymbol{B}(\mathcal{H})$ is quasi-invertible, then so is A^*. An operator $A \in \boldsymbol{B}(\mathcal{H})$ is called a *partial isometry* if $\|Ax\| = \|x\|$ for all $x \in \mathcal{N}(A)^\perp$.

We define a Hilbert space $\ell_2(\mathbb{Z}, \mathcal{H})$ as the space $\oplus_{n\in\mathbb{Z}}\mathcal{H}$ equipped with the inner product defined by $\langle x, y\rangle = \sum_{n=-\infty}^{\infty} \langle x_n, y_n\rangle_{\mathcal{H}}$ for $x, y \in \ell_2(\mathbb{Z}, \mathcal{H})$. This space consists of all vectors $x = (\dots, x_{-1}, \boxed{x_0}, x_1, \dots)$ satisfying $\sum_{n=-\infty}^{\infty} \|x_n\|_{\mathcal{H}}^2 < \infty$, where $\boxed{\cdot}$ denotes the 0th element of x. An operator $U \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ can be expressed as an infinite matrix $[U_{i,j}]_{i,j\in\mathbb{Z}}$, where $U_{i,j} \in \boldsymbol{B}(\mathcal{H})$ for all $i, j \in \mathbb{Z}$.

We say that $S \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ is a *diagonal operator* if there exists a two-sided sequence of operators $\{S_n\}_{n\in\mathbb{Z}} \subseteq \boldsymbol{B}(\mathcal{H})$ such that the sequence $\{\|S_n\|\}_{n\in\mathbb{Z}}$

is bounded and

$$S(\ldots, x_{-1}, \boxed{x_0}, x_1, \ldots) = (\ldots, S_{-1}x_{-1}, \boxed{S_0 x_0}, S_1 x_1, \ldots), \quad x \in \ell_2(\mathbb{Z}, \mathcal{H}).$$

Let $\{S_n\}_{n\in\mathbb{Z}} \subseteq \boldsymbol{B}(\mathcal{H})$ be a two-sided sequence of nonzero operators such that the sequence $\{\|S_n\|\}_{n\in\mathbb{Z}}$ is bounded. We define $S \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ by

$$S(\ldots, x_{-1}, \boxed{x_0}, x_1, \ldots) = (\ldots, S_{-1}x_{-2}, \boxed{S_0 x_{-1}}, S_1 x_0, \ldots), \quad x \in \ell_2(\mathbb{Z}, \mathcal{H}).$$

The operator S is called a *bilateral operator valued weighted shift* with operator weights $\{S_n\}_{n\in\mathbb{Z}}$ defined on $\mathcal{H}$ and it is denoted by $S \sim \{S_n\}_{n\in\mathbb{Z}}$. It is worth noting that, as opposed to [11] and [15], we do not assume that weights of S are invertible.

Denote by F the unitary bilateral operator valued weighted shift with all weights being identity operators on $\mathcal{H}$. We say that an operator $S \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ is *of diagonal form* if there exist $k \in \mathbb{Z}$ and a diagonal operator $T \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ such that $S = F^k T$.

In this paper we focus on the problem of unitary equivalence of bilateral operator valued weighted shifts with quasi-invertible weights. The paper is organized as follows. In Section 2 we investigate unitary equivalence given by operators of diagonal form. Corollary 2.4 establishes a characterization of unitary equivalence of bilateral operator valued weighted shifts with quasi-invertible weights given by an operator of diagonal form. In Theorem 2.5 we prove that each bilateral operator valued weighted shift with quasi-invertible weights is unitarily equivalent to a bilateral weighted shift having positive weights. We conclude this section with proving that a bilateral operator valued weighted shift having normal and commuting weights defined on $\mathbb{C}^m$ for $m \geq 2$ is unitarily equivalent to a bilateral weighted shift with weights being diagonal operators (see Proposition 2.9).

Section 3 is devoted to investigation of unitary operators on $\ell_2(\mathbb{Z}, \mathcal{H})$ that can give unitary equivalence of bilateral weighted shifts. We begin it with Example 3.1 that shows two bilateral operator valued weighted shifts with weights defined on $\mathbb{C}^2$ which are unitarily equivalent, but the unitary equivalence is not given by any operator of diagonal form. Proposition 3.2 states that, if a unitary operator $U \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ contains exactly two nonzero diagonals and all other elements of U are zero operators, then the operators on these diagonals are partial isometries.

Finally, Section 4 concludes the paper with remarks and open problems related to unitary equivalence of bilateral operator valued weighted shifts.

2. UNITARY EQUIVALENCE GIVEN BY AN OPERATOR OF DIAGONAL FORM

In this section we present results related to unitary equivalence of bilateral operator valued weighted shifts given by an operator of diagonal form. It contains also some general facts which usage is not limited to this section.

We begin with stating the following key lemma required for further references, which is a two-sided counterpart of [13, Lemma] (see also [15, Lemma 4]). Its proof is left to the reader.

Lemma 2.1. *Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ and S_n, T_n be quasi-invertible for each $n \in \mathbb{Z}$. Assume that $A \in \boldsymbol{B}(\ell_2(\mathbb{Z},\mathcal{H}))$. Then the following are equivalent:*

(i) $AS = TA$,
(ii) $A_{i+1,j+1}S_j = T_i A_{i,j}$ *for all* $i, j \in \mathbb{Z}$.

There is a significant difference between [13, Lemma] and Lemma 2.1. In the case of unilateral weighted shifts zero is the first element of every vector in the range. Hence, each operator intertwining two unilateral weighted shifts has a triangular matrix. On the other hand, in the case of bilateral operator valued weighted shifts the equality $AS = TA$ does not imply triangularity of A (see Example 3.1 below).

Lemma 2.1 yields the following corollary.

Corollary 2.2. *Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ and S_n, T_n be quasi-invertible for each $n \in \mathbb{Z}$. Assume that $A \in \boldsymbol{B}(\ell_2(\mathbb{Z},\mathcal{H}))$ be such that $AS = TA$. If $A_{i,j} \neq 0$ for some $i, j \in \mathbb{Z}$, then $A_{i+n,j+n} \neq 0$ for all $n \in \mathbb{Z}$.*

Next theorem gives a necessary and sufficient condition for two bilateral operator valued weighted shifts with quasi-invertible weights to be unitarily equivalent by an operator of diagonal form. The idea of the proof is partially based on a construction from the proof of a similar result for unilateral operator valued weighted shifts from [13, Theorem 1] (see also [1, Theorem 2.3]).

Theorem 2.3. *Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ and $m \in \mathbb{Z}$ be such that S_{m+n}, T_n, S^*_{m-n-1} and T^*_{-n-1} have dense ranges for each $n \in \mathbb{N}$. Then the following are equivalent:*

(i) *there exists a unitary operator $U \in \boldsymbol{B}(\ell_2(\mathbb{Z},\mathcal{H}))$ of diagonal form such that $S \cong_U T$ and $U_{0,m} \neq 0$,*
(ii) *there exists a unitary operator $U_{0,m} \in \boldsymbol{B}(\mathcal{H})$ such that the following hold:*
  (a) $\|S_{m+n-1}\dots S_m x\| = \|T_{n-1}\dots T_0 U_{0,m}x\|$ *for all $x \in \mathcal{H}$ and $n \in \mathbb{N}_+$,*
  (b) $\|S^*_{m-n}\dots S^*_{m-1}x\| = \|T^*_{-n}\dots T^*_{-1}U_{0,m}x\|$ *for all $x \in \mathcal{H}$ and $n \in \mathbb{N}_+$.*

Proof. (i) $\Rightarrow$ (ii). Assume that $S \cong_U T$, where $U \in \boldsymbol{B}(\ell_2(\mathbb{Z},\mathcal{H}))$ is of diagonal form. Let $n \in \mathbb{N}_+$. Then, by Lemma 2.1,

$$U_{n,m+n}S_{m+n-1}\dots S_m = T_{n-1}\dots T_0 U_{0,m}$$

which implies (a). Let us now check that (b) also holds. Let $n \in \mathbb{N}_+$. Again, by Lemma 2.1,

$$U_{0,m}S_{m-1}\dots S_{m-n} = T_{-1}\dots T_{-n}U_{-n,m-n}$$

which is equivalent to the following

$$S_{m-1}\dots S_{m-n}U^*_{-n,m-n} = U^*_{0,m}T_{-1}\dots T_{-n}.$$

After taking adjoints we get that for all $x \in \mathcal{H}$ and $n \in \mathbb{N}_+$ it is true that

$$\|S^*_{m-n}\dots S^*_{m-1}x\| = \|T^*_{-n}\dots T^*_{-1}U_{0,m}x\|,$$

which proves (b).

(ii) $\Rightarrow$ (i). We need to construct $U \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ of diagonal form with unitary operators $U_{n,m+n} \in \boldsymbol{B}(\mathcal{H})$ on its only nonzero diagonal, which satisfy the following

$$U_{n+1,m+n+1} S_{m+n} = T_n U_{n,m+n}, \quad n \in \mathbb{Z}. \tag{2.1}$$

In order to simplify formulas we introduce notation $V_n := U_{n,m+n}$ for $n \in \mathbb{Z}$.

We begin with constructing operators V_n for $n \in \mathbb{N}_+$. Since S_m and $T_0 V_0$ have dense ranges and (a) holds with $n = 1$, it is well known that there exists a unitary operator V_1 such that $V_1 S_m = T_0 V_0$. Now, assume that $n > 1$ and unitary operators $V_1, \ldots, V_n$ are already defined to be such that $V_{i+1} S_{m+i} = T_i V_i$ for $i \in \{1, \ldots, n-1\}$. Again, since operators $S_{m+n} \ldots S_m$ and $T_n \ldots T_0 V_0$ have dense ranges, there exists a unitary operator V_{n+1} such that

$$V_{n+1} S_{m+n} \ldots S_m = T_n \ldots T_0 V_0.$$

By the above we see that

$$(V_{n+1} S_{m+n} - T_n V_n) S_{m+n-1} \ldots S_m = V_{n+1} S_{m+n} \ldots S_m - T_n \ldots T_0 V_0 = 0. \tag{2.2}$$

Since $S_{m+n-1} \ldots S_m$ has a dense range, (2.2) implies that $V_{n+1} S_{m+n} = T_n V_n$.

We now focus on finding operators V_{-n} for $n \in \mathbb{N}_+$. We begin with the definition of V_{-1}. Since S^*_{m-1} and $T^*_{-1} V_0$ have dense ranges and (b) holds for $n = 1$, there exists a unitary operator V_{-1} such that $V_{-1} S^*_{m-1} = T^*_{-1} V_0$. This implies that $V_0 S_{m-1} = T_{-1} V_{-1}$. Let $n > 1$. Assume that $V_{-1}, \ldots, V_{-n+1}$ are already defined unitary operators on $\mathcal{H}$ such that $V_{-i+1} S_{m-i} = T_{-i} V_{-i}$ for $i \in \{1, \ldots, n-1\}$. We construct V_{-n} such that $V_{-n+1} S_{m-n} = T_{-n} V_{-n}$. It is enough to find V_{-n} so that the following holds

$$V_{-n} S^*_{m-n} \ldots S^*_{m-1} = T^*_{-n} \ldots T^*_{-1} V_0,$$

because then we get the following equality

$$V_0 S_{m-1} \ldots S_{m-n} = T_{-1} \ldots T_{-n} V_{-n}.$$

The existence of unitary V_{-n} is guaranteed by the fact that operators $S^*_{m-n} \ldots S^*_{m-1}$ and $T^*_{-n} \ldots T^*_{-1} V_0$ have dense ranges. Now, we need to show that $V_{-n+1} S_{m-n} = T_{-n} V_{-n}$. We do this by proving that $V_{-n} S^*_{m-n} = T^*_{-n} V_{-n+1}$. Let us observe that

$$(V_{-n} S^*_{m-n} - T^*_{-n} V_{-n+1}) S^*_{m-n+1} \ldots S^*_{m-1} = V_{-n} S^*_{m-n} \ldots S^*_{m-1} - T^*_{-n} \ldots T^*_{-1} V_0 = 0.$$

Since $S^*_{m-n+1} \ldots S^*_{m-1}$ has a dense range, it follows that $V_{-n} S^*_{m-n} = T^*_{-n} V_{-n+1}$.

We constructed the sequence $\{U_{n,m+n}\}_{n \in \mathbb{Z}}$ of unitary operators such that (2.1) holds. By Lemma 2.1, it is true that $S \cong_U T$, where U is of diagonal form. This completes the proof. □

It is worth noting that, if we additionally assume that S and T have quasi-invertible weights in Theorem 2.3, then we can choose any other operator $U_{k,m+k}$ instead of $U_{0,m}$ for $k \in \mathbb{Z}$ and modify the statement. In this way we get the following result.

Corollary 2.4. *Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ have quasi-invertible weights and let $m \in \mathbb{Z}$. Then the following are equivalent:*

(i) *there exists $U \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ of diagonal form such that $S \cong_U T$ and $U_{0,m} \neq 0$,*
(ii) *there exist $k \in \mathbb{Z}$ and a unitary operator $U_{k,m+k} \in \boldsymbol{B}(\mathcal{H})$ such that the following hold:*
  (a) *$\|S_{m+n+k-1} \dots S_{m+k}x\| = \|T_{n+k-1} \dots T_k U_{k,m+k}x\|$ for all $x \in \mathcal{H}$ and $n \in \mathbb{N}_+$,*
  (b) *$\|S^*_{m-n+k} \dots S^*_{m-1+k}x\| = \|T^*_{-n+k} \dots T^*_{-1+k} U_{k,m+k}x\|$ for all $x \in \mathcal{H}$ and $n \in \mathbb{N}_+$.*
(iii) *for all $k \in \mathbb{Z}$ there exists a unitary operator $U_{k,m+k} \in \boldsymbol{B}(\mathcal{H})$ such that (a) and (b) hold.*

Shields showed that each bilateral weighted shift with weights $\{a_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ is unitarily equivalent to the shift with weights $\{|a_n|\}_{n\in\mathbb{Z}}$. This fact follows from [16, Theorem 1]. Pietrzycki used it to prove that each bounded injective classical bilateral weighted shift S satisfying $S^{*n}S^n = (S^*S)^n$ for any $n \geq 2$ is quasinormal (see [14, Theorem 3.3]). Jabłoński, Jung and Stochel generalized Shields' result to the class of weighted shifts on directed trees (see [10, Theorem 3.2.1]).

Let us now focus on the bilateral case with operator weights. Guyker proved in [6, Theorem 1] that, if $S \sim \{S_n\}_{n\in\mathbb{Z}}$ has normal weights that can be divided into two certain sequences of commuting operators, then S is unitarily equivalent to the bilateral operator valued weighted shift with weights of the form $(S_n^* S_n)^{\frac{1}{2}}$. This result is similar to the one of Shields for shifts with classical weights. Pilidi provided a proof of the fact that a bilateral operator valued weighted shift with invertible weights is unitarily equivalent to a shift with positive weights (see [15, Lemma 2, Lemma 3]). Now, we prove a counterpart of this assuming only quasi-invertibility of weights and using argument based on the proofs of similar results from [13, Theorem 3.4] and [11, Theorem 2] for unilateral operator valued weighted shifts.

Theorem 2.5. *Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$ and S_n be quasi-invertible for all $n \in \mathbb{Z}$. Then $S \cong T$, where $T \sim \{T_n\}_{n\in\mathbb{Z}}$ and T_n is positive for each $n \in \mathbb{Z}$.*

Proof. It follows from the polar decomposition that for each $n \in \mathbb{Z}$ there exist unitary U_n and positive P_n such that $S_n = U_n P_n$. Let $\tilde{P}$ and $\tilde{U}$ be diagonal operators on $\ell_2(\mathbb{Z}, \mathcal{H})$ such that $\tilde{P_n} = P_{n+1}$ and $\tilde{U_n} = U_{n+1}$ for all $n \in \mathbb{Z}$. A simple calculation can prove that $S = F\tilde{U}\tilde{P}$. It is easy to verify that condition (ii) from Theorem 2.3 is satisfied with $U_{0,0} = I$ as F and $F\tilde{U}$ have unitary weights. Thus, there exists a diagonal unitary operator $V \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ such that $VF = F\tilde{U}V$. It is true that

$$S = F\tilde{U}\tilde{P} = VV^*F\tilde{U}\tilde{P} = VFV^*\tilde{P} = V(FV^*\tilde{P}V)V^*.$$

Observe that $V^*\tilde{P}V$ is a diagonal operator. This implies that $FV^*\tilde{P}V$ is a bilateral operator valued weighted shift. Since unitary equivalence preserves positivity and the elements of $V^*\tilde{P}V$ are unitarily equivalent to the elements of $\tilde{P}$, the proof is completed. □

Now, we state a useful fact which gives necessary conditions for unitary equivalence of bilateral operator valued weighted shifts given by an operator of diagonal form.

Lemma 2.6. *Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ have quasi-invertible weights. Suppose that $S \cong_U T$, where U is of diagonal form and $U_{0,k} \neq 0$ for some $k \in \mathbb{Z}$. Then $\|S_{n+k}\| = \|T_n\|$ for each $n \in \mathbb{Z}$.*

Proof. Define $V_n = U_{n,n+k}$ for all $n \in \mathbb{Z}$. By Lemma 2.1, $V_{n+1}S_{n+k} = T_nV_n$ for each $n \in \mathbb{Z}$, where operators V_n are unitary. Therefore, we see that $T_n = V_{n+1}S_{n+k}V_n^*$ and $S_{n+k} = V_{n+1}^*T_nV_n$ for each $n \in \mathbb{Z}$. This completes the proof. □

In the following proposition we provide a necessary condition of unitary equivalence given by a diagonal operator when $\mathcal{H} = \mathbb{C}^2$.

Proposition 2.7. *Suppose $\mathcal{H} = \mathbb{C}^2$. Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ have invertible and normal weights. Assume that $S \cong_U T$, where U is a diagonal operator. Then the moduli of eigenvalues of corresponding weights are equal.*

Proof. Since all weights are normal matrices, then they are diagonalizable. Therefore, it is easy to see that we can diagonalize (using a unitary operator) one of the weights in each of the shifts. Let $n \in \mathbb{Z}$. By the above we can assume that S_n and T_n are diagonal matrices. By Corollary 2.4, there exists unitary $V \in \boldsymbol{B}(\mathbb{C}^2)$ such that $\|S_nx\| = \|T_nVx\|$ for all $x \in \mathbb{C}^2$. Let us now assume that

$$S_n = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, \quad T_n = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}.$$

Taking $x = (1, 0)$ and $y = (0, 1)$, by the previous property, we get the following system of equations:

$$|s_1|^2 = |v_1t_1|^2 + |v_3t_2|^2,$$
$$|s_2|^2 = |v_2t_1|^2 + |v_4t_2|^2.$$

We see that both equations are convex combinations. Also, by Lemma 2.6, it is true that $\max\{|s_1|, |s_2|\} = \max\{|t_1|, |t_2|\}$. Since V is unitary, it must be true that

$$\begin{cases} |s_1| = |t_1| \\ |s_2| = |t_2| \end{cases} \quad \text{or} \quad \begin{cases} |s_1| = |t_2| \\ |s_2| = |t_1| \end{cases}$$

which is exactly our claim. □

We can now use the above result to determine whether two bilateral operator valued weighted shifts with invertible weights defined on $\mathbb{C}^2$ are unitarily equivalent by a diagonal operator. First, we use Theorem 2.5 to transform both shifts to their forms with positive weights. Then we compare the eigenvalues of the corresponding weights and check whether their moduli are equal. If there is at least one pair of two corresponding weights with at least one different eigenvalue, then it means that eventual unitary equivalence of the shifts cannot be given by a diagonal operator. It is important to note that moving to the form with positive weights is achieved by using a diagonal operator and, therefore, the argument presented above is correct.

Unfortunately, there is no clear dependency between spectra of weights of original shift and the one with positive weights. Another problem is that the condition provided in Proposition 2.7 is not sufficient. To see this let us consider the following

Example 2.8. Let $\mathcal{H} = \mathbb{C}^2$. We set $S_n = T_n = I$ to be identity operators on $\mathcal{H}$ for $n \in \mathbb{Z} \setminus \{0,1\}$. For $n \in \{0,1\}$ we define

$$S_n = \begin{bmatrix} s_{1,n} & 0 \\ 0 & s_{2,n} \end{bmatrix}, \quad T_n = \begin{bmatrix} t_{1,n} & 0 \\ 0 & t_{2,n} \end{bmatrix}.$$

Let us fix $|s_{1,0}| = |t_{2,0}|$ and $|s_{2,0}| = |t_{1,0}|$ and $|s_{1,0}| > |s_{2,0}| > 0$. For S_1 and T_1 we choose $|s_{1,1}| = |t_{1,1}|$ and $|s_{2,1}| = |t_{2,1}|$ and $|s_{1,1}| > |s_{2,1}| > 1$. Now, by Corollary 2.4, for S and T to be unitarily equivalent by a diagonal operator we need a unitary operator $U \in \boldsymbol{B}(\mathcal{H})$ such that:

$$\|S_0x\| = \|T_0Ux\|, \quad \|S_1S_0x\| = \|T_1T_0Ux\|, \quad x \in \mathcal{H}.$$

But the above cannot be true as the first equation implies that

$$U = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix},$$

for some $|u| = 1$ and $|v| = 1$. In this case, the second equation is not satisfied.

Li, Ji and Sun proved in [12, Theorem 2.1] that a bilateral operator valued weighted shift with weights defined on $\mathbb{C}^m$ for $m \geq 1$ is unitarily equivalent to a shift with upper triangular weights. Let us see that, under some additional assumptions, it is possible to prove that some bilateral operator valued weighted shifts are unitarily equivalent to shifts with diagonal weights.

Proposition 2.9. *Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$ be defined on $\ell_2(\mathbb{Z}, \mathbb{C}^m)$ for $m \geq 2$ with normal and commuting weights. Then there is a $D \sim \{D_n\}_{n\in\mathbb{Z}}$ such that $S \cong D$ and D_n is a diagonal matrix for each $n \in \mathbb{Z}$.*

Proof. It is a well-known fact that any set of normal matrices $\{T_a\}_{a\in A}$ which commutes with each other can be simultaneously diagonalized, i.e., there exists a unitary matrix V such that VT_aV^* is diagonal for each $a \in A$ (see [7, Theorem 1.3.19]). Now, we see that a diagonal operator consisting of operators V on its diagonal gives unitary equivalence between S and $D \sim \{D_n\}_{n\in\mathbb{Z}}$, where D_n is a diagonal matrix for every $n \in \mathbb{Z}$. □

3. UNITARY OPERATORS GIVING UNITARY EQUIVALENCE OF BILATERAL OPERATOR VALUED WEIGHTED SHIFTS

In this section we focus on investigation of unitary operators that can give unitary equivalence of bilateral operator valued weighted shifts. Some of the results concern only finite-dimensional Hilbert spaces. Note that, in regard to investigating unitary operators, Corollary 2.2 enables us to focus on number of nonzero diagonals in unitary operators.

Let U be a unitary operator acting on $\ell_2(\mathbb{Z}, \mathcal{H})$ with two nonzero diagonals. Then there exist $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \neq k_2$ and operators $U_{n,n+k_1}, U_{n,n+k_2}$ are nonzero elements from these diagonals for all $n \in \mathbb{Z}$. From now on, we identify nonzero diagonals of U with k_1 and k_2 and denote $U_n^{(1)} := U_{n,n+k_1}$, $U_n^{(2)} := U_{n,n+k_2}$ for all $n \in \mathbb{Z}$. Without loss of generality we can assume that $k_2 > k_1$. We generalize this notation to an arbitrary number of diagonals in U.

In [16, Theorem 1] one can find a proof of the fact that in the case of classical bilateral weighted shifts unitary equivalence is always given by an operator of diagonal form. Let us now see that there are bilateral operator valued weighted shifts which are unitarily equivalent, but the unitary equivalence can not be given by any operator of diagonal form.

Example 3.1. Assume $\mathcal{H} = \mathbb{C}^2$, $w = \frac{1}{2} - \frac{1}{2}\mathrm{i}$ and define

$$s_n = \begin{cases} 1, & \text{if } n = 0, \\ \frac{1}{n}, & \text{otherwise}. \end{cases} \tag{3.1}$$

Let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ have weights

$$S_n := \begin{bmatrix} s_n & s_n \\ -s_n & s_n \end{bmatrix}, \quad T_n := \begin{bmatrix} s_{n-1}w + s_{n+1}\bar{w} & s_{n-1}\bar{w} + s_{n+1}w \\ -s_{n-1}\bar{w} - s_{n+1}w & s_{n-1}w + s_{n+1}\bar{w} \end{bmatrix}$$

for $n \in \mathbb{Z}$. It is easy to see that weights of S and T are invertible and normal. Now, we construct a unitary operator with two nonzero diagonals determined by $k_1 = -1$ and $k_2 = 1$ which gives unitary equivalence of S and T. Let us define the following operators

$$A = \frac{1}{2}\begin{bmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}.$$

Both A and B are orthogonal projections onto one-dimensional subspaces. Moreover, $A + B = I$. Define $U_n^{(1)} = B$ and $U_n^{(2)} = A$ for all $n \in \mathbb{Z}$. This implies that

$$U = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & A & 0 & \ddots \\ \ddots & B & \boxed{0} & A & \ddots \\ \ddots & 0 & B & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The reader can check that U is unitary and $US = TU$.

Now, our aim is to show that it is not possible to find a unitary operator of diagonal form which would give unitary equivalence of S and T. First, one can easily verify that $\|S_n\| = \sqrt{2}|s_n|$ for each $n \in \mathbb{Z}$. Let us now compute the norms of operators T_n. We find the eigenvalues of $T_n^*T_n$ using the characteristic polynomial

$$W(\lambda) = \lambda^2 - 2\lambda(s_{n-1}^2 + s_{n+1}^2) + 4s_{n-1}^2 s_{n+1}^2.$$

The roots of W are $2s_{n+1}^2$ and $2s_{n-1}^2$, hence $\|T_n\| = \max\{\sqrt{2}|s_{n-1}|, \sqrt{2}|s_{n+1}|\}$. Now, it follows from (3.1) that

$$\begin{cases} \|S_i\| = \sqrt{2} \text{ if and only if } i \in \{-1, 0, 1\}, \\ \|T_i\| = \sqrt{2} \text{ if and only if } i \in \{-2, -1, 0, 1, 2\}. \end{cases} \tag{3.2}$$

Suppose that, contrary to our claim, S and T are unitarily equivalent by an operator of diagonal form. By Lemma 2.6, there exists $k \in \mathbb{Z}$ that $\|S_{n+k}\| = \|T_n\|$ for all $n \in \mathbb{Z}$. This contradicts (3.2).

Example 3.1 exhibits another interesting property, but before stating it let us first recall some known results. Shields showed in [16, Theorem 1] that, if two bilateral weighted shifts with complex weights are unitarily equivalent, then there exists $k \in \mathbb{Z}$ such that $|s_n| = |t_{n+k}|$ for each $n \in \mathbb{Z}$. Moreover, it follows from [13, Theorem 1] that, if two unilateral shifts S, T with quasi-invertible weights denoted by $\{S_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$, respectively, are unitarily equivalent, then the unitary equivalence is given by a diagonal operator. It follows from a similar argument as in Lemma 2.6 that $\|S_n\| = \|T_n\|$ for each $n \in \mathbb{N}$. As we can see in Example 3.1, this is not true for bilateral operator valued weighted shifts with weights defined on a Hilbert space of dimension greater than one.

Next proposition states that, if there are exactly two nonzero diagonals in a unitary operator, then both diagonals contain only partial isometries.

Proposition 3.2. *Suppose that $U \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ is a unitary operator that has exactly two nonzero diagonals and all other elements are zero operators. Then the elements on these diagonals are partial isometries such that elements in each row of U have orthogonal ranges.*

Proof. Let us fix $k = k_2 - k_1 > 0$. Note that U is unitary if and only if conditions

$$I = U_n^{(1)}(U_n^{(1)})^* + U_n^{(2)}(U_n^{(2)})^*, \tag{3.3}$$

$$I = (U_{n+k}^{(1)})^* U_{n+k}^{(1)} + (U_n^{(2)})^* U_n^{(2)}, \tag{3.4}$$

$$0 = U_{n+k}^{(1)}(U_n^{(2)})^*, \tag{3.5}$$

$$0 = (U_n^{(1)})^* U_n^{(2)}, \tag{3.6}$$

hold for all $n \in \mathbb{Z}$. Now, we can multiply (3.3) by $U_n^{(1)}$ from the right and get

$$U_n^{(1)} = U_n^{(1)}(U_n^{(1)})^* U_n^{(1)} + U_n^{(2)}(U_n^{(2)})^* U_n^{(1)}. \tag{3.7}$$

Now, by (3.6), we see that $U_n^{(2)}(U_n^{(2)})^* U_n^{(1)} = 0$ and thus, by (3.7) and the characterization of partial isometries (see [4, Exercise VIII.3.15]), $U_n^{(1)}$ is a partial isometry for all $n \in \mathbb{Z}$. It is clear that operators $U_n^{(2)}$ are also partial isometries. From (3.6) we deduce that $\mathcal{R}(U_n^{(1)})$ is orthogonal to $\mathcal{R}(U_n^{(2)})$ for all $n \in \mathbb{Z}$. This completes the proof. □

It is worth noting that, using property (3.5), we can deduce that $\mathcal{R}((U_n^{(2)})^*) \perp \mathcal{R}((U_{n+k}^{(1)})^*)$ for all $n \in \mathbb{Z}$.

The unitary operator in Example 3.1 consists only of orthogonal projections. Next example shows that, in general, sequences $\{U_n^{(1)}\}_{n\in\mathbb{Z}}$, $\{U_n^{(2)}\}_{n\in\mathbb{Z}}$ do not need to be sequences of orthogonal projections.

Example 3.3. Let $\mathcal{H} = \mathbb{C}^2$. We define the following

$$U_n^{(1)} = \begin{bmatrix} 0 & u_n^{(1)} \\ 0 & 0 \end{bmatrix}, \quad U_n^{(2)} = \begin{bmatrix} 0 & 0 \\ u_n^{(2)} & 0 \end{bmatrix},$$

where $|u_n^{(1)}| = |u_n^{(2)}| = 1$ for all $n \in \mathbb{Z}$. The reader can verify that these operators satisfy conditions (3.3) - (3.6) from the proof of Proposition 3.2 and form a unitary operator U with nonzero diagonals determined by $k_1 = -1$ and $k_2 = 1$. Now we define $S \sim \{S_n\}_{n\in\mathbb{Z}}$ in the following way

$$S_n = \begin{bmatrix} s_{1,n} & 0 \\ 0 & s_{2,n} \end{bmatrix},$$

where $s_{1,n}s_{2,n} \neq 0$ for all $n \in \mathbb{Z}$. It is easy to check that USU^* is a bilateral operator valued weighted shift and neither $U_n^{(1)}$ nor $U_n^{(2)}$ are orthogonal projections for any $n \in \mathbb{Z}$.

The next result states that there cannot be more than m nonzero diagonals which contain partial isometries in a unitary operator giving unitary equivalence of bilateral operator valued weighted shifts with weights defined on a m-dimensional Hilbert space for $m \geq 2$.

Proposition 3.4. *Let $\mathcal{H}$ be a m-dimensional Hilbert space for $m \geq 2$ and let $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$ have quasi-invertible weights. Assume $U \in \boldsymbol{B}(\ell_2(\mathbb{Z}, \mathcal{H}))$ is unitary and its matrix representation consists of partial isometries only. If $US = TU$, then U has at most m nonzero diagonals and all other elements of U are zero operators.*

Proof. By the fact that $UU^* = I$ we get that $\sum_{j\in\mathbb{Z}} U_n^{(j)}(U_n^{(j)})^* = I$ for all $n \in \mathbb{Z}$. Fix $n \in \mathbb{Z}$. It follows from [4, Exercise VIII.3.15] that $P_n^{(j)} := U_n^{(j)}(U_n^{(j)})^*$ is an orthogonal projection for all $j \in \mathbb{Z}$. It is a well-known fact that, if $\sum_{j\in\mathbb{Z}} P_n^{(j)}$ is an orthogonal projection, then $\mathcal{R}(P_n^{(i)}) \perp \mathcal{R}(P_n^{(j)})$ for all $i, j \in \mathbb{Z}$ such that $i \neq j$. The rest follows directly from the fact that $\mathcal{H}$ is m-dimensional and from Corollary 2.2. □

Pilidi stated in [15, Theorem 5] that in the case $\dim \mathcal{H} = 2$ unitary equivalence is given by an operator with at most two nonzero diagonals and all operator entries in matrix representation of this unitary operator are partial isometries.

Unfortunately, this result was not provided with proof and it is difficult to see how it can be verified. If it was true, then the problem of characterization of unitary equivalence when $\dim \mathcal{H} = 2$ would be solved.

4. FURTHER REMARKS

It is an open question, whether for $S \sim \{S_n\}_{n\in\mathbb{Z}}$, $T \sim \{T_n\}_{n\in\mathbb{Z}}$, where $\mathcal{H}$ is a m-dimensional Hilbert space, $S \cong T$ implies that there exists U that has at most m nonzero diagonals with all other elements of U being zero operators such that $S \cong_U T$ for $m \geq 2$. If one proves that any unitary equivalence of bilateral operator valued weighted shifts with weights defined on finite-dimensional Hilbert space can be given by an operator consisting only of partial isometries, then Proposition 3.4 gives the positive answer.

Another interesting problem for further investigation, which comes up naturally, is the problem of characterization of unitary equivalence of bilateral shifts with weights defined on an arbitrary Hilbert space. Corollary 2.4 gives the characterization of unitary equivalence given only by an operator of diagonal form. Example 3.1 shows that there is a rich class of unitary operators in $\ell_2(\mathbb{Z}, \mathbb{C}^m)$ for $m \geq 2$ which are not of diagonal form and can give unitary equivalence of bilateral weighted shifts.

Acknowledgements

The author would like to thank Zenon Jabłoński for insightful discussions concerning the subject of the paper. The author also thanks the referee for thorough review and comments which improved the final text.

The research was supported by Grant No. PSP: K/DSC/004434.

REFERENCES

[1] A. Anand, S. Chavan, Z.J. Jabłoński, J. Stochel, *Complete systems of unitary invariants for some classes of 2-isometries*, Banach J. Math. Anal. **13** (2019) 2, 359–385.

[2] A. Athavale, *On completely hyperexpansive operators*, Proc. Amer. Math. Soc. **124** (1996), 3745–3752.

[3] A. Bourhim, C.E. Chidume, *The single-valued extension property for bilateral operator weighted shifts*, Proc. Amer. Math. Soc., **133** (2004), 485–491.

[4] J.B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1990.

[5] G.P. Gehér, *Bilateral weighted shift operators similar to normal operators*, Oper. Matrices **10** (2016) 2, 419–423.

[6] J. Guyker, *On reducing subspaces of normally weighted bilateral shifts*, Houston J. Math. **11** (1985) 4, 515–521.

[7] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[8] N. Ivanovski, *Similiarity and quasisimiliarity of bilateral operator valued weighted shifts*, Mat. Bilten **17** (1993), 33–37.

[9] Z.J. Jabłoński, *Hyperexpansive operator valued unilateral weighted shifts*, Glasg. Math. J. **46** (2004), 405–416.

[10] Z.J. Jabłoński, I.B. Jung, J. Stochel, *Weighted Shifts on Directed Trees*, Mem. Amer. Math. Soc. **216** (2012) 1017.

[11] A. Lambert, *Unitary equivalence and reducibility of invertibly weighted shifts*, Bull. Aust. Math. Soc. **5** (1971), 157–173.

[12] J.X. Li, Y.Q. Ji, S.L. Sun, *The essential spectrum and Banach reducibility of operator weighted shifts*, Acta Math. Sin., English Series, **17** (2001) 3, 413–424.

[13] M. Orovčanec, *Unitary equivalence of unilateral operator valued weighted shifts with quasi-invertible weights*, Mat. Bilten **17** (1993), 45–50.

[14] P. Pietrzycki, *The single equality $A^{*n}A^n = (A^*A)^n$ does not imply the quasinormality of weighted shifts on rootless directed trees*, J. Math. Anal. Appl. **435** (2016), 338–348.

[15] V.S. Pilidi, *On unitary equivalence of multiple weighted shift operators*, Teor. Funkts. Funkts. Anal. Prilozh. **34** (1980), 96–103 [in Russian].

[16] A. Shields, *Weighted shift operators, analytic function theory*, Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc., Providence, R. I. (1974), 49–128.

Jakub Kośmider
jakub.kosmider@im.uj.edu.pl

Instytut Matematyki
Uniwersytet Jagielloński
ul. Łojasiewicza 6, 30-348 Kraków, Poland

Received: March 4, 2019.
Revised: March 19, 2019.
Accepted: March 20, 2019.