Opuscula Math. 43, no. 3 (2023), 409–428
https://doi.org/10.7494/OpMath.2023.43.3.409 Opuscula Mathematica

EXISTENCE AND ASYMPTOTIC STABILITY FOR GENERALIZED ELASTICITY EQUATION WITH VARIABLE EXPONENT

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Communicated by P.A. Cojuhari

Abstract. In this paper we propose a new mathematical model describing the deformations of an isotropic nonlinear elastic body with variable exponent in dynamic regime. We assume that the stress tensor $\sigma^{p(\cdot)}$ has the form

$$\sigma^{p(\cdot)}(u) = \left(2\mu + |d(u)|^{p(\cdot)-2}\right) d(u) + \lambda Tr\left(d(u)\right) I_3,$$

where u is the displacement field, μ, λ are the given coefficients $d(\cdot)$ and I_3 are the deformation tensor and the unit tensor, respectively. By using the Faedo–Galerkin techniques and a compactness result we prove the existence of the weak solutions, then we study the asymptotic behaviour stability of the solutions.

Keywords: asymptotic stability, variable exponent Lebesgue and Sobolev spaces, generalized elasticity equation.

Mathematics Subject Classification: 35B37, 35L55, 35L70, 46E30.

1. INTRODUCTION

In recent years, the study of the problems with variable exponent is a new and important topic. These problems are motivated by the applications of electrofluids, non-Newtonian fluid dynamics, applications related to image processing, Poisson equation and elasticity equations see [5, 6, 11, 13, 25, 28]. Moreover, the variable exponent spaces are involved in studies that provide other types of applications, like the contact mechanics [4].

Recently, the parabolic and elliptic equations which involve variable exponents have been intensively studied in the literature. For the questions of the existence and the uniqueness, we mention: Antontsev and Shmarev in [2] proved the existence and uniqueness of weak solutions of the Dirichlet problem for the nonlinear degenerate parabolic equation. In the article [1] Antontsev proved the existence and blow up for the weak solution of a wave equation with $p(\cdot;t)$-Laplacian and damping terms. Boureanu in [3] studied the existence of solutions for a class of quasilinear elliptic equations involving the anisotropic $p(\cdot)$-Laplace operator, on a bounded domain with smooth boundary. Stegliński in the work [27], used the Dual Fountain Theorem to obtain some infinite existence for many solutions of local and nonlocal elliptic equations with a variable exponent. Simsen *et al.* [26], studied the asymptotic behavior of coupled systems of $p(\cdot)$-Laplacian differential inclusions; they obtained that the generalized semiflow generated by the coupled system has a global attractor, and proved the continuity of the solutions with respect to initial conditions. Otmani *et al.* in [22], they focus on the numerical side of the problem of the parabolic equations with variable exponent. A comprehensive analysis of nonlinear partial differential equations with variable exponent can be found in [24].

For the stability of solutions of the hyperbolic problems with nonlinearities of variable-exponent type, there are some interesting works, for instance, Messaoudi and Talahmeh [19], proved the finite-time blow up of solutions of the following equation

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(|\nabla u|^{r(\cdot)-2}\nabla u\right) + \alpha\left|\frac{\partial u}{\partial t}\right|^{m(\cdot)-1}\frac{\partial u}{\partial t} = \beta|u|^{p(\cdot)-1}u. \tag{1.1}$$

Messaoudi *et al.* [20], studied (1.1) with $\beta = 0$ and proved decay estimates for the solution under suitable assumptions on the variable exponents $m(\cdot)$, $r(\cdot)$ and the initial data. Ghegal *et al.* [12], considered the problem (1.1) with $r(\cdot) = 2$, $m(\cdot) = l(\cdot) - 1$, and $\alpha = \beta = 1$, in a bounded domain, with Dirichlet-boundary conditions, and proved a global existence and a stability result. Lian *et al.* [16] were interested in a fourth-order wave equation with strong and weak damping terms. They obtained the local solution, the global existence, asymptotic behavior, and blow-up of solutions for the arbitrarily positive initial energy case. Rahmoune in [23], considered a semi-linear generalized hyperbolic boundary value problem associated to the linear elastic equations with general damping term and nonlinearities of variable exponent type, where he proved the existence, uniqueness and stability of weak solutions.

On the other hand, several authors studied the system of elasticity with laws of particular behavior. For example, in [8] Duvant and Lions studied the existence and uniqueness of the classical and frictional problems of elasticity in the boundary domain. Oden in [21], proved the existence theorems for a class of problems in nonlinear elasticity. Lagnese in [15], proved some uniform stability results of elasticity systems with linear dissipative term. In the paper [7] Dilmi *et al.* investigated the asymptotic behavior of an elasticity problem with a nonlinear dissipative term in a bidimensional thin domain Ω^ε and proved some convergence results when the thickness tends to zero.

In this work we generalize a hyperbolic boundary value problem associated to the nonlinear elastic equations involving variable exponents with variable exponents nonlinear source and linear dissipative term. More precisely, we are interested to study the existence and stability behavior of the solutions of the following initial-boundary value problem.

Let Ω is a bounded domain in $\mathbb{R}^3$, the boundary $\partial\Omega$ of Ω is assumed to be regular and is composed of two parts $\partial\Omega_1$ and $\partial\Omega_2$. For $x \in \Omega$ and $t \in]0, T[$, we denote $u(x, t)$ to be the displacement field, we consider the law of the nonlinear elasticity behavior with the variable exponents given by

$$\sigma_{ij}^{p(\cdot)}(u) = \left(2\mu + |d(u)|^{p(\cdot)-2}\right) d_{ij}(u) + \lambda \sum_{k=1}^{3} d_{kk}(u)\delta_{ij}, \quad 1 \le i, j \le 3,$$

where

$$d_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Here δ_{ij} is the Krönecker symbol, λ, μ are the Lamé constants and $d_{ij}(\cdot)$ the deformation tensor.

The equation which governs the deformations of an isotropic nonlinear elastic body with variable exponent and a nonlinear source and a linear dissipative term in dynamic regime is the following

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(\sigma^{p(\cdot)}(u)\right) + \alpha|u|^{p(\cdot)-2}u + \beta\frac{\partial u}{\partial t} = f, \quad \text{in } \Omega \times]0, T[\,, \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^3$, f represents a force density, $p(\cdot)$ is the variable exponent such that $2 \le p(\cdot)$ and α, $\beta \in \mathbb{R}_+$.

To describe the boundary conditions we use the usual notation

$$u_n = u.n, \quad u_\tau = u - u.n, \quad \sigma_n^{p(\cdot)} = \left(\sigma^{p(\cdot)}.n\right).n, \quad \sigma_\tau^{p(\cdot)} = \sigma^{p(\cdot)}.n - \left(\sigma_n^{p(\cdot)}\right).n,$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal to $\partial\Omega$.

The displacement is known on $\partial\Omega_1 \times]0, T[$:

$$u(x, t) = 0 \quad \text{on} \quad \partial\Omega_1 \times]0, T[. \tag{1.3}$$

On $\partial\Omega_2$ the stress tensor satisfies the following condition

$$\sigma^{p(\cdot)}(u).n = 0 \quad \text{on} \quad \partial\Omega_2 \times]0, T[. \tag{1.4}$$

The problem consists in finding u satisfying (1.2)–(1.4) and the following initial conditions:

$$u(x, 0) = \vartheta_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = \vartheta_1(x), \quad \forall x \in \Omega. \tag{1.5}$$

The paper is organized as follows: In Section 2, we introduce some basic properties of the variable exponent spaces and notations, also we give the weak formulation of the problem. In Section 3, we prove the existence of the weak solutions by Faedo–Galerkin methods. Finally, the global existence and the stability of solution are established in Section 4.

2. PRELIMINARIES AND WEAK FORMULATION

In this section, we present some materials needed in the proof of our results.

For an open $\Omega \subset \mathbb{R}^n$, let $p : \Omega \to [1, +\infty[$ a measurable function, called the variable exponent. We say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|\log(|x-y|)|}, \quad \text{for all } x, y \in \bar{\Omega} \text{ such that } |x-y| < \frac{1}{2}, \tag{2.1}$$

with a possible different constant C. We denote

$$C_+\left(\bar{\Omega}\right) = \{p : p \text{ is log-Hölder continuous function with } \\ 1 < p^- \leq p(x) \leq p^+ < \infty \text{ for all } x \in \bar{\Omega}\},$$

where

$$p^- = \operatorname*{ess\,inf}_{x \in \Omega} (p(x)), \quad p^+ = \operatorname*{ess\,sup}_{x \in \Omega} (p(x)).$$

We define the variable exponent Lebesgue space for $p \in C_+\left(\bar{\Omega}\right)$ by

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} : \ u \text{ is measurable}, \int\limits_{\Omega} |u|^{p(x)} dx < \infty\right\},$$

the space $L^{p(\cdot)}(\Omega)$ under the norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{q(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

Now, we define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\Omega)^3 = \left\{u \in L^{p(\cdot)}(\Omega)^3 : \nabla u \in L^{p(\cdot)}(\Omega)^{3\times 3}\right\}.$$

The space $W^{1,p(\cdot)}(\Omega)^3$ with the norm

$$\|u\|_{W^{1,p(\cdot)}} = \|u\|_{L^{p(\cdot)}(\Omega)^3} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)^{3\times 3}},$$

is a Banach space.

We denote by $W_0^{1,p(\cdot)}(\Omega)^3$ the closure of $\mathcal{D}(\Omega)^3$ in $W^{1,p(\cdot)}(\Omega)^3$, and $W^{-1,q(\cdot)}(\Omega)^3$ is the dual of the space $W_0^{1,p(\cdot)}(\Omega)^3$. Additionally to the Lebesgue and Sobolev spaces and their generalizations, we introduce the closed convex set

$$K^{p(\cdot)} = \left\{v \in W^{1,p(\cdot)}(\Omega)^3 : v = 0 \text{ on } \partial\Omega_1\right\},$$

to get a weak formulation, and for the treatment of the generalized elasticity problems with variable exponent, we will need the following results.

Proposition 2.1 ([9,10])**.** *Let* u_n*,* $u \in L^{p(\cdot)}(\Omega)$ *and* $p^+ < +\infty$*, then:*

(1) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ *(resp.,* $= 1, > 1$*)* $\Longleftrightarrow \int_\Omega |u|^{p(x)}dx < 1$ *(resp.,* $= 1$*,* > 1*),*

(2) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \int_\Omega |u|^{p(x)}dx \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$*,*

(3) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \int_\Omega |u|^{p(x)}dx \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$*,*

(4) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to 0 \Longleftrightarrow \int_\Omega |u_n|^{p(x)}\, dx \to 0$*.*

We will also need the following inequalities.

(i) Hölder's inequality [9]**:** For any functions $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have

$$\int\limits_\Omega |u|.\, |v|\, dx \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)}, \tag{2.2}$$

where

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

(ii) Poincaré's inequality [6]: For every $u \in K^{p(\cdot)}$ there exists a constant $c^* > 0$ depending only on the dimension of Ω, p^- and p^+ such that

$$\|u\|_{L^{p(\cdot)}(\Omega)^3} \leq c^* \mathrm{diam}(\Omega)\, \|\nabla u\|_{L^{p(\cdot)}(\Omega)^{3\times 3}}\,, \tag{2.3}$$

for all $u \in K^{p(\cdot)}$.

(iii) Korn's inequality [6]**:** Let $p \in C_+(\Omega)$ with $1 < p^- \leq p^+ < \infty$. Then

$$C_K\, \|\nabla u\|_{L^{p(\cdot)}(\Omega)^{3\times 3}} \leq \|d(u)\|_{L^{p(\cdot)}(\Omega)^{3\times 3}}\,, \tag{2.4}$$

for all $u \in K^{p(\cdot)}$.

(iv) Young's inequality [9]**:** For all $u, v \in \mathbb{R}_+$, we have

$$u.v \leq \frac{(u)^{p(\cdot)}}{p^-} + \frac{(v)^{q(\cdot)}}{q^-}. \tag{2.5}$$

If the condition (2.1) is fulfilled, and Ω has a finite measure and $q(\cdot)$, $p(\cdot)$ are variable exponents so that $q(\cdot) \leq p(\cdot)$ almost everywhere in Ω, then the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous.

If $p : \Omega \longrightarrow [1, +\infty[$ is a measurable function and $p^\star > \mathrm{esssup}_{x\in\Omega} p(x)$ with $p^\star \leq \frac{2n}{n-2}$, then the embedding $W^{1,2}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact (see [6]).

For convenience, we note

$$\frac{1}{p(\cdot)} \|v\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)} = \int\limits_\Omega \frac{1}{p(x)}\, |v(x)|^{p(x)}\, dx,$$

and

$$\|v\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)} = \int\limits_\Omega |v(x)|^{p(x)}\, dx,$$

for all $v \in L^{p(\cdot)}(\Omega)$.

Now, we derive the variational formulation of the problem.

By multiplying equation (1.2) by a test-function φ, then integrating over Ω and using the Green formula, we get the following variational formulation:

$$\begin{aligned}
&\text{Find } u \in K^{p(\cdot)}, \forall t \in]0,T[\text{ such that}\\
&\left(\frac{\partial^2 u}{\partial t^2}, \varphi\right) + a_{p(\cdot)}(u,\varphi) + \alpha\left(|u|^{p(\cdot)-2}u, \varphi\right) + \beta\left(\frac{\partial u}{\partial t}, \varphi\right) = (f,\varphi), \quad \forall \varphi \in K^{p(\cdot)},\\
&u(x,0) = \vartheta_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \vartheta_1(x),
\end{aligned} \tag{2.6}$$

where

$$a_{p(\cdot)}(u,\varphi) = \int_\Omega \left(2\mu + |d(u)|^{p(\cdot)-2}\right) d(u) : d(\varphi)dx + \lambda \int_\Omega \operatorname{div}(u)\operatorname{div}(\varphi)dx,$$

with

$$d(u) : d(\varphi) = \sum_{i,j=1}^{3} d_{ij}(u).d_{ij}(\varphi).$$

Also, we denote by $\mathcal{A}$ the nonlinear operator

$$\begin{aligned}
\mathcal{A} : W_0^{1,p(\cdot)}(\Omega)^3 &\longrightarrow W^{-1,q(\cdot)}(\Omega)^3,\\
u &\longrightarrow \mathcal{A}(u),
\end{aligned}$$

where

$$(\mathcal{A}(u), v) = a_{p(\cdot)}(u,v), \quad \text{for all } v \in W_0^{1,p(\cdot)}(\Omega)^3.$$

3. EXISTENCE OF A WEAK SOLUTION

In this part, we are interested the local existence of the solution for the problem (1.2)–(1.5).

Theorem 3.1. *Under the assumptions*

$$\begin{aligned}
&f, \frac{\partial f}{\partial t} \in L^{q(\cdot)}\left(0,T, L^{q(\cdot)}(\Omega)^3\right),\\
&\vartheta_0 \in W^{1,p(\cdot)}(\Omega)^3, \quad \vartheta_1 \in L^2(\Omega)^3,
\end{aligned} \tag{3.1}$$

there exists a weak solution u of (2.6) such that

$$u \in L^\infty\left(0,T, W^{1,p(\cdot)}(\Omega)^3\right), \tag{3.2}$$

$$\frac{\partial u}{\partial t} \in L^\infty\left(0,T, L^2(\Omega)^3\right). \tag{3.3}$$

Proof. We use the standard Faedo–Galerkin method to prove our result.

We introduce a sequence of functions (v_j) having the following properties:

(i) for every $j \in \{1, \ldots, m\}$, $v_j \in K^{p(\cdot)}$,
(ii) the family $\{v_1, v_2, \ldots, v_m\}$ is linearly independent,
(iii) the space $K_m = [v_j]_{1\le j\le m}$ generated by the family $\{v_1, v_2, \ldots, v_m\}$ is dense in $K^{p(\cdot)}$.

Let $u_m = u_m(t)$ be an approached solution of the problem (1.2)–(1.5) such that

$$u_m(t) = \sum_{i=1}^{m} \eta_{jm}(t) v_j, \quad m = 1, 2, \ldots,$$

verifies the system of equations

$$\begin{aligned}&\left(\frac{\partial^2 u_m}{\partial t^2}, v_j\right) + a_{p(\cdot)}\left(u_m, v_j\right) + \alpha\left(\left|u_m\right|^{p(\cdot)-2} u_m, v_j\right) + \beta\left(\frac{\partial u_m}{\partial t}, v_j\right)\\ &= (f, v_j), \quad 1 \le j \le m,\end{aligned} \tag{3.4}$$

which is a nonlinear system of ordinary deferential equations and will be completed by the following initial conditions:

$$u_m(x,0) = \vartheta_{0m} = \sum_{i=1}^{m} \omega_{jm} v_j \to \vartheta_0 \quad \text{when} \quad m \to \infty \text{ in } W^{1,p(\cdot)}(\Omega)^3, \tag{3.5}$$

and

$$\frac{\partial u_m}{\partial t}(x,0) = \vartheta_{1m} = \sum_{i=1}^{m} \chi_{jm} v_j \to \vartheta_1 \quad \text{when} \quad m \to \infty \text{ in } L^2(\Omega)^3. \tag{3.6}$$

From the general results on systems of differential equations, we are assured of the existence of a solution of (3.4) (note that $\det(v_i, v_j) \neq 0$) thanks to the linear independence of $v_1, v_2, \ldots, v_m$ in an interval $[0, t_m]$, (see [17]).

Multiplying the equation (3.4) by $\eta'_{jm}(t)$ and performing the summation over $j = 1$ to m, we find

$$\begin{aligned}&\left(\frac{\partial^2 u_m}{\partial t^2}, \frac{\partial u_m}{\partial t}\right) + a_{p(x)}\left(u_m, \frac{\partial u_m}{\partial t}\right) + \alpha\left(|u_m|^{p(\cdot)-2} u_m, \frac{\partial u_m}{\partial t}\right)\\ &+ \beta\left(\frac{\partial u_m}{\partial t}, \frac{\partial u_m}{\partial t}\right) = \left(f, \frac{\partial u_m}{\partial t}\right).\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\left(|u_m|^{p(\cdot)-2} u_m, \frac{\partial u_m}{\partial t}\right) = \frac{1}{p(\cdot)} \frac{d}{dt} \|u_m(t)\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}, \tag{3.8}$$

also

$$\begin{aligned}a_{p(\cdot)}\left(u_m, \frac{\partial u_m}{\partial t}\right) = \frac{d}{dt}\Bigg[&\frac{1}{p(\cdot)} \|d(u_m(t))\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^{3\times 3}} + \mu \|d(u_m(t))\|^2_{L^2(\Omega)^{3\times 3}}\\ &+ \frac{\lambda}{2} \|\operatorname{div}(u_m(t))\|^2_{L^2(\Omega)}\Bigg].\end{aligned} \tag{3.9}$$

By using Eqs. (3.8)–(3.9) in Eq. (3.7), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u_m(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}+\frac{d}{dt}\left[\frac{1}{p(\cdot)}\left\|d\left(u_m(t)\right)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^{3\times 3}}\right.\\
&\quad\left.+\mu\left\|d\left(u_m(t)\right)\right\|^2_{L^2(\Omega)^{3\times 3}}+\frac{\lambda}{2}\left\|\operatorname{div}\left(u_m(t)\right)\right\|^2_{L^2(\Omega)}\right]\\
&\quad+\frac{\alpha}{p(\cdot)}\frac{d}{dt}\left\|u_m(t)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\beta\left\|\frac{\partial u_m(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}\\
&=\left(f,\frac{\partial u_m(t)}{\partial t}\right).
\end{aligned}
$$

By integrating the last equation on $]0,t[$ and applying Hölder and Young inequalities, we deduce

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}+\mu\left\|d\left(u_m(t)\right)\right\|^2_{L^2(\Omega)^{3\times 3}}+\frac{1}{p(\cdot)}\left\|d\left(u_m(t)\right)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^{3\times 3}}\\
&+\frac{\alpha}{p(\cdot)}\left\|u_m(t)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\beta\int_0^t\left\|\frac{\partial u_m(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}\\
&\leq\left\|\vartheta_{1m}\right\|^2_{L^2(\Omega)^3}+\frac{1}{p^-}\left\|\vartheta_{0m}\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}+\left(\frac{2\mu+\lambda}{2}\right)\left\|\vartheta_{0m}\right\|^2_{W^{1,2}(\Omega)^3}\\
&\quad+\frac{\alpha}{p^-}\left\|\vartheta_{0m}\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\int_0^t\left\|u_m(s)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}ds+\frac{\alpha}{2p^+}\left\|u_m(s)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}\\
&\quad+\left(\frac{2p^+}{\alpha}\right)^{\frac{q^+}{p^-}}\left\|f(t)\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}+\int_0^t\left\|\frac{\partial f(s)}{\partial t}\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}ds\\
&\quad+\left\|f(0)\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}+\left\|\vartheta_{0m}\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}.
\end{aligned}
\tag{3.10}
$$

Now, using Korn's inequality (2.4) and $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)$ we have

$$
\begin{gathered}
\frac{C_K}{p(\cdot)}\left\|u_m(t)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}\leq\frac{1}{p(\cdot)}\left\|d\left(u_m(t)\right)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^{3\times 3}},\\
\left\|u_m(s)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}\leq c_{p^+}\left\|u_m(s)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}.
\end{gathered}
$$

Then the inequality (3.10) will be

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}+\frac{\mu C_K}{2}\|u_m(t)\|^2_{W^{1,2}(\Omega)^3}+\frac{C_K}{p^+}\|u_m(t)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}\\
&+\frac{\alpha}{2p^+}\|u_m(t)\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\beta\int_0^t\left\|\frac{\partial u_m(s)}{\partial t}\right\|^2_{L^2(\Omega)^3}ds\\
&\leq c_{p^+}\int_0^t\|u_m(s)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}ds+\int_0^t\left\|\frac{\partial f(s)}{\partial t}\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}ds+\|f(0)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}\\
&\quad+\left(\frac{2p^+}{\alpha}\right)^{\frac{q^+}{p^-}}\|f(t)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}+\|\vartheta_{1m}\|^2_{L^2(\Omega)^3}\\
&\quad+\left(1+\frac{1+\alpha c_{p^+}}{p^-}+\frac{2\mu+\lambda}{2}\right)\|\vartheta_{0m}\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3},
\end{aligned}
$$

as

$$
\begin{aligned}
&\int_0^t\left\|\frac{\partial f(s)}{\partial t}\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}ds+\left(\frac{2p^+}{\alpha}\right)^{\frac{q^+}{p^-}}\|f(t)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}+\|f(0)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}\\
&+\left(1+\frac{1+\alpha c_{p^+}}{p^-}+\frac{2\mu+\lambda}{2}\right)\|\vartheta_{0m}\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}+\|\vartheta_{1m}\|^2_{L^2(\Omega)^3}\leq C,
\end{aligned}
$$

for all $m\in\mathbb{N}$, where C is a constant independent of m. So, we get

$$
\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u_m(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}+\frac{\mu C_K}{2}\|u_m(t)\|^2_{W^{1,2}(\Omega)^3}+\frac{C_K}{p^+}\|u_m(t)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}\\
&+\frac{\alpha}{2p^+}\|u_m(t)\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\beta\int_0^t\left\|\frac{\partial u_m(s)}{\partial t}\right\|^2_{L^2(\Omega)^3}ds\\
&\leq C+c_{p^+}\int_0^t\|u_m(s)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}ds.
\end{aligned}
\tag{3.11}
$$

By using the Gronwall inequality, we obtain

$$
\|u_m(t)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)}\leq C_T. \tag{3.12}
$$

Therefore, (3.11) gives

$$
\left\|\frac{\partial u_m(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}+\|u_m(t)\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}\leq C'. \tag{3.13}
$$

The estimates (3.12) and (3.13) imply

$$u_m \text{ bounded in } L^\infty\left(0,T;W^{1,p(\cdot)}(\Omega)^3\right),$$
$$\frac{\partial u_m}{\partial t} \text{ bounded in } L^\infty\left(0,T;L^2(\Omega)^3\right),$$

from this, we deduce that we can extract a subsequence u_m such that

$$\begin{aligned} u_m &\rightharpoonup u && \text{in } L^\infty\left(0,T;W^{1,p(\cdot)}(\Omega)^3\right), \\ \frac{\partial u_m}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{in } L^\infty\left(0,T;L^2(\Omega)^3\right), \\ |u_m|^{p(\cdot)-2}u_m &\rightharpoonup \chi && \text{in } L^\infty\left(0,T;L^{q(\cdot)}(\Omega)^3\right), \\ \mathcal{A}(u_m) &\rightharpoonup \theta && \text{in } L^\infty\left(0,T;W^{-1,q(\cdot)}(\Omega)^3\right). \end{aligned} \tag{3.14}$$

We have the sequences u_m, $\frac{\partial u_m}{\partial t}$ are bounded in $L^2\left(0,T;L^2(\Omega)^3\right) = L^2(Q)$, then by the compactness lemma of Lions [17], we can deduce

$$u_m \overset{\text{strongly}}{\longrightarrow} u \quad \text{in } L^2\left(0,T;L^2(\Omega)^3\right).$$

On the other hand, we have

$$\int\limits_\Omega \left||u_m|^{p(x)-2}u_m\right|^{q(x)} dx = \int\limits_\Omega |u_m|^{p(x)}\, dx \le C.$$

So $|u_m|^{p(\cdot)-2}u_m$ is bounded in $L^\infty\left(0,T;L^{q(\cdot)}(\Omega)^3\right)$.

As $u_m \overset{\text{strongly}}{\longrightarrow} u$ in $L^2\left(0,T;L^2(\Omega)^3\right)$ we get

$$|u_m|^{p(\cdot)-2}u_m \rightharpoonup \chi = |u|^{p(\cdot)-2}u \quad \text{in} \quad L^\infty\left(0,T;L^{q(\cdot)}(\Omega)^3\right). \tag{3.15}$$

As the operator $\mathcal{A}(\cdot)$ is bounded, monotone and hemicontinuous, we can prove that (see, for example [18])

$$0 \le \int\limits_0^t \left(\theta(s) - \mathcal{A}(u(s)), w(s)\right) ds, \quad \forall w \in L^2\left(0,T;W_0^{1,p(\cdot)}(\Omega)^3\right).$$

From this we conclude that $\theta = \mathcal{A}(u)$.

Now, let j be fixed and $l > j$. Then, using (3.4), we get

$$\begin{aligned} &\left(\frac{\partial^2 u_l}{\partial t^2}, v_j\right) + a_{p(\cdot)}(u_l, v_j) + \alpha\left(|u_l|^{p(\cdot)-2}u_l, v_j\right) + \beta\left(\frac{\partial u_l}{\partial t}, v_j\right) \\ &= (f, v_j), \quad 1 \le j \le l. \end{aligned} \tag{3.16}$$

From (3.14) and (3.15), it results that

$$\begin{aligned} \left(|u_l|^{p(\cdot)-2}\,u_l, v_j\right) &\overset{\text{weak star}}{\rightharpoonup} \left(|u|^{p(\cdot)-2}u, v_j\right) \quad \text{in} \quad L^\infty(0,T), \\ \left(\frac{\partial u_l}{\partial t}, v_j\right) &\overset{\text{weak star}}{\rightharpoonup} \left(\frac{\partial u}{\partial t}, v_j\right) \qquad\quad \text{in} \quad L^2(0,T), \\ a_{p(\cdot)}\left(u_l, v_j\right) &\overset{\text{weak star}}{\rightharpoonup} a_{p(\cdot)}\left(u, v_j\right) \qquad\quad \text{in} \quad L^\infty(0,T), \end{aligned}$$

therefore

$$\left(\frac{\partial^2 u_l}{\partial t^2}, v_j\right) \rightharpoonup \left(\frac{\partial^2 u}{\partial t^2}, v_j\right) \quad \text{in} \quad \mathcal{D}'(0,T).$$

Then (3.16) as $l \longrightarrow \infty$ takes the form

$$\left(\frac{\partial^2 u}{\partial t^2}, v_j\right) + a_{p(\cdot)}(u, v_j) + \alpha\left(|u|^{p(\cdot)-2}u, v_j\right) + \beta\left(\frac{\partial u}{\partial t}, v_j\right) = (f, v_j).$$

Now, using the density of K_m in $K^{p(\cdot)}$, we obtain

$$\left(\frac{\partial^2 u}{\partial t^2}, \varphi\right) + a_{p(\cdot)}\left(u, \varphi\right) + \alpha\left(|u|^{p(\cdot)-2}u, \varphi\right) + \beta\left(\frac{\partial u}{\partial t}, \varphi\right) = (f, \varphi), \quad \forall \varphi \in K^{p(\cdot)}.$$

Thus, u satisfies (1.2)–(1.4).

To handle the initial conditions, we note that

$$\begin{aligned} &u \in L^2\left(0, T; W^{1,p(\cdot)}(\Omega)^3\right), \\ &\frac{\partial u}{\partial t} \in L^2\left(0, T; L^2\left(\Omega\right)^3\right). \end{aligned}$$

Thus, using Lion's Lemma [17] and Eq. (3.5), we easily obtain

$$u(x,0) \rightharpoonup \vartheta_0(x).$$

For the second condition, we have

$$\begin{aligned} &\int_0^T \left|\left(\frac{\partial^2 u(s)}{\partial t^2}, \varphi(s)\right)\right| ds \\ &\leq \int_0^T \left|a_{p(\cdot)}\left(u\left(s\right), \varphi(s)\right)\right| ds + \alpha \int_0^T \left|\left(|u(s)|^{p(\cdot)-2}u(s), \varphi(s)\right)\right| ds \\ &\quad + \beta \int_0^T \left|\left(\frac{\partial u\left(s\right)}{\partial t}, \varphi(s)\right)\right| ds + \int_0^T \left(f(s), \varphi(s)\right) ds, \quad \forall \varphi(s) \in L^2\left(0, T; K^{p(\cdot)}\right). \end{aligned}$$

This implies

$$\int_0^T \left| \left(\frac{\partial^2 u(s)}{\partial t^2}, \varphi(s) \right) \right| ds \leq c \int_0^T \left(\|u(s)\|_{W^{1,p(\cdot)}(\Omega)^3} + \left\| \frac{\partial u(s)}{\partial t} \right\|_{L^2(\Omega)^3} \right.$$

$$\left. + \|f(s)\|_{L^{q(\cdot)}(\Omega)^3} \right) \|\varphi(s)\|_{W^{1,p(\cdot)}(\Omega)^3} \, ds,$$

$$\leq c \|\varphi\|_{L^2\left(0,T;W^{1,p(\cdot)}(\Omega)^3\right)}, \quad \forall \varphi(s) \in L^2\left(0,T;K^{p(\cdot)}\right),$$

it means that

$$\frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T;W^{-1,q(\cdot)}(\Omega)^3\right).$$

Recalling that $\frac{\partial u}{\partial t} \in L^2\left(0,T;L^2(\Omega)^3\right)$, we obtain

$$\frac{\partial u}{\partial t} \in C\left(0,T;W^{-1,q(\cdot)}(\Omega)^3\right).$$

So, $\frac{\partial u_m(x,0)}{\partial t}$ makes sense and

$$\frac{\partial u_m(x,0)}{\partial t} \rightharpoonup \frac{\partial u(x,0)}{\partial t} \quad \text{in } W^{-1,q(\cdot)}(\Omega)^3.$$

But

$$\frac{\partial u_m(x,0)}{\partial t} \to \vartheta_1(x) \quad \text{in } L^2(\Omega)^3,$$

hence

$$\frac{\partial u(x,0)}{\partial t} = \vartheta_1(x). \qquad \square$$

4. STABILITY BEHAVIOR

We will now show a stability behavior of the solution of the problem (1.2)–(1.5) with $f = 0$. To this aim, we introduce the "modified" energy associated to the problem by the formula

$$\mathcal{E}(t) = \frac{1}{2} \left\| \frac{\partial u(t)}{\partial t} \right\|^2_{L^2(\Omega)^3} + \mu \|u(t)\|^2_{W^{1,2}(\Omega)^3} + \frac{1}{p(\cdot)} \|u(t)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}$$

$$+ \frac{\lambda}{2} \|\mathrm{div}(u(t))\|^2_{L^2(\Omega)} + \frac{\alpha}{p(\cdot)} \|u(t)\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}.$$

Lemma 4.1. *The energy $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function for all $t \geq 0$.*

Proof. Choosing $\varphi = \frac{\partial u(s)}{\partial t}$ in (2.6), we get

$$\mathcal{E}(t) - \mathcal{E}(0) = -\beta \int_0^t \left\| \frac{\partial u(s)}{\partial t} \right\|^2_{L^2(\Omega)^3} ds.$$

This means that

$$\mathcal{E}'(t) = -\beta \left\| \frac{\partial u(t)}{\partial t} \right\|^2_{L^2(\Omega)} \leq 0, \quad \text{for all } t \geq 0. \tag{4.1}$$

□

Theorem 4.2 (Global existence)**.** *Under the hypotheses of Theorem 3.1, the solution u of the problem* (1.2)–(1.5) *satisfies*

$$u \in C\left(\mathbb{R}_+, W^{1,p(\cdot)}(\Omega)^3\right), \quad \frac{\partial u}{\partial t} \in C\left(\mathbb{R}_+, L^2(\Omega)^3\right).$$

Proof. We have u and $\frac{\partial u}{\partial t}$ verify the identity (4.1). Then

$$\begin{aligned}&\frac{1}{2}\left\| \frac{\partial u(t)}{\partial t} \right\|^2_{L^2(\Omega)^3} + \frac{\mu}{2}\|u(t)\|^2_{W^{1,2}(\Omega)^3} + \frac{1}{p(\cdot)}\|u(t)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}\\ &+ \frac{\lambda}{2}\|\mathrm{div} u(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{p(\cdot)}\|u(t)\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}\\ &\leq \mathcal{E}(0),\end{aligned}$$

for all $t \geq 0$. This estimate holds independently of t. □

Next, we establish several technical lemmas for proof the main result of stability behavior.

Lemma 4.3 ([14, Theorem 8.1])**.** *Let $\mathcal{E} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nonincreasing function verifying the estimate*

$$\int_t^{\infty} \mathcal{E}^{\nu+1}(s)ds \leq K\mathcal{E}^{\nu}(0)\mathcal{E}(t), \quad \forall t \in \mathbb{R}_+,$$

then

$$\mathcal{E}(t) \leq \mathcal{E}(0)\left(\frac{K + \nu K}{K + \nu t}\right)^{\frac{1}{\nu}}, \quad \forall t \in \mathbb{R}_+, \quad \text{if} \quad \nu > 0,$$

and

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{1-\frac{1}{K}t}, \quad \forall t \in \mathbb{R}_+, \quad \text{if} \quad \nu = 0,$$

where $\nu \geq 0$ and $K > 0$ are two constants.

Lemma 4.4. *The energy functional $\mathcal{E}(\cdot)$ satisfies the following estimate for all $T > T_0 \geq 0$:*

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt \leq -\left[\mathcal{E}^{\frac{p(\cdot)}{2}}(t)\int_{\Omega}\frac{\partial u}{\partial t}udx\right]_{T_0}^{T} + \frac{p(\cdot)-2}{2}\int_{T_0}^{T}\left(\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\mathcal{E}'(t)\int_{\Omega}\frac{\partial u}{\partial t}udx\right)dt + \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left(2\left|\frac{\partial u}{\partial t}\right|^2 - u\frac{\partial u}{\partial t}\right)dxdt. \tag{4.2}$$

Proof. By multiplying Eq. (1.2) by $\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t).u$ and integrating over $\Omega\times]T_0,T[$, we get

$$0 = \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}u\left[\frac{\partial^2 u}{\partial t^2} - \mathrm{div}\sigma^{p(x)}(u) + \alpha|u|^{p(x)-2}u + \beta\frac{\partial u}{\partial t}\right]dxdt,$$

using the fact that

$$\int_{\Omega}\frac{\partial^2 u}{\partial t^2}udx = \frac{d}{dt}\int_{\Omega}\frac{\partial u}{\partial t}udx - \int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^2 dx,$$

we easily obtain

$$0 = \left[\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\frac{\partial u}{\partial t}udx\right]_{T_0}^{T} - \frac{p(\cdot)-2}{2}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\mathcal{E}'(t)\int_{\Omega}\frac{\partial u}{\partial t}udxdt + \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left[-u\mathrm{div}\sigma^{p(x)}(u) + \alpha|u|^{p(x)} + \beta u\frac{\partial u}{\partial t} - \left|\frac{\partial u}{\partial t}\right|^2\right]dxdt.$$

On the other side, we have

$$\int_{\Omega}\left[-u\mathrm{div}\,\sigma^{p(x)}(u) + \alpha|u|^{p(x)}\right]dx \geq 2\mathcal{E}(t) - \int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^2 dx,$$

thus, we get

$$0 \geq \left[\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\frac{\partial u}{\partial t}udx\right]_{T_0}^{T} - \frac{p(\cdot)-2}{2}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\mathcal{E}'(t)\int_{\Omega}\frac{\partial u}{\partial t}udxdt + \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left[2\mathcal{E}(t) - \left|\frac{\partial u}{\partial t}\right|^2 + \beta u\frac{\partial u}{\partial t} - \left|\frac{\partial u}{\partial t}\right|^2\right]dxdt,$$

then

$$0 \geq \left[\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\frac{\partial u}{\partial t}udx\right]_{T_0}^{T} - \frac{p(\cdot)-2}{2}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\mathcal{E}'(t)\int_{\Omega}\frac{\partial u}{\partial t}udxdt$$
$$+2\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)}{2}}(t) - \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left(2\left|\frac{\partial u}{\partial t}\right|^2 - \beta u\frac{\partial u}{\partial t}\right)dxdt.$$

The proof of the lemma is finished. □

In what follows, we denote by c generic positive constant, which may have different values at different occurrences.

Lemma 4.5. *There exist a positive constant c independent of $\mathcal{E}(0)$, T_0 and of T such that the energy $\mathcal{E}(\cdot)$ verifies the following estimate:*

$$\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt \leq c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0) + \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left(2\left|\frac{\partial u}{\partial t}\right|^2 - \beta u\frac{\partial u}{\partial t}\right)dxdt, \tag{4.3}$$

for all $T > T_0 \geq 0$.

Proof. We know that there exist a positive constant c_1 such that

$$\int_{\Omega} -u\,\mathrm{div}\sigma^{p(x)}(u)dx \geq c_1\left[\|u\|^2_{W^{1,2}(\Omega)^3} + \|u\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}\right] \geq c_1\int_{\Omega}|u|^2dx.$$

The use of the Young inequality gives

$$\begin{aligned}\left|\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}u\frac{\partial u}{\partial t}dx\right| &\leq c\,\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left(\left|\frac{\partial u}{\partial t}\right|^2 + |u|^2\right)dx\\ &\leq c\,\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left(\left|\frac{\partial u}{\partial t}\right|^2 - u\mathrm{div}\sigma^{p(x)}(u)\right)dx\\ &\leq c\,\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left|\frac{p(\cdot)-2}{2}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\mathcal{E}'(t)\int_{\Omega}u\frac{\partial u}{\partial t}dxdt\right| &\leq c\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\left(-\mathcal{E}'(t)\right)\mathcal{E}(t)dt\\ &\leq c\left[\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0) - \mathcal{E}^{\frac{p(\cdot)}{2}}(T)\right]\\ &\leq c\,\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).\end{aligned}$$

Then, we replace these two estimates in (4.2) to find (4.3). □

Lemma 4.6. *For all* $\varsigma > 0$*, we have*

$$\begin{aligned}&\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \\ &\leq \varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c(\varsigma)\mathcal{E}\left(T_0\right) + c\mathcal{E}^{\frac{p(\cdot)}{2}}\left(T_0\right), \quad \text{for all } T > T_0 \geq 0.\end{aligned} \tag{4.4}$$

Proof. For $t \in \mathbb{R}_+$ fixed, we see that

$$\int_{\Omega} \left|\frac{\partial u}{\partial t}\right|^2 dx = \int_{\left|\frac{\partial u}{\partial t}\right| \leq 1} \left|\frac{\partial u}{\partial t}\right|^2 dx + \int_{\left|\frac{\partial u}{\partial t}\right| > 1} \left|\frac{\partial u}{\partial t}\right|^2 dx.$$

Also, there exists a constant $c \geq 0$ such that

$$\int_{\left|\frac{\partial u}{\partial t}\right| \leq 1} \left|\frac{\partial u}{\partial t}\right|^2 dx \leq c \left(\int_{\left|\frac{\partial u}{\partial t}\right| \leq 1} \left|\frac{\partial u}{\partial t}\right|^2 dx \right)^{\frac{2}{p(x)}}.$$

Then

$$\begin{aligned}\int_{\Omega} \left|\frac{\partial u}{\partial t}\right|^2 dx &\leq c \left(\int_{\left|\frac{\partial u}{\partial t}\right| \leq 1} \left|\frac{\partial u}{\partial t}\right|^2 dx \right)^{\frac{2}{p(x)}} + c \int_{\left|\frac{\partial u}{\partial t}\right| > 1} \left|\frac{\partial u}{\partial t}\right|^2 dx \\ &\leq c\left(-\mathcal{E}'(t)\right)^{\frac{2}{p(\cdot)}} - c\mathcal{E}'(t).\end{aligned}$$

Therefore

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \leq c \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\left(-\mathcal{E}'(t)\right)^{\frac{2}{p(\cdot)}} dt - c \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\mathcal{E}'(t)dt,$$

using the Young inequality, we get

$$\begin{aligned}c \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\left(-\mathcal{E}'(t)\right)^{\frac{2}{p(\cdot)}} dt &\leq c\frac{p(\cdot)-2}{p(\cdot)} \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)+1}{2}}(t)dt + c\frac{2}{p(\cdot)} \int_{T_0}^{T} \left(-\mathcal{E}'(t)\right) dt \\ &\leq \varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c(\varsigma)\mathcal{E}(T_0).\end{aligned}$$

So, we find

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \le \varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c(\varsigma)\mathcal{E}(T_0) + c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0),$$

thus (4.4) holds. □

Lemma 4.7. *The energy $\mathcal{E}(\cdot)$ satisfies the following estimate, for all $\varsigma > 0$:*

$$\left|\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} u\frac{\partial u}{\partial t} dxdt\right| \le \varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c(\varsigma)\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0). \tag{4.5}$$

Proof. By applying the Young inequality, for all $\varsigma > 0$, we have

$$\begin{aligned}\int_{\Omega} u\frac{\partial u}{\partial t} dx &\le \varsigma \int_{\Omega} |u|^2 dx + c(\varsigma) \int_{\Omega} \left|\frac{\partial u}{\partial t}\right|^2 dx \\ &\le \varsigma \int_{\Omega} -u \operatorname{div}\left(\sigma^{p(x)}(u)\right) dx + c(\varsigma) \int_{\Omega} \left|\frac{\partial u}{\partial t}\right|^2 dx \\ &\le \varsigma\mathcal{E}(t) + c(\varsigma)\left(-\mathcal{E}'(t)\right).\end{aligned}$$

Then we conclude that, for any $T > T_0 \ge 0$

$$\left|\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} u\frac{\partial u}{\partial t} dxdt\right| \le \varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c(\varsigma)\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0),$$

which finishes the proof. □

Lemma 4.8. *For all $T > T_0 \ge 0$, we have the estimate*

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt \le c\left(1 + \mathcal{E}^{\frac{p(\cdot)-2}{2}}(0)\right)\mathcal{E}(T_0).$$

Proof. By (4.4) and (4.5), we obtain

$$\begin{aligned}&\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(2\left|\frac{\partial u}{\partial t}\right|^2 - \beta u\frac{\partial u}{\partial t}\right) dxdt \\ &\le 2\varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c(\varsigma)\mathcal{E}(T_0) + c(\varsigma)\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).\end{aligned}$$

Choosing $\varsigma = \frac{1}{4}$, we get

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(2\left|\frac{\partial u}{\partial t}\right|^2 - \beta u \frac{\partial u}{\partial t} \right) dxdt \leq \frac{1}{2} \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c\mathcal{E}(T_0) + c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0). \tag{4.6}$$

Now, we use the inequality (4.6) in (4.3), we get

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt \leq \frac{1}{2} \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt + c\mathcal{E}(T_0) + c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0), \quad 0 \leq T_0 < T.$$

This implies that

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)dt \leq c\left(1 + \mathcal{E}^{\frac{p(\cdot)-2}{2}}(T_0)\right)\mathcal{E}(T_0) \leq c\left(1 + \mathcal{E}^{\frac{p(\cdot)-2}{2}}(0)\right)\mathcal{E}(T_0).$$

This completes the proof. □

Lemmas 4.1 and 4.8 imply that $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function and verify the following inequalities

$$\int_{t}^{\infty} \mathcal{E}^{\frac{p^+}{2}}(s)ds \leq c\mathcal{E}^{\frac{p^+-2}{2}}(0)\mathcal{E}(t), \quad \forall t > 0. \tag{4.7}$$

Theorem 4.9 (Stability of the solution)**.** *There exists two positives constants $\mathcal{A}$ and $\mathcal{B}$ such that the solution of the problem* (1.2)–(1.5) *verifies the following estimates:*

$$\begin{aligned} &\mathcal{E}(t) \leq \mathcal{A}t^{\frac{-2}{p^+-2}}, \quad \forall t \geq 0, \quad \textit{if} \quad p^+ > 2, \\ &\mathcal{E}(t) \leq \mathcal{E}(0)e^{1-\mathcal{B}t}, \quad \forall t \geq 0, \quad \textit{if} \quad p^+ = 2, \end{aligned}$$

where the constant $\mathcal{A}$ depends on the initial energy $\mathcal{E}(0)$ and the constant $\mathcal{B}$ independent of $\mathcal{E}(0)$.

Proof. The proof follows directly by application of Lemma 4.3 and the inequality (4.7). □

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Received: February 21, 2022.
Revised: February 24, 2023.
Accepted: February 25, 2023.