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THE HEAT EQUATION ON TIME SCALES

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Abstract. We present the use of a Fourier transform on time scales to solve a dynamic heat IVP. This is done by inverting a certain exponential function via contour integral. We include some specific examples and directions for further study.

Keywords: heat equation, time scales, Fourier transform.

Mathematics Subject Classification: 39A14, 34N05, 42A38.

1. INTRODUCTION

The time scales calculus is a well established theory [5] and primarily focuses on the relationship between continuous and discrete mathematical models. In this work, we demonstrate how to use a time scales Fourier transform to solve the dynamic heat equation $u^{\Delta_t} = u^{\Delta^2_{xx}}$, which was first investigated in [19] imposed with initial and boundary conditions. Its continuous counterpart is the classical heat equation which is well-known to be solvable by Fourier transform methods [12, Section 4.3.1]. In particular, we augment the dynamic heat equation with an initial function, impose that $t \in [0, \infty)$, and x lies in some time scale $\mathbb{T}$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u^{\Delta^2_{xx}}, \quad x \in \mathbb{T}, t \in (0, \infty), \\ u(0, x) = g(x). \end{cases} \tag{1.1}$$

When $\mathbb{T} = \mathbb{Z}$, the IVP (1.1) is a special case of a more general diffusion-type equation with connections to random walks and stochastic processes [14]. Other related works include [25, 26] which study and find explicit solutions to a problem similar to, but different from, (1.1). More broadly, the book [6] is a good resource for the related area of partial difference equations and [22] investigates the lattice differential equations, which are a class of semidiscrete systems of differential-difference equations whose spatial variable lies in a lattice.

Many Fourier transforms have been investigated on time scales, including the original by Hilger [17], those defined on time scales that also have a group structure [18], Davis *et al.*'s discrete Fourier transform [11], and the recent transform defined in [8] by the formula

$$\mathcal{F}\{f\}(z;t_0) = \int_{\mathbb{T}} f(t)e_{\ominus iz}(\sigma(t),t_0)\Delta t. \tag{1.2}$$

The transform (1.2) has desirable operational properties similar to the unilateral and bilateral Laplace transforms [4,10], making it well-suited for solving partial dynamic equations. A significant part of our work here demonstrates how to compute the time scales inverse Fourier transform via contour integral.

The heat IBVP in [19] is solved with the bilateral Laplace transform, but a general convolution theorem was not known at the time, leading to only a finite number of specific functions g being considered. Thus, the other major contribution of our manuscript is to provide the explicit solution to the problem (1.1) for a certain class of spatial time scales $\mathbb{T}$. To the best of our knowledge, there is no other solution of (1.1) solved in such generality in the literature, although a modified version of (1.1) whose spatial variable is $\mathbb{T} = \mathbb{Z}$ and a shifted right-hand side is solved explicitly in [25].

This work is organized as follows: in Section 2 we introduce some concepts and preliminary results that are needed. In Section 3 we specifically deal with the time scales Fourier transform, which will be used in Section 4 in order to solve the heat equation with a space variable belonging to certain time scales. We finally conclude this work with some possible research directions.

2. PRELIMINARIES AND DEFINITIONS

We refer readers to the monograph [5] for the elementary definitions of time scales calculus, and in particular the papers [9,23] for information on the exponential function on time scales with nonregressive subscripts. We now recall the convolution theorem for the unilateral Laplace transform on time scales. The shifting problem was originally introduced in [3, Definition 2.1] in order to define convolution on all time scales for the unilateral Laplace transform. We now consider a modified shift problem appropriate for the time scales Fourier transform by

$$\begin{cases} u^{\Delta_t}(t,\sigma(s)) = -u^{\Delta_s}(t,s), & t,s \in \mathbb{T}, \\ u(t,t_0) = f(t), & t \in \mathbb{T}, \end{cases}$$

where $f\colon [t_0,\infty)\cap\mathbb{T} \to \mathbb{C}$. The solution of the shifting problem is denoted by $\hat{f}$, i.e. $\hat{f}(t,s) = u(t,s)$. A convolution for all time scales is defined using the shift by [3, Definition 2.5]

$$(f*g)(t) = \int_{t_0}^{t} \hat{f}(t,\sigma(s))g(s)\Delta s. \tag{2.1}$$

With the convolution (2.1), the convolution theorem for unilateral Laplace transforms on time scales is given by

$$\mathscr{L}\{f*g\}(z;t_0) = \mathscr{L}\{f\}(z;t_0)\mathscr{L}\{g\}(z;t_0)$$

(see [3, Theorem 3.2]).

We make use of the Hilger complex plane

$$\mathbb{C}_h = \begin{cases} \mathbb{C}, & h = 0, \\ \mathbb{C} \setminus \left\{-\frac{1}{h}\right\}, & h > 0 \end{cases}$$

and the Hilger real part of a number $z \in \mathbb{C}$ given by

$$\mathrm{Re}_h(z) = \begin{cases} \mathrm{Re}(z), & h = 0, \\ \frac{|1+hz|-1}{h}, & h > 0. \end{cases}$$

We additionally define the minimal graininess function $\mu_* \colon \mathbb{T} \to [0, \infty)$ by the formula

$$\mu_*(s) = \inf_{t \in [s,\infty)\cap\mathbb{T}} \mu(t),$$

and the maximal graininess function $\mu^* \colon \mathbb{T} \to [0, \infty)$ by the formula

$$\mu^*(s) = \sup_{t \in (-\infty,s]\cap\mathbb{T}} \mu(t),$$

where μ denotes the graininess (or stepsize) function associated to the time scale $\mathbb{T}$. The Fourier transform (1.2) is defined on the set $R_{s,\alpha,\gamma}$, defined for real-valued positive regressive constants α and γ and $s \in \mathbb{T}$ by

$$\begin{aligned} R_{s,\alpha,\gamma} = \Big\{ z \in \mathbb{C} \colon \mathrm{Re}_{\mu^*(s)}(iz) < \gamma, \quad & \mathrm{Re}_{\mu_*(s)}(iz) > \alpha, \\ & 1 + \overline{\overline{\mu}}(s,z)\mathrm{Re}_{\overline{\overline{\mu}}(s)}(iz) \neq 0 \Big\}, \end{aligned} \tag{2.2}$$

where $\overline{\overline{\mu}}$ is given by

$$\overline{\overline{\mu}}(s,z) := \begin{cases} \mu^*(s), & \mathrm{Re}_{\mu(s)}(z) \leq 0, \\ \overline{\mu}(s), & \mathrm{Re}_{\overline{\mu}(s)}(z) > 0, \end{cases}$$

and

$$\overline{\mu}(s) = \inf_{\tau \in (-\infty,s]\cap\mathbb{T}} \mu(\tau).$$

If $z \in R_{s,\alpha,\gamma}$, and λ is a positively regressive constant, then [4, Theorem 3.4] shows for $t \in [s, \infty)$,

$$|e_{\lambda \ominus z}(t,s)| \leq e_{\lambda \ominus \mathrm{Re}_{\mu_*(s)}(z)}(t,s), \tag{2.3}$$

and [8, Theorem 2] shows for $t \in (-\infty, s]$,

$$|e_{\lambda \ominus z}(t,s)| \leq e_{\lambda \ominus \mathrm{Re}_{\mu^*(s)}(z)}(t,s). \tag{2.4}$$

3. FOURIER TRANSFORM RESULTS

We modify the convolution (2.1) to be appropriate for the Fourier transform as

$$(f * g)(t) = \int_{\mathbb{T}} \hat{f}(t, \sigma(s)) g(s) \Delta s.$$

The function $\psi(s) = \int_{\mathbb{T}} \hat{f}(t,s)e_{\ominus iz}(\sigma(t),s)\Delta t$ is useful in the exploration of the convolution theorem for the time scales Fourier transform. The convolution theorem has been proven in [15], but its proof relied on the Δ-derivative of ψ commuting with the integral over $\mathbb{T}$ without proof. We fill that gap in the next lemma.

Lemma 3.1. *If $z \in R_{s,0,0}$ and the map $\tau \mapsto \hat{f}^{\Delta_s}(\sigma(t),\tau)$ is continuous, then the Δ-derivative of ψ commutes with its defining integral, i.e.,*

$$\psi^{\Delta}(s) = \int\limits_{\mathbb{T}} \left(\hat{f}(t,s)e_{\ominus iz}(\sigma(t),s)\right)^{\Delta_s} \Delta t.$$

Proof. If $\sigma(s) > s$, then the linearity of the Δ-integral shows

$$\begin{aligned}
\psi^{\Delta}(s) &= \frac{\psi(\sigma(s)) - \psi(s)}{\mu(s)} \\
&= \int\limits_{\mathbb{T}} \frac{\hat{f}(t,\sigma(s))e_{\ominus iz}(\sigma(t),\sigma(s)) - \hat{f}(t,s)e_{\ominus iz}(\sigma(t),s)}{\mu(s)} \Delta\tau,
\end{aligned}$$

completing this case. If $\sigma(s) = s$, then let $s_n \to s$ be a sequence with each $s_n \in \mathbb{T} \cap [s-1, s+1]$ for all n. We compute

$$\begin{aligned}
\psi^{\Delta}(s) = \psi'(s) &= \lim_{n\to\infty} \frac{\psi(s_n) - \psi(s)}{s_n - s} \\
&= \lim_{n\to\infty} \frac{1}{s_n - s} \int\limits_{\mathbb{T}} \hat{f}(\sigma(t),s_n)e_{\ominus iz}(\sigma(t),s_n) - \hat{f}(t,s)e_{\ominus iz}(\sigma(t),s)\Delta t \\
&= \lim_{n\to\infty} \int\limits_{\mathbb{T}} \frac{g(s_n) - g(s)}{s_n - s} e_{\ominus iz}(\sigma(t),s)\Delta t,
\end{aligned}$$

where $g(\tau) := \hat{f}(\sigma(t),\tau)e_{\ominus iz}(s,\tau)$. Define the sets

$$U_n = [\min\{s_n, s\}, \max\{s_n, s\}] \quad \text{and} \quad U = [s-1, s+1].$$

By a corollary to the mean value theorem on time scales [16, Corollary 3.3], we know that

$$|g(s_n) - g(s)| \leq \left\{ \sup_{t \in U_n \cap \mathbb{T}} |g^{\Delta}(t)| \right\} |s_n - s|. \tag{3.1}$$

So we get for

$$f_n(s) := \frac{g(s_n) - g(s)}{s_n - s} e_{\ominus iz}(\sigma(t),s)$$

that f_n converges pointwise to $\left(\hat{f}(t,s)e_{\ominus iz}(\sigma(t),s)\right)^{\Delta_s}$ and by (3.1), we conclude that

$$|f_n| = \left| \frac{g(s_n) - g(s)}{s_n - s} e_{\ominus iz}(\sigma(t),s) \right| \leq \left\{ \sup_{\tau \in U_n \cap \mathbb{T}} \hat{f}^{\Delta_s}(\sigma(t),\tau) \right\} |e_{\ominus iz}(\sigma(t),s)|.$$

Let

$$\alpha := \sup_{\tau \in U \cap \mathbb{T}} \hat{f}^{\Delta_s}(\sigma(t), \tau).$$

Note that the constant α is independent of n and

$$\alpha \geq \sup_{\tau \in U_n \cap \mathbb{T}} \hat{f}^{\Delta_s}(\sigma(t), \tau)$$

for all n. Using (2.4) and (2.3), we obtain

$$\begin{aligned}
|\psi^{\Delta}(s)| &\leq \lim_{n\to\infty} \int_{\mathbb{T}} \left| \frac{g(s_n) - g(s)}{s_n - s} e_{\ominus iz}(\sigma(t), s) \right| \Delta t \\
&\leq \alpha \int_{\mathbb{T}} |e_{\ominus iz}(\sigma(t), s)| \Delta t \\
&= \alpha \left[\int_{-\infty}^{s} |e_{\ominus iz}(\sigma(t), s)| \, \Delta t + \int_{s}^{\infty} |e_{\ominus iz}(\sigma(t), s)| \, \Delta t \right] \\
&\leq \alpha \left[\int_{-\infty}^{s} e_{\ominus \mathrm{Re}_{\mu^*(s)}(iz)}(\sigma(t), s) \Delta t + \int_{s}^{\infty} e_{\ominus \mathrm{Re}_{\mu_*(s)}(iz)}(\sigma(t), s) \Delta t \right].
\end{aligned}$$

Since $e_p^{\sigma} = (1 + \mu p) e_p$, we simplify

$$\begin{aligned}
e_{\ominus \mathrm{Re}_{\mu^*(s)}(iz)}(\sigma(t), s) &= \Big(1 + \mu(t) \left(\ominus \mathrm{Re}_{\mu^*(s)}(iz)\right) \Big) e_{\ominus \mathrm{Re}_{\mu^*(s)}(iz)}(t, s) \\
&= \frac{1}{1 + \mu(t) \mathrm{Re}_{\mu^*(s)}(iz)} e_{\ominus \mathrm{Re}_{\mu^*(s)}(iz)}(t, s) \\
&= \left(-\frac{1}{\mathrm{Re}_{\mu^*(s)}(iz)} \right) \left(\ominus \mathrm{Re}_{\mu^*(s)}(iz)\right) e_{\ominus \mathrm{Re}_{\mu^*(s)}(iz)}(t, s) \\
&= \left(-\frac{1}{\mathrm{Re}_{\mu^*(s)}(iz)} \right) e^{\Delta}_{\ominus \mathrm{Re}_{\mu^*(s)}(iz)}(t, s),
\end{aligned}$$

and similarly,

$$e_{\ominus \mathrm{Re}_{\mu_*(s)}(iz)}(\sigma(t), s) = \left(-\frac{1}{\mathrm{Re}_{\mu_*(s)}(iz)} \right) e^{\Delta}_{\ominus \mathrm{Re}_{\mu_*(s)}(iz)}(t, s).$$

We know that the quantities $\mathrm{Re}_{\mu^*(s)}(iz)$ and $\mathrm{Re}_{\mu_*(s)}(iz)$ are nonzero because this is explicitly disallowed by the definition of $R_{s,0,0}$, see (2.2). Thus applying the decay properties of the exponentials from [8, Theorem 2] and [4, Theorem 3.4], we arrive at

$$\begin{aligned}|\psi^\Delta(s)| &\leq \frac{-\alpha}{\mathrm{Re}_{\mu^*(s)}(iz)} \int\limits_{-\infty}^{s} e^{\Delta}_{\ominus \mathrm{Re}_{\mu^*(s)}(iz)}(t,s)\Delta t \\ &\quad + \frac{-\alpha}{\mathrm{Re}_{\mu_*(s)}(iz)} \int\limits_{-\infty}^{s} e^{\Delta}_{\ominus \mathrm{Re}_{\mu_*(s)}(iz)}(t,s)\Delta t \\ &= \frac{-\alpha}{\mathrm{Re}_{\mu^*(s)}(iz)}\Big[1-0\Big] + \frac{-\alpha}{\mathrm{Re}_{\mu_*(s)}(iz)}\Big[0-1\Big],\end{aligned}$$

which is finite. Therefore f_n converges pointwise to $g^\Delta(s)e_{\ominus iz}(\sigma(t),s)$ and $|f_n| \leq \alpha |e_{\ominus iz}(\sigma(t),s)|$ which we have just shown is integrable. Hence, by the Lebesgue dominated convergence theorem,

$$\psi^\Delta(s) = \psi'(s) = \lim_{n\to\infty} \int\limits_{\mathbb{T}} f_n = \int\limits_{\mathbb{T}} \lim_{n\to\infty} f_n = \int\limits_{\mathbb{T}} \Big(\hat{f}(t,s)e_{\ominus iz}(\sigma(t),s)\Big)^{\Delta_s} \Delta t,$$

completing the proof. □

Using Lemma 3.1, the proof that ψ is constant and ultimately the Fourier convolution theorem $\mathcal{F}\{f * g\}(z;t_0) = \mathcal{F}\{f\}(z;t_0)\mathcal{F}\{g\}(z;t_0)$ follows as in [15].

The $\mathbb{T}=\mathbb{Z}$ Fourier transform is

$$\mathcal{F}\{f\}(z;0) := \sum_{k\in\mathbb{Z}} \frac{f(k)}{(1+iz)^{k+1}}.$$

We now present its contour integral inversion. The fundamental observation for the following result is that the reciprocal of exponential function used the Fourier transform is used as the kernel of the inverse transformation. Since the exponential function on the integers is an exponential with base $1+iz$, a contour integral is a natural form for the inversion because it leads to residues which are not difficult to compute. The contour is chosen as a circle around $z=i$ since the kernel of the inversion integral has poles at $z=i$ and arranging for its radius to be between 0 and 1 guarantees the interior of the contour is analytic when F has a typical branch cut.

Theorem 3.2. *If $\mathbb{T}=\mathbb{Z}$ and F is analytic, except possibly on a branch cut $(-\infty,0]$, then*

$$\mathcal{F}^{-1}\{F\}(t;0) := \frac{1}{2\pi} \oint\limits_{C} F(z)(1+iz)^t \mathrm{d}z,$$

where C is a circle with center $z=i$ and radius $0<r<1$.

Proof. We will show that $\mathcal{F}^{-1}\Big\{\mathcal{F}\{f\}(\cdot;0)\Big\}(t;0) = f(t)$. Calculate

$$\begin{aligned}\mathcal{F}^{-1}\Big\{\mathcal{F}\{f\}(\cdot;0)\Big\}(t;0) &= \frac{1}{2\pi}\oint\limits_C \mathcal{F}\{f\}(z;0)(1+iz)^t \mathrm{d}z \\ &= \frac{1}{2\pi}\oint\limits_C \sum_{k\in\mathbb{Z}} f(k)(1+iz)^{t-k-1}\mathrm{d}z \\ &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} f(k)\oint\limits_C (1+iz)^{t-k-1}\mathrm{d}z \\ &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} f(k) i^{t-k-1}\oint\limits_C (z-i)^{t-k-1}\mathrm{d}z.\end{aligned}$$

By residue theorem,

$$\oint\limits_C (z-i)^{t-k-1}\mathrm{d}z = \begin{cases} 0, & t-k-1 \neq -1, \\ 2\pi i, & t-k-1 = -1 \leftrightarrow t = k.\end{cases}$$

Therefore, $\mathcal{F}^{-1}\Big\{\mathcal{F}\{f\}(\cdot;0)\Big\}(t,0) = f(t)$, completing the proof. □

With modification to the contour C, the same principle can be applied to a much larger class of time scales, as we now show. In the case that follows, the poles of the exponential function in the inversion integral are at most simple poles due to injectivity of μ, meaning the residues are trivial to compute. In general, this result can be extended to non-injective μ at the cost of higher order poles introducing derivatives with respect to z to the calculation of the residues.

Theorem 3.3. *Let $\mathbb{T}$ be a countable time scale of the form $\{\dots, t_{-1}, t_0, t_1, \dots\}$ such that for all m, n, $t_m < t_n$ if and only if $m < n$. If F is analytic, except possibly on a branch cut $(-\infty, 0]$ and the graininess function $\mu\colon \mathbb{T} \to [0,\infty)$ associated with $\mathbb{T}$ is injective, then*

$$\mathcal{F}^{-1}\{F\}(t;s) = \frac{1}{2\pi}\oint\limits_C F(z)e_{iz}(t,s)\mathrm{d}z,$$

where $\oint\limits_C$ is understood as $\lim\limits_{m\to\infty}\oint\limits_{C_m}$, where C_m is a contour surrounding the points $\Big\{\frac{i}{t_k}\colon -m \le k \le m\Big\}$.

Proof. Calculate directly

$$\begin{aligned}\mathcal{F}^{-1}\left\{\mathcal{F}\{f\}(\cdot;s)\right\}(t;s) &= \frac{1}{2\pi}\oint_C \mathcal{F}\{f\}(z;s)e_{iz}(t,s)\mathrm{d}z \\ &= \frac{1}{2\pi}\oint_C\left(\int_{\mathbb{T}} f(\tau)e_{\ominus iz}(\sigma(\tau),s)\Delta\tau\right)e_{iz}(t,s)\mathrm{d}z \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(\tau)\oint_C \frac{e_{iz}(t,\tau)}{1+i\mu(\tau)z}\mathrm{d}z\Delta\tau \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} f(\tau)\frac{1}{i\mu(\tau)}\oint_C \frac{e_{iz}(t,\tau)}{z-\frac{i}{\mu(\tau)}}\mathrm{d}z\Delta\tau.\end{aligned}$$

Since the graininess function is injective, we know that $e_{\ominus iz}(t,\tau)$ has no factors of the form $1+i\mu(\tau)z$. Therefore, by the residue theorem,

$$\begin{aligned}\mathcal{F}^{-1}\{\mathcal{F}\{f\}(\cdot;s)\}(t;s) &= \frac{1}{2\pi i}\int_{\mathbb{T}} f(\tau)\frac{1}{\mu(\tau)}\oint_C \frac{e_{iz}(t,\tau)}{z-\frac{i}{\mu(\tau)}}\mathrm{d}z\Delta\tau \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} f(\tau)\frac{2\pi i}{\mu(\tau)}e_{-\frac{1}{\mu(\tau)}}(t,\tau)\mathrm{d}z\Delta\tau.\end{aligned}$$

Since the map $\tau\mapsto -\frac{1}{\mu(\tau)}$ violates regressivity at all $\tau\in\mathbb{T}$, $e_{-\frac{1}{\mu(\tau)}}(\tau,t)=0$ for all $t\neq\tau$ and $e_{-\frac{1}{\mu(\tau)}}(\tau,t)=1$ when $t=\tau$. Therefore, the integral reduces to an integral over the singleton $\{t\}$ and we obtain $\mathcal{F}^{-1}\{\mathcal{F}\{f\}(\cdot;s)\}(t;s)=f(t)$, completing the proof. □

4. HEAT EQUATION

Taking the Fourier transform of (1.1) with respect to the x variable yields the initial value problem

$$\frac{\partial\hat{u}}{\partial t} = (iz)^2u = -z^2u, \quad \hat{u}(0,z)=\hat{g}(z), \tag{4.1}$$

which has unique solution

$$\hat{u}(t,z)=\hat{g}(z)e^{-z^2t}. \tag{4.2}$$

Inversion of (4.2) is the crucial for the solution of (1.1). We will do so via the convolution theorem; first, a technical lemma involving the Hermite polynomials. Recall [1, p. 280] the recurrence

$$2zH_n(z)-H_n'(z)=H_{n+1}(z). \tag{4.3}$$

Lemma 4.1. *The following formula holds:*

$$e^{z^2t}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[e^{-z^2t}\right]=(-1)^nt^{\frac{n}{2}}H_n(z\sqrt{t}). \tag{4.4}$$

Proof. For the base case, let $n = 1$. Then the left-hand side of (4.4) is

$$e^{z^2t}\frac{\mathrm{d}}{\mathrm{d}z}e^{-z^2t} = -2zt.$$

On the other hand, the right-hand side of (4.4) is

$$(-1)\sqrt{t}H_1\left(z\sqrt{t}\right) = (-1)\sqrt{t}(2z\sqrt{t}) = -2zt,$$

completing the base case. Now assume (4.4) holds for $n = N$. Then using (4.3),

$$\begin{aligned}
e^{z^2t}\frac{\mathrm{d}^{N+1}}{\mathrm{d}z^{N+1}} = e^{z^2t}\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}^N}{\mathrm{d}z^N}e^{-z^2t}\right] &= e^{z^2t}\frac{\mathrm{d}}{\mathrm{d}z}\left[(-1)^N t^{\frac{N}{2}} H_N(z\sqrt{t})e^{-z^2t}\right] \\
&= (-1)^N t^{\frac{N}{2}} e^{z^2t}\frac{\mathrm{d}}{\mathrm{d}z}\left[H_N(z\sqrt{t})e^{-z^2t}\right] \\
&= (-1)^N t^{\frac{N}{2}} e^{z^2t}\left[\sqrt{t}H_N'(z\sqrt{t}) + H_N(z\sqrt{t})(-2zt)\right]e^{-z^2t} \\
&= (-1)^{N+1} t^{\frac{N+1}{2}}\left[2z\sqrt{t}H_N(z\sqrt{t}) - H_N'(z\sqrt{t})\right] \\
&\overset{(4.3)}{=} (-1)^{N+1} t^{\frac{N+1}{2}} H_{N+1}(z\sqrt{t}),
\end{aligned}$$

completing the proof. □

Now we invert the exponential function.

Lemma 4.2. *If $x \in \mathbb{Z}$ with $x < 0$, then*

$$\mathcal{F}^{-1}\{v_t\}(x;0) = \frac{(-1)^{|x|-1}t^{\frac{|x|-1}{2}}H_{|x|-1}(i\sqrt{t})e^t}{i^{|x|-1}(|x|-1)!},$$

where $v_t(z) = e^{-z^2t}$.

Proof. Compute

$$\mathcal{F}^{-1}\{v_t\}(x;0) = \frac{1}{2\pi}\oint_C e^{-z^2t}(1+iz)^x\mathrm{d}z.$$

The integrand is an analytic function of z when $x \geq 0$, so the integral becomes identically zero for such x. So assume that $x < 0$. Then we obtain by the residue theorem

$$\begin{aligned}
\mathcal{F}^{-1}\{v_t\}(x;0) &= \frac{1}{2\pi}\oint_C \frac{e^{-z^2t}}{(1+iz)^{|x|}}\mathrm{d}z = \frac{1}{2\pi i^{|x|}}\oint_C \frac{e^{-z^2t}}{(z-i)^{|x|}}\mathrm{d}z \\
&= \frac{1}{i^{|x|-1}}\frac{1}{(|x|-1)!}\frac{\mathrm{d}^{|x|-1}}{\mathrm{d}z^{|x|-1}}\left[e^{-z^2t}\right|_{z=i}.
\end{aligned}$$

We compute with (4.4) taking $n = |x| - 1$ to obtain

$$\mathcal{F}^{-1}\{v_t\}(x;0) = \frac{1}{i^{|x|-1}(|x|-1)!}\left[(-1)^{|x|-1}t^{\frac{|x|-1}{2}}H_{|x|-1}(z\sqrt{t})e^{-z^2t}\right]_{z=i}$$
$$= \frac{(-1)^{|x|-1}t^{\frac{|x|-1}{2}}H_{|x|-1}(i\sqrt{t})e^{t}}{i^{|x|-1}(|x|-1)!},$$

completing the proof. □

With the inversion from Lemma 4.2, the convolution theorem completes the solution of (1.1) on the time scale $\mathbb{T} = \mathbb{Z}$. We recall what it means for an rd-continuous function f to be of exponential order α: there exists a constant K and $t_0 \in \mathbb{T}$ such that $|f(t)| \leq K e_\alpha(t, t_0)$. The more general idea of double exponential order (α, γ) [8, Definition 3.2] says there is $s \in \mathbb{T}$ and $\alpha, \gamma \in \mathbb{R}$ so that the restrictions $f\big|_{(-\infty,s]\cap\mathbb{T}}$ and $f\big|_{[s,\infty)\cap\mathbb{T}}$ are of exponential order α and γ, respectively. It is known that if f is of double exponential order (α, γ), then its time scales Fourier transform exists.

Theorem 4.3. *If $x \in \mathbb{T} = \mathbb{Z}$, $t \in [0, \infty)$, and the initial function g is of double exponential order (α, γ), then (1.1) has solution*

$$u(x,t) = \left(\mathcal{F}^{-1}\{v_t\} * g\right)(x;0),$$

where $v_t(z) = e^{-z^2t}$.

Proof. Let $\hat{g} = \mathcal{F}\{g\}$. By the convolution theorem,

$$u(x,t) = \mathcal{F}^{-1}\{\hat{g}v_t\}(x) = \frac{1}{2\pi}\oint_C \hat{g}(\xi)e^{-\xi^2t}(1+i\xi)^x \mathrm{d}\xi.$$

Therefore, compute

$$\frac{\partial u}{\partial t} = \frac{1}{2\pi}\oint_C \hat{g}(\xi)\left(-\xi^2\right)e^{-\xi^2t}(1+i\xi)^x \mathrm{d}\xi.$$

On the other hand,

$$\Delta_x(1+i\xi)^x = (1+i\xi)^{x+1} - (1+i\xi)^x = i\xi(1+i\xi)^x,$$

and hence $\Delta_{xx}(1+i\xi)^x = -\xi^2(1+i\xi)^x$. Thus,

$$\Delta_{xx}u(x,t) = \frac{1}{2\pi}\oint_C \hat{g}(\xi)e^{-\xi^2t}\Delta_{xx}(1+i\xi)^x \mathrm{d}\xi$$
$$= \frac{1}{2\pi}\oint_C \hat{g}(\xi)e^{-\xi^2t}(-\xi^2)(1+i\xi)^x \mathrm{d}\xi = \frac{\partial u}{\partial t},$$

completing the proof. □

Example 4.4. We approximate the time derivative with

$$\frac{\partial u}{\partial t}(t,x) \approx \frac{u(t+h,x)-u(t,x)}{h} \tag{4.5}$$

for some $h > 0$ and when $\mathbb{T} = \mathbb{Z}$ we express

$$u^{\Delta^2_{xx}}(t,x) = u(t,x+2) - 2u(t,x+1) + u(t,x).$$

Solving the resulting equation for $u(t+h,x)$ yields

$$u(t+h,x) = u(t,x) + h\Big(u(t,x+2) - 2u(t,x+1) + u(t,x)\Big),$$

which we use to approximate the values in the t-direction in Figure 1. As $|x| \to \infty$ this scheme potentially introduces accumulations of errors that grow large, so it is not optimized for solutions that involve large values of x. The purpose of the numerical solution here is to visualize near the initial value.

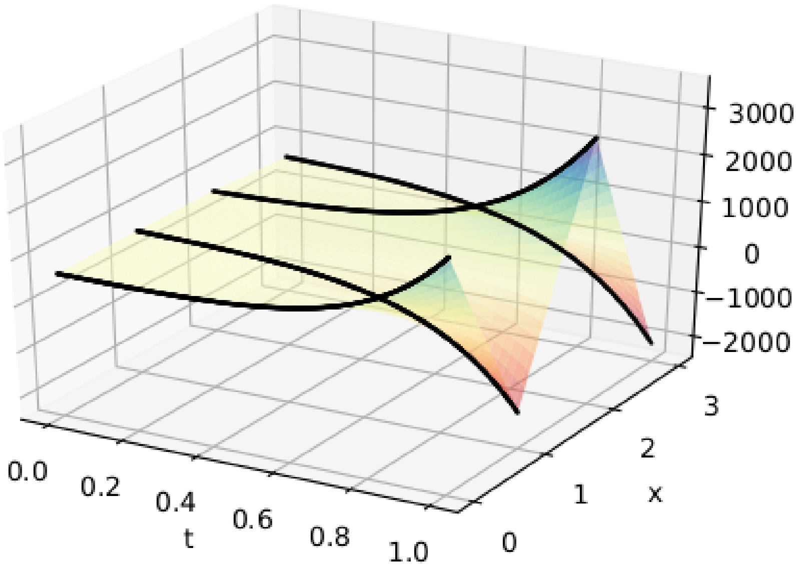

Fig. 1. The $\mathbb{T} = \mathbb{Z}$ case with $h = 0.001$. The black lines are solutions of (1.1) at each $x \in \mathbb{T}$ for $0 \le t \le 1.0$. The initial data is a function $g\colon \mathbb{T} \to [500, 700]$ whose values were randomly sampled from the uniform distribution on that interval. The curved surface between them is linearly interpolated to help visualize the structure of the solution

Let $\mathbb{T}$ be a time scale whose graininess map $\mu\colon \mathbb{T} \to [0,\infty)$ is an injective map, i.e. for all $t, s \in \mathbb{T}$, if $\mu(t) = \mu(s)$, then $t = s$. We further simplify assumptions by assuming $\mathbb{T}$ can be expressed as a unilateral or bilateral sequence. Throughout, we interpret the contour C as in Theorem 3.3. We will use the bijection π defined by $\pi(x_k) = k$ so $\pi^{-1}(k) = x_k$, where k is an integer.

Lemma 4.5. *Let $\mathbb{T} = \{x_0, x_1, \ldots\}$ with $x_0 < x_1 < \ldots$. If $x^*, x \in \mathbb{T}$ with $x < x^*$, then*

$$\mathcal{F}^{-1}\left\{v_t\right\}(x;x^*) = \prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{i\mu(x_k)}\right) \times \sum_{j=\pi(x)}^{\pi(x^*)-1} e^{\frac{t}{\mu(x_j)^2}} \prod_{k=\pi(x),k\neq j}^{\pi(x^*)-1}\left(\frac{1}{\frac{i}{\mu(x_j)}-\frac{i}{\mu(x_k)}}\right)\mathrm{d}z,$$

where $v_t(z) = e^{-z^2 t}$.

Proof. Since exponential functions on this type of time scale reduce to a product, we compute

$$\begin{aligned}
\mathcal{F}^{-1}\left\{v_t\right\}(x;x^*) &= \frac{1}{2\pi i}\int_C e^{-\cdot^2 t} e_{iz}(x,x^*)\mathrm{d}z = \frac{1}{2\pi i}\int_C \frac{e^{-z^2 t}}{e_{iz}(x^*,x)}\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_C e^{-z^2 t}\prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{1+\mu(x_k)iz}\right)\mathrm{d}z \\
&= \frac{1}{2\pi i}\prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{i\mu(x_k)}\right)\int_C e^{-z^2 t}\prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{z-\frac{i}{\mu(x_k)}}\right)\mathrm{d}z.
\end{aligned}$$

Since the graininess is injective, the integrand contains a product of simple poles, so we algebraically rearrange as

$$\begin{aligned}
&\mathcal{F}^{-1}\left\{v_t\right\}(x;x^*) \\
&= \prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{i\mu(x_k)}\right)\sum_{j=\pi(x)}^{\pi(x^*)-1}\operatorname*{Res}_{z=\frac{i}{\mu(x_j)}} e^{-z^2 t}\prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{z-\frac{i}{\mu(x_k)}}\right)\mathrm{d}z \\
&= \prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{i\mu(x_k)}\right)\sum_{j=\pi(x)}^{\pi(x^*)-1}\operatorname*{Res}_{z=\frac{i}{\mu(x_j)}} \frac{e^{-z^2 t}\prod_{k=\pi(x),k\neq j}^{\pi(x^*)-1}\left(\frac{1}{z-\frac{i}{\mu(x_k)}}\right)\mathrm{d}z}{z-\frac{i}{\mu(x_j)}}.
\end{aligned}$$

With this form, we easily resolve the residues in the sum by substituting $z = \frac{i}{\mu(x_j)}$ into the numerators, yielding

$$\begin{aligned}
&\mathcal{F}^{-1}\left\{v_t\right\}(x;x^*) \\
&= \prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{i\mu(x_k)}\right)\sum_{j=\pi(x)}^{\pi(x^*)-1} e^{-\left(\frac{i}{\mu(x_j)}\right)^2 t}\prod_{k=\pi(x),k\neq j}^{\pi(x^*)-1}\left(\frac{1}{\frac{i}{\mu(x_j)}-\frac{i}{\mu(x_k)}}\right)\mathrm{d}z \\
&= \prod_{k=\pi(x)}^{\pi(x^*)-1}\left(\frac{1}{i\mu(x_k)}\right)\sum_{j=\pi(x)}^{\pi(x^*)-1} e^{\frac{t}{\mu(x_j)^2}}\prod_{k=\pi(x),k\neq j}^{\pi(x^*)-1}\left(\frac{1}{\frac{i}{\mu(x_j)}-\frac{i}{\mu(x_k)}}\right)\mathrm{d}z,
\end{aligned}$$

completing the proof. □

Corollary 4.6. *If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then for $x < x^*$,*

$$\mathcal{F}^{-1}\left\{v_t\right\}(x;x^*) = \prod_{k=\log_q(x)}^{\log_q(x^*)-1} \left(\frac{1}{i(q-1)q^k}\right)$$
$$\times \sum_{j=\log_q(x)}^{\log_q(x^*)-1} e^{\frac{t}{(q-1)^2q^{2j}}} \prod_{k=\log_q(x),k\neq j}^{\pi(x^*)-1} \left(\frac{1}{\frac{i}{(q-1)q^j} - \frac{i}{(q-1)q^k}}\right) \mathrm{d}z,$$

where $v_t(z) = e^{-z^2t}$ and $\log_q$ denotes the logarithm with base q.

The proof of the following theorem is the same as the proof of Theorem 4.3.

Theorem 4.7. *If $\mathbb{T}$ is a countable time scale with injective graininess representable by a bilateral or unilateral sequence, $t \in [0,\infty)$, and g is of double exponential order (α,γ), then (1.1) has solution $u(x,t) = \left(\mathcal{F}^{-1}\left\{v_t\right\} * g\right)(x;x^*)$, where $v_t(z) = e^{-z^2t}$.*

We now present the special case of $\mathbb{T}$ being a quantum time scale.

Example 4.8. From the definition of the Δ-derivative, we express

$$\begin{aligned}
u^{\Delta^2_{xx}}(t,x) &= \frac{\mu(x)\mu(q^2x)u(t,q^2x) - \big(\mu(x)+\mu(qx)\big)u(t,qx) + \mu(qx)u(t,x)}{\mu(x)^2\mu(qx)} \\
&= \frac{q^2(q-1)xu(t,q^2x) - \big((q-1)x + q(q-1)x\big)u(t,qx) + q(q-1)xu(t,x)}{q(q-1)^2x^2} \\
&= \frac{q^2u(t,q^2x) - \big(1+q\big)u(t,qx) + qu(t,x)}{q(q-1)x}.
\end{aligned}$$

Simplifying this expression with the definition of μ on this time scale yields

$$u^{\Delta^2_{xx}}(t,x) = \frac{u(t,q^2x) - (1+q)u(t,qx) + qu(t,x)}{q(q-1)^2x^2}.$$

So again approximating $\dfrac{\partial u}{\partial t}$ with (4.5), we ultimately arrive at the recurrence

$$u(t+h,x) = u(t,x) + \frac{h}{q(q-1)^2x^2}\Big[u(t,q^2x) - (1+q)u(t,qx) + qu(t,x)\Big], \tag{4.6}$$

which we use to create Figure 2. Our comment on the limitation of this scheme from Example 4.4 also applies here.

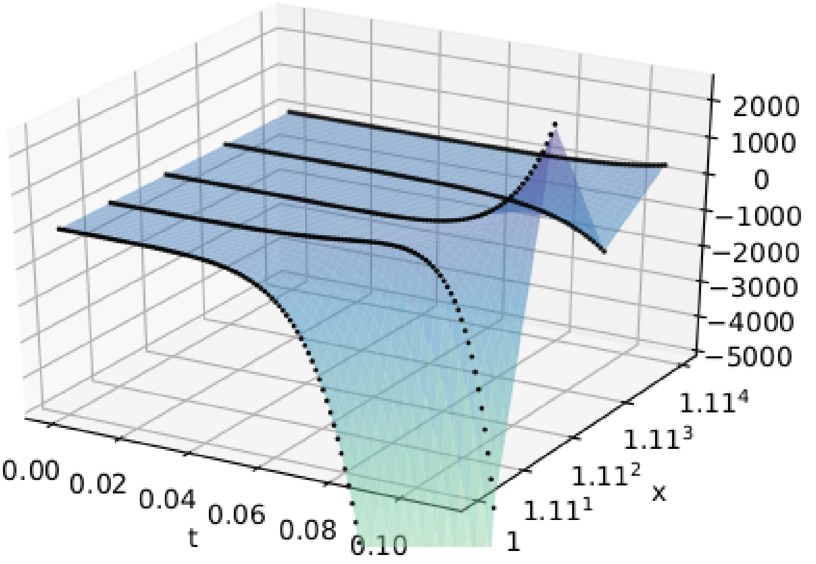

Fig. 2. The q-case with $q = 1.11$ and $h = 0.001$. The black lines use (4.6) to approximate a solution of (1.1) at each $x \in \mathbb{T}$ for $0 \leq t \leq 0.1$. The initial data is a function $g \colon \mathbb{T} \to [-0.3, 0.3]$ whose values were randomly sampled from the uniform distribution on that interval. The colored regions between them is a surface made of linear interpolations to help visualize the structure of the solution

5. CONCLUSIONS

In this work we used operator techniques to find explicit solutions to the heat equation with a space variable defined on certain time scales. A major goal of future research will be to extend these techniques to arbitrary time scales. The calculations here give insight into the main hurdles of extending our technique further: understanding the contour integral for an arbitrary time scale, for which the recent work [20] gives hints of the proper approach and loosening the injectivity requirement leads to residues that are not simple substitutions but rather require differentiation with respect to the frequency variable for each element of the range of μ.

The question of generalizing the time variable in (1.1) to more general time scales is certainly of interest – in that case, applying the Fourier transform to the spatial variable would lead to a dynamic equation instead of (4.1), meaning inversion would involve a function of form $e_{-z^2}(t, s)$. Calculation of residues related to this function would generally involve differentiation of it with respect to z, which can be approached using techniques from [21]. Of course different types of problems from the IVP (1.1) to boundary value problems and IBVPs like in [19] are also of interest. Even generalizing the time scale for the space variable to allow limit points could be of some interest in the wider literature. Finally, extending our technique to other types of equations such as wave equations or more general lattice differential equations could be pursued.

More broadly, a better understanding of the inversion contour integral for Fourier and related transforms is likely to give insight to other topics in the time scales. We highlight two such topics here: first, fractional calculus on time scales has a variety of foundations but two of the most popular are the axiomatic framework defining a general class of h_α monomials [27] and the other defines the fractional operator directly as a certain inverse Laplace transform [2]. In the former case, the existence of functions satisfying the axioms could be established with contour integral techniques and in the latter case, the connection is direct. The second topic could be special functions on time scales, for which a certain contour integral inversion was proposed in the thesis [7] for extending large classes of special functions, such as hypergeometric series, to arbitrary time scales.

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