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SOLUTIONS FOR A NONHOMOGENEOUS $p\&q$-LAPLACIAN PROBLEM VIA VARIATIONAL METHODS AND SUB-SUPERSOLUTION TECHNIQUE

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Abstract. In this paper it is obtained, through variational methods and sub-supersolution arguments, existence and multiplicity of solutions for a nonhomogeneous problem which arise in several branches of science such as chemical reactions, biophysics and plasma physics. Under a general hypothesis it is proved an existence result and multiple solutions are obtained by considering an additional natural condition.

Keywords: $p\&q$-Laplacian operator, nonhomogeneous operator, sub-supersolutions, existence, multiplicity.

Mathematics Subject Classification: 35A15, 35J60.

1. INTRODUCTION

The goal of this manuscript, which is motivated by [7, 8, 12, 17], consists in study nonnegative solutions for the nonhomogeneous $p\&q$-Laplacian type problem

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = b(x)u^{\alpha-1} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $2 \leq p < N, N \geq 3$, $f : \Omega \times [0,+\infty) \to \mathbb{R}$ is a continuous function, $1 \leq \alpha$ and $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function with:

(A_1) there are constants $k_i > 0$, $i = 1,\ldots,4$, and $2 \leq p \leq q < N$ satisfying

$$k_1 t^p + k_2 t^q \leq a(t^p)t^p \leq k_3 t^p + k_4 t^q, \quad \text{for all } t \geq 0,$$

(A_2) the function $t \mapsto a(t^p)t^{p-2}$ is increasing,
(A_3) the function $t \mapsto A(t^p)$ is strictly convex, where $A(t) = \int_0^t a(s)\,ds$,

(A_4) there are constants θ and μ such that $\theta \in (q, q^*)$, $\mu p < q < \theta$ and

$$a(t)\, t \leq \mu A(t), \quad \text{for all} \quad t \geq 0.$$

Regarding the nonlinearity in (1.1) it will be considered the hypotheses below.

(H) $b \in L^\infty(\Omega)$ and $b(x) > 0$ a.e. in Ω;
(f_1) There is $\delta > 0$ with $f(x,t) \geq b(x)(1 - t^{\alpha-1})$, for all $0 \leq t \leq \delta$, a.e. in Ω;
(f_2) There exists $r > 1$ such that $|f(x,t)| \leq b(x)(1 + |t|^{r-1})$, for all $t \geq 0$, a.e. in Ω.

Since Ω is bounded and $p < q$ we have that $W_0^{1,q}(\Omega) \cap W_0^{1,p}(\Omega) = W_0^{1,q}(\Omega)$. Thus, in order to obtain solutions for (1.1) it will be considered the space $W_0^{1,q}(\Omega)$ equipped with the norm

$$\|u\|_{1,q} = \Big(\int_\Omega |\nabla u|^q dx \Big)^{1/q}.$$

We will consider that $u \in W_0^{1,q}(\Omega)$ is a weak solution for (1.1) if $u(x) \geq 0$ a.e. in Ω and

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u \nabla \varphi = \int_\Omega a(x) u^{\alpha-1}\varphi + \int_\Omega f(x,u)\varphi, \quad \text{for all } \varphi \in W_0^{1,q}(\Omega).$$

By considering $\|\cdot\|_\infty$ the norm in the space $L^\infty(\Omega)$ it is possible, via sub-supersolutions and minimization arguments in convex sets, to prove the existence result below.

Theorem 1.1. *Suppose that the hypotheses* (H), (A_1)–(A_4) *and* (f_1)–(f_2) *hold. The problem* (1.1) *has a solution for* $\|b\|_\infty$ *small enough.*

If one consider the additional condition

(f_3) There are $r < q^* := \dfrac{Nq}{(N-q)}$ and $\alpha \leq q$ or it holds simultaneously that $q < \alpha$ and there is $t_0 > 0$ with

$$0 < \theta F(x,t) \leq t f(x,t), \quad \text{a.e. in } \Omega \text{ for all } t \geq t_0,$$

where θ is provided in (A_4) and $F(x,t) = \int_0^t f(x,\tau)\, d\tau$.

It is possible to obtain the next result.

Theorem 1.2. *Suppose that the hypotheses* (H), (A_1)–(A_4) *and* (f_1)–(f_3) *hold. The problem* (1.1) *admits at least two nonnegative weak solutions for* $\|b\|_\infty$ *small enough.*

Partial Differential Equations with the nonhomogeneous $p\&q$-Laplacian type operator in (1.1) arose due do its applicability in several relevant models in chemical reaction design, biophysics and plasma physics which are driven by the parabolic reaction-diffusion problem

$$u_t = \operatorname{div}[(|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u] + c(x,u).$$

The relevance of these equations stems from their ability to accurately describe complex phenomena that occur in these fields. For example, chemical reaction design involves the study of how chemical reactions occur and how they can be manipulated to achieve desired outcomes. Similarly, biophysics deals with the study of the physical processes that occur within living organisms. Many of these processes involve the diffusion of particles, such as molecules and ions, which can be described using parabolic reaction-diffusion systems. Plasma physics involves the study of ionized gases and their interactions with electromagnetic fields. Such field has applications in a wide range of areas, from fusion energy to space propulsion. Lastly, plasma physics involves the study of ionized gases and their interactions with electromagnetic fields. This field has applications in a wide range of areas, from fusion energy to space propulsion.

In the described applications, the function u provides the concentration, the divergent term allows to obtain information regarding the diffusion considered in the model; the term c describes mathematically the reaction and provide information related to the source and the the loss in the process. Typically in applications in Chemistry and Biology, the reaction function $c(x,u)$ has polynomial behavior in the term u and has variable coefficients. For more details on p&q-Laplacian problems we refer to [2–6, 9, 13–16, 20].

Regarding the mathematical viewpoint, the main motivations for (1.1) are [12] and [18] where it was considered the operator in (1.1) and an equation was studied in the case $\alpha \equiv 2$ with an anisotropic operator respectively. We also mention the references [7, 17]. With respect to the obtained results, we highlight that Theorem 1.1 permits to consider nonlinearities with arbitrary growth. Such result allows to consider problems with supercritical and critical growth, which are particularly challenging aspects in the study of nonlinear elliptic equations. Theorem 1.2 asserts the existence of multiple solutions for a wide class of nonlinearities problems, subject to certain natural additional conditions. The results mentioned represent an improvement, particularly for the Laplacian operator, over the recent reference [18]. The approach considered to study (1.1) is based on sub-supersolution arguments and variational methods. An important mathematical difficulty that arise in (1.1) is the lack of homogeneity of the operator which imply that sub-supersolution techniques cannot be directly applied. We point out that that the results of this manuscript allow to consider the nonlinearity

$$m(x,t) = \begin{cases} (1 - t^{\alpha-1})b(x), & 0 \le t \le s_0, \\ ((1 - s_0^{\alpha-1}) + (t - s_0)^{r-1})b(x), & t > s_0, \end{cases}$$

in the right-hand side of (1.1), where $0 < s_0 < 1$ is fixed and $1 \le \alpha < q^\star$ with $\alpha \neq q$ and $\mu p < q < r$ where μ, p and q are provided in (A_4).

The operator considered in (1.1) allows to consider a wide class of problems.

Example 1.3. If $a \equiv 1, q = p$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 1$ the problem (1.1) becomes

$$\begin{cases} -\Delta_p u = b(x)u^{\alpha-1} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

Example 1.4. Considering $a(t) = 1 + t^{\frac{q-p}{p}}$ and $k_1 = k_2 = k_3 = k_4 = 1$ we have

$$\begin{cases} -\Delta_p u - \Delta_q u = b(x)u^{\alpha-1} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Example 1.5. Choosing $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, $q = p$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 2$, we obtain the problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right) = b(x)u^{\alpha-1} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Example 1.6. In the case $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, $k_1 = k_2 = k_4 = 1$ and $k_3 = 2$ we obtain

$$\begin{cases} -\Delta_p u - \Delta_q u - \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right) = b(x)u^{\alpha-1} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2. AUXILIARY RESULTS

In what follows we provide some results which will be needed for our purposes.

Lemma 2.1 ([8, Lemma 2.5])**.** *If (A_1) holds and $u \in W_0^{1,q}(\Omega)$ is a solution of the problem*

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $f \in L^r(\Omega)$ with $r > q^*/(q^* - q)$, then it holds that $u \in L^\infty(\Omega)$ with $\|u\|_\infty \le C\|f\|_r^{\frac{1}{q-1}}|\Omega|^\iota$, where $|\cdot|$ is the Lebesgue measure and ι and C are constants which does not depend on the function u.

Lemma 2.2 ([10, Lemma 2.1])**.** *Suppose that (A_1)–(A_2) hold. If $h \in (W_0^{1,q}(\Omega))'$, then the problem*

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has an unique solution $u \in W_0^{1,q}(\Omega)$.

Lemma 2.3 ([11, Lemma 2.2])**.** *Suppose that (A_1)–(A_2) hold. If $u, v \in W_0^{1,q}(\Omega)$ satisfy*

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) \le -\operatorname{div}(a(|\nabla v|^p)|\nabla u|^{p-2}\nabla u) & \text{in } \Omega, \\ u \le v & \text{on } \partial\Omega, \end{cases}$$

then $u(x) \le v(x)$ a.e. in Ω.

3. PROOF OF THEOREM 1.1

Below we present the notion of sub-supersolution that will be considered and a lemma related with such functions.

We say that $(\underline{u}, \overline{u}) \in (W_0^{1,q}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,q}(\Omega) \cap L^\infty(\Omega))$ is a sub-supersolution pair for (1.1) if $\underline{u}(x) \leq \overline{u}(x)$ a.e in Ω and the inequalities

$$\begin{aligned} \int_\Omega a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\varphi &\leq \int_\Omega b(x)\underline{u}^{\alpha-1}\varphi + \int_\Omega f(x,\underline{u})\varphi, \\ \int_\Omega a(|\nabla \overline{u}|^p)|\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla\varphi &\geq \int_\Omega b(x)\overline{u}^{\alpha-1}\varphi + \int_\Omega f(x,\overline{u})\varphi, \end{aligned} \tag{3.1}$$

hold for all $\varphi \in W_0^{1,q}(\Omega)$, with $\varphi(x) \geq 0$ a.e in Ω.

The result below assures the existence of a sub-supersolution pair for (1.1) with $\|b\|_\infty$ small enough.

Lemma 3.1. *Consider the hypotheses* (H) *and* (f_1)–(f_2). *Then, there is* $\eta > 0$ *such that* (1.1) *has a sub-supersolution pair* $(\underline{u}, \overline{u}) \in (W_0^{1,q}(\Omega) \cap L^\infty(\Omega)) \times (W_0^{1,q}(\Omega) \cap L^\infty(\Omega))$, *satisfying* $\|\underline{u}\|_\infty \leq \delta$, *where* δ *was provided in* (f_1), *for* $\|b\|_{L^\infty(\Omega)} < \eta$.

Proof. From Lemmas 2.1 and 2.2 there are unique nonnegative functions $\underline{u}, \overline{u} \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ for the problems

$$\begin{cases} -\operatorname{div}(a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}) = b(x) & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \partial\Omega \end{cases}$$
$$\begin{cases} -\operatorname{div}(a(|\nabla \overline{u}|^p)|\nabla \overline{u}|^{p-2}\nabla \overline{u}) = 1 + b(x) & \text{in } \Omega, \\ \overline{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

respectively such that $\|\underline{u}\|_\infty \leq C\|b\|_\infty^{\frac{1}{q-1}}|\Omega|^\iota$, $\|\overline{u}\|_\infty \leq C\|1+b\|_\infty^{\frac{1}{q-1}}|\Omega|^\iota$, where the constants C and ι are provided by Lemma 2.1. Moreover, it is possible to choose $\eta > 0$, depending only on C and ι such that $\|\underline{u}\|_\infty \leq \delta/2$ and $\|b\|_\infty \max\{\|\overline{u}\|_\infty^{\alpha-1}, \|\overline{u}\|_\infty^{r-1}\} \leq 1$ for $\|b\|_\infty < \eta$. By using Lemma 2.3 and (3.2) we obtain that $0 < \underline{u}(x) \leq \overline{u}(x)$ a.e. in Ω.

Consider a function $\varphi \in W_0^{1,q}(\Omega)$ with $\varphi(x) \geq 0$ a.e. in Ω. Combining (f_1) and (3.2) it follows that

$$\begin{aligned} &\int_\Omega a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla\varphi - \int_\Omega b(x)\underline{u}^{\alpha-1}\varphi - \int_\Omega f(x,\underline{u})\varphi \\ &\leq \int_\Omega b(x)\varphi - \int_\Omega b(x)\underline{u}^{\alpha-1}\varphi - \int_\Omega (1-\underline{u}^{\alpha-1})b(x)\varphi = 0. \end{aligned}$$

From (f_2), (3.2) and the choice of η we have

$$\begin{aligned}
&\int_\Omega a(|\nabla \overline{u}|^p)|\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla \varphi - \int_\Omega b(x)\overline{u}^{\alpha-1}\varphi - \int_\Omega f(x,\overline{u})\varphi \\
&\geq \int_\Omega (1+b(x))\varphi - b(x)\overline{u}^{\alpha-1}\varphi - b(x)(1+|\overline{u}|^{r-1})\varphi \\
&\geq \int_\Omega (1-\|b\|_\infty \max\{\|\overline{u}\|_\infty^{\alpha-1}, \|\overline{u}\|_\infty^{r-1}\})\varphi \geq 0,
\end{aligned}$$

By using Lemma for all $\varphi \in W_0^{1,q}(\Omega)$ with $\varphi(x) \geq 0$ a.e. in Ω. $\square$

Proof of Theorem 1.1. Consider $\underline{u}, \overline{u} \in W_0^{1,q}(\Omega)$ the functions given in Lemma 3.2. Define the truncated function below

$$w(x,t) := \begin{cases} f(x,\overline{u}(x)) + \overline{u}^{\alpha-1}b(x), & t > \overline{u}(x), \\ f(x,t) + t^{\alpha-1}b(x), & \underline{u}(x) \leq t \leq \overline{u}(x), \\ f(x,\underline{u}(x)) + \underline{u}^{\alpha-1}b(x), & t < \underline{u}(x), \end{cases} \tag{3.3}$$

for $(x,t) \in \Omega \times \mathbb{R}$ and the equation

$$\begin{cases} -\mathrm{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = w(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

The C^1 energy functional associated to the previous problem is

$$J(u) := \frac{1}{p}\int_\Omega A(|\nabla u|^p) - \int_\Omega W(x,u), \quad u \in W_0^{1,q}(\Omega),$$

with $W(x,t) := \int_0^t w(x,\tau)d\tau$. From (A_3), (A_5) and the boundedness of h we ontain that J is a sequentially weakly lower semicontinuous and coercive functional. Define

$$K := \{u \in W_0^{1,q}(\Omega); \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e in } \Omega\}.$$

We have that K is convex and closed, which provides that it is weakly closed in $W_0^{1,q}(\Omega)$. Therefore $J\big|_{\mathcal{K}}$ attains its infimum in some $u_0 \in K$. A similar reasoning with respect to [19, Theorem 2.4] provides that $J'(u_0) = 0$, which finishes the proof. $\square$

4. PROOF OF THEOREM 1.2

Consider $\underline{u} \in W_0^{1,q}(\Omega)$ the function provided in Lemma 3.2, the function

$$z(x,t) = \begin{cases} f(x,t) + t^{\alpha-1}b(x), & t \geq \underline{u}(x), \\ f(x,\underline{u}(x)) + \underline{u}(x)^{\alpha-1}b(x), & t < \underline{u}(x) \end{cases}$$

and the problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)u = z(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

whose the associated C^1 energy functional is

$$L(u) := \frac{1}{p}\int_\Omega A(|\nabla u|^p) - \int_\Omega Z(x,u), \quad u \in W_0^{1,q}(\Omega), \tag{4.2}$$

with $Z(x,t) := \int_0^t z(x,\tau)d\tau$.

Lemma 4.1. *If $q < \alpha$ or it holds simultaneously that $\alpha \le q$ and $\|b\|_\infty$ is small enough, then the functional L satisfies the Palais–Smale condition.*

Proof. Let $(u_n) \subset W_0^{1,q}(\Omega)$ be a sequence with $L'(u_n) \to 0$ and such that $L(u_n) \to c$ for some $c \in \mathbb{R}$.

Initially it will be considered the case $q < \alpha$. Note that (f_3) is also true with $\theta' > 0$ satisfying $q < \theta' < \min\{\theta, \alpha\}$. Thus, from (f_2)–(f_3), the boundedness of $\underline{u}$, the Sobolev embeddings $W_0^{1,q}(\Omega) \hookrightarrow L^1(\Omega), W_0^{1,q}(\Omega) \hookrightarrow L^\alpha(\Omega)$ and $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$, we obtain positive constants C_i, $i = 1, 2, 3$, such that

$$\begin{aligned} C_1 + C_2\|u_n\|_{1,q} &\ge L(u_n) - \frac{1}{\theta'}L'(u_n)u_n \\ &\ge \left(\frac{1}{\mu p} - \frac{1}{\theta'}\right)k_2\|u_n\|_{1,q}^q + \int_{\{u_n \ge \underline{u}\}} \left(\frac{1}{\theta'} - \frac{1}{\alpha}\right) b(x)u_n^\alpha - C_3\|u_n\|_{1,q} \\ &\ge \left(\frac{1}{\mu p} - \frac{1}{\theta'}\right)k_2\|u_n\|_{1,q}^q - C_3\|u_n\|_{1,q}, \end{aligned}$$

which imply the boundedness of (u_n) in $W_0^{1,q}(\Omega)$.

With respect to the case $\alpha < q$ note that (f_2), (f_3), the boundedness of $\underline{u}$ and the Sobolev embeddings provide

$$\begin{aligned} C_1 + C_2\|u_n\|_{1,q} &\ge L(u_n) - \frac{1}{\theta}L'(u_n)u_n \\ &\ge \left(\frac{1}{\mu p} - \frac{1}{\theta}\right)k_2\|u_n\|_{1,q}^q - C_4\int_\Omega |u_n|^\alpha - C_3\|u_n\|_{1,q} \\ &\ge \left(\frac{1}{\mu p} - \frac{1}{\theta}\right)k_2\|u_n\|_{1,q}^q - C_5\|b\|_\infty\|u_n\|_{1,q}^\alpha - C_3\|u_n\|_{1,q}, \end{aligned} \tag{4.3}$$

for positive constants C_i, $i = 1, \dots, 5$, which do not depend on $n \in \mathbb{N}$. Since $\alpha < q$, it follows the boundedness of (u_n) in $W_0^{1,q}(\Omega)$. If $\alpha = q$, the boundedness of the sequence (u_n) in $W_0^{1,q}(\Omega)$ follows by considering $\|b\|_\infty$ small enough such that $((\mu p)^{-1} - \theta^{-1} - C_5\|b\|_\infty) > 0$.

Thus in both cases we obtain up to a subsequence, that

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,q}(\Omega), \\ u_n(x) \to u(x) & \text{a.e. in } \Omega, \\ u_n \to u & \text{in } L^s(\Omega), \quad 1 \le s < q^* \end{cases} \tag{4.4}$$

for some $u \in W_0^{1,q}(\Omega)$. Applying (4.4), the Lebesgue Dominated Convergence Theorem and the inequality (see [12, p. 515])

$$C|x-y|^p \le \langle a(|x|^p)x - a(|y|^p)y, x-y\rangle \quad \text{for all } x, y \in \mathbb{R}^N, \tag{4.5}$$

where $C > 0$ is a constant which does not depend on $x, y \in \mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$, we get

$$C\|\nabla u_n - \nabla u\|_{1,p}^p \le \int_\Omega \langle a(|\nabla u|^p)\nabla u - a(|\nabla u_n|^p)\nabla u_n, \nabla u_n - \nabla u\rangle = o_n(1),$$

which imply $u_n \to u$ in $W_0^{1,p}(\Omega)$. From (A_1) we obtain that

$$k_1 t^p + \frac{p}{q} k_2 t^q \le A(t^p) \le k_3 t^p + \frac{p}{q} k_4 t^q, \quad \text{for all } t \ge 0. \tag{4.6}$$

We have, up to a subsequence, that $|\nabla u_n| \le h$ for some $h \in L^p(\Omega)$. From (A_2)–(A_3) it follows that

$$A(|\nabla u_n - \nabla u|^p) \le C(A(|h|^p) + A(|\nabla u|^p)$$

for some constant $C > 0$ and all $n \in \mathbb{N}$. Thus, from Lebesgue Dominated Convergence Theorem we obtain that

$$\int_\Omega A(|\nabla u_n - \nabla u|^p) \to 0. \tag{4.7}$$

We have from (4.6) that

$$k_1 \int_\Omega |\nabla u_n - \nabla u|^p + \frac{p}{q} k_2 \int_\Omega |\nabla u_n - \nabla u|^q \le \int_\Omega A(|\nabla u_n - \nabla u|^p).$$

Thus, it follows from (4.7) that $u_n \to u$ in $W_0^{1,q}(\Omega)$, which proves the result. □

The Mountain Pass Geometry is obtained below for the functional L defined in (4.2).

Lemma 4.2. *For $\|b\|_{L^\infty(\Omega)}$ small enough the claims below are true.*

(i) *There are constants σ and R with $R > \|\underline{u}\|$, such that*

$$L(\underline{u}) < 0 < \sigma \le \inf_{u \in \partial B_R(0)} L(u).$$

(ii) *There is $e \in W_0^{1,q}(\Omega) \setminus \overline{B_{2R}(0)}$ satisfying $L(e) < \sigma$.*

Proof. Since $p > 1$, it follows that $L(\underline{u}) < 0$. Let $u \in W_0^{1,q}(\Omega)$ be a function. From the Sobolev embeddings $W_0^{1,q}(\Omega) \hookrightarrow L^\alpha(\Omega), W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ and $W_0^{1,q}(\Omega) \hookrightarrow L^1(\Omega)$ we have

$$L(u) \geq K_1\|u\|_{1,q}^q - K_2\|u\|_{1,q} - K_3\|b\|_\infty(\|u\|_{1,q}^\alpha + \|u\|_{1,q}^r),$$

for constants $K_i > 0, i = 1, 2, 3$. If necessary, consider a smaller $\|b\|_\infty$ such that $\|\underline{u}\|_{1,q} < 1$, which is possible by applying the function $\varphi = \underline{u}$ in the first inequality of (3.1) and using Lemma 2.1.

Fix $\sigma > 0$ and consider $R \geq 1 > \|\underline{u}\|$ large enough satisfying $K_1R^q - K_2R \geq 2\sigma$. Decreasing $\|b\|_\infty$ such that $K_2\|b\|_\infty(R^\alpha + R^r) \leq \sigma$, we obtain $L(u) \geq \sigma$ for all $u \in W_0^{1,q}(\Omega)$ with $\|u\| = R$, which provides (i).

With respect to the second part note that (f_3) and the inequality $q < \theta$ imply that

$$L(t\underline{u}) \leq C_1t^q - C_2t^\alpha - C_3t^\theta + C_4 < 0$$

for some positive constants C_i, $i = 1, \dots, 4$, and $t > 0$ large enough. □

Proof of Theorem 1.2. Consider $\underline{u}, \overline{u} \in W_0^{1,q}(\Omega)$ the functions given in Lemma 3.2 and consider $u_1 \in W_0^{1,q}(\Omega)$ the solution of the problem (1.1) obtained in Theorem 1.1. Recall that u_1 minimizes $J\big|_{\mathcal{K}}$, where

$$K = \{u \in W_0^{1,q}(\Omega); \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. in } \Omega\}. \tag{4.8}$$

Applying Lemma 4.1 and 4.2 we obtain that the conditions of the Mountain Pass Theorem [1, Theorem 2.1] are satisfied. Therefore,

$$l := \inf_{\gamma\in\Lambda} \max_{t\in[0,1]} L(\lambda(t)),$$

where

$$\Lambda := \{\lambda \in C([0, 1], W_0^{1,q}(\Omega)); \lambda(0) = \underline{u}, \lambda(1) = e\},$$

is a critical value of the functional L, i.e., $L'(u_2) = 0$ and $L(u_2) = l$, for $u_2 \in W_0^{1,q}(\Omega)$. The definition of w provided in (3.3) imply $J(u) = L(u)$ for

$$u \in \{h \in W_0^{1,q}(\Omega); 0 \leq h(x) \leq \overline{u}(x) \text{ a.e. in } \Omega\}.$$

Hence $L(\underline{u}) = J(\underline{u})$ and $J(u_1) = L(u_1) = \inf_{u\in\mathcal{K}} J(u)$. Recall from the proof of Lemma 4.2 that $L(\underline{u}) < 0$. Thus, it follows that if $u_2(x) \geq \underline{u}(x)$ a.e. in Ω, then we will have that (1.1) has two solutions $u_1, u_2 \in W_0^{1,q}(\Omega)$ with $L(u_1) \leq L(\underline{u}) < 0 < \beta \leq l = L(u_2)$, with $\sigma > 0$ provided in Lemma 4.2.

Note that the inequality $u_2(x) \geq \underline{u}(x)$ a.e. in Ω holds. In fact, by applying in (4.1) the test function $(\underline{u} - u_2)^+ \in W_0^{1,q}(\Omega)$ we get

$$\begin{aligned}
&\int_\Omega a(|\nabla u_2|^p)|\nabla u_2|^{p-2}\nabla u_2 \nabla(\underline{u} - u_2)^+ \\
&= \int_{\{u_2<\underline{u}\}} z(x,u_2)(\underline{u} - u_2)^+ \\
&= \int_{\{u_2<\underline{u}\}} (f(x,\underline{u}(x)) + b(x)\underline{u}(x)^{\alpha-1})(\underline{u} - u_2)^+ \\
&\geq \int_\Omega a(|\nabla \underline{u}|^p)|\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla(\underline{u} - u_2)^+.
\end{aligned}$$

Using (4.5), (4.6) and the previous inequality we have $(\underline{u} - u_2)^+(x) = 0$ a.e. in Ω, which finishes the proof of the claim. □

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