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RADIAL SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATION WITH NONLINEAR NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this article, we prove existence of radial solutions for a nonlinear elliptic equation with nonlinear nonlocal boundary conditions. Our method is based on some fixed point theorem in a cone.

Keywords: nonlocal boundary value problem, radial solutions, elliptic equation, the Krasnosielskii fixed point theorem in cone.

Mathematics Subject Classification: 34B10, 34B15, 47H11.

1. INTRODUCTION

The paper is concerned with the existence of positive radial solutions for the nonlinear Poisson equation with some some nonlocal condition, namely

$$-\Delta u = f(|x|, u), \quad u|_{\partial\Omega} = \int\limits_{\Omega} K(|x|, |y|)h\big(u(y)\big)\,\mathrm{d}y, \tag{1.1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, $\Omega = B(0,b) \setminus \overline{B}(0,a) \subset \mathbb{R}^n$, $0 < a < b$, $B(x_0, r)$ is the open ball of radius $r > 0$ with center at point $x_0 \in \mathbb{R}^n$ corresponding to the Euclidean norm. Moreover, $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ is Carathédory function (i.e. $f(\cdot, u)$ is measurable for every $u \in \mathbb{R}_+$ and $f(r, \cdot)$ is continuous for almost every $r \in [a,b]$), $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and $K : [a,b] \times [a,b] \to \mathbb{R}_+$ is integrable with respect to the first variable. Throughout this paper, $\mathbb{R}_+$ denotes $[0, +\infty)$.

The Poisson equation has been investigated for many years due to its significance in many applications, especially in physics. Many phenomena which are modelled by Poisson equation and possess the property of physical symmetry, require searching for the so-called radial solutions. A lot of papers ([1, 3, 6–8, 10, 18, 22–24]) have been published on the existence of radial solutions for non-linear elliptic equations with classic boundary conditions (e.g. those of Dirichlet, Neumann or Robin). Typically, the methods used in this research include the shooting method, the mountain pass theorem and other topological-order methods, e.g. based on the cone expansion and compression theorem.

While investigating the existence of radial solutions using an appropriate substitution, the boundary conditions become two-point conditions, i.e. a special case of multiple-point conditions of the form $F(u(t_1),\dots,u(t_k)) = 0$. If the boundary conditions refer to the values of the unknown function on the whole domain (integral conditions), then they are called the nonlocal conditions. One of the first papers investigating the nonlocal-type conditions was the one of Whyburn [30]. The integral nonlocal conditions have proved to be applicable in many areas of physic such as thermoelasticity (compare in [4] and [5]).

The research in the area of nonlocal conditions for ordinary and partial differential equations has been continuously expanding. Many papers have been published, focusing both on ordinary ([2,14,15,19,20,25–29]) and partial ([3,9,12,13,21]) equations. Radial solutions for nonlocal elliptic equations have been investigated in [6,7] and [8], however, in these papers the boundary conditions were the classic ones and the nonlocality underlay just the equation itself.

Nonlinear nonlocal conditions can in turn be found in [12] and [13]. In [12], the following problem is investigated:

$$L_i u_i(x) = \lambda_i f_i(x,u(x)),\ x\in\Omega,\quad B_i u(x) = \eta_i h_i[u],\ x\in\partial\Omega,\quad i=1,\dots,n,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain with sufficiently regular boundary and L_i is a strongly uniformly elliptic operator. The nonlocal conditions studied in above problem are general ones, but the author looks for weak solutions of this problem. Therefore our results are different kind and are not comparable to mentioned above.

2. AUXILIARY PROBLEM

Since we are looking for radial solutions $u(x) = U(|x|)$, where $U\colon [a,b] \to \mathbb{R}$, the problem (1.1) is reduced to the following boundary value problem for ordinary differential equation

$$\begin{cases} -U''(r) - \dfrac{n-1}{r}U'(r) = f(r,U(r)), \\ U(a) = \omega_n \displaystyle\int_a^b K(a,r)r^{n-1}h(U(r))\,\mathrm{d}r, \\ U(b) = \omega_n \displaystyle\int_a^b K(b,r)r^{n-1}h(U(r))\,\mathrm{d}r, \end{cases} \tag{2.1}$$

where ω_n stands for the measure of the unit sphere in $\mathbb{R}^n$.

To eliminate the term with the first derivative one can use the following substitution

$$\tau = \Phi(r) = \Big((n-2)r^{n-2}\Big)^{-1},\quad r\in[a,b]$$

in case $n \geq 3$. Then (2.1) can be rewritten as

$$\begin{cases} -v''(t) = \tilde{f}(\tau, v(\tau)), \\ v(\alpha) = \int\limits_{\alpha}^{\beta} \tilde{K}(\alpha, \sigma) h(v(\sigma))\, \mathrm{d}\sigma, \\ v(\beta) = \int\limits_{\alpha}^{\beta} \tilde{K}(\beta, \sigma) h(v(\sigma))\, \mathrm{d}\sigma, \end{cases} \tag{2.2}$$

where $\alpha = \Phi(b)$, $\beta = \Phi(a)$ and

$$\tilde{f}(\tau, v) = \big[(n-2)\tau\big]^{-\frac{2n-2}{n-2}} f\Big(\big[(n-2)\tau\big]^{-\frac{1}{n-2}}, v\Big),$$
$$\tilde{K}(\tau, \sigma) = \omega_n \big[(n-2)\sigma\big]^{-\frac{2n-2}{n-2}} K\Big(\big[(n-2)\tau\big]^{-\frac{1}{n-2}}, \big[(n-2)\sigma\big]^{-\frac{1}{n-2}}\Big).$$

As for $n = 2$, in terms of variables

$$t = \Phi(r) = \log r, \tag{2.3}$$

the problem (2.1) can also be rewritten as (2.2) with

$$\tilde{f}(\tau, v) = \mathrm{e}^{2\tau} f(\mathrm{e}^{\tau}, v), \quad \tilde{K}(\tau, \sigma) = 2\pi \mathrm{e}^{\sigma} K(\mathrm{e}^{\tau}, \mathrm{e}^{\sigma}), \quad \alpha = \log a, \quad \beta = \log b.$$

Finally, to obtain the boundary value problem on $[0, 1]$, we use the affine mapping

$$t = A(\tau) = \frac{\tau - \alpha}{\beta - \alpha}.$$

Then, above substitution yields

$$\begin{cases} -w''(t) = g(t, w(t)), \\ w(0) = \int\limits_{0}^{1} k_0(s) h(w(s))\, \mathrm{d}s, \\ w(1) = \int\limits_{0}^{1} k_1(s) h(w(s))\, \mathrm{d}s, \end{cases} \tag{2.4}$$

where

$$g(t, w) = \tilde{f}\left((\beta - \alpha)t + \alpha, w\right),$$
$$k_0(s) = \tilde{K}\left(\alpha, (\beta - \alpha)s + \alpha\right),$$
$$k_1(s) = \tilde{K}\left(\beta, (\beta - \alpha)s + \alpha\right).$$

By definitions of $\tilde{f}$ and $\tilde{K}$ we have:

— for $n \geq 3$,

$$g(t,w) = \big\{(n-2)[(\beta-\alpha)t+\alpha]\big\}^{-\frac{2n-2}{n-2}} \cdot f\Big(\big\{(n-2)[(\beta-\alpha)t+\alpha]\big\}^{-\frac{1}{n-2}}, w\Big),$$

$$k_0(s) = \omega_n\big\{(n-2)[(\beta-\alpha)s+\alpha]\big\}^{-\frac{2n-2}{n-2}} \cdot K\Big(b, \big\{(n-2)[(\beta-\alpha)s+\alpha]\big\}^{-\frac{1}{n-2}}\Big),$$

$$k_1(s) = \omega_n\big\{(n-2)[(\beta-\alpha)s+\alpha]\big\}^{-\frac{2n-2}{n-2}} \cdot K\Big(a, \big\{(n-2)[(\beta-\alpha)s+\alpha]\big\}^{-\frac{1}{n-2}}\Big),$$

where $\alpha = \Big((n-2)b^{n-2}\Big)^{-1}$, $\beta = \Big((n-2)a^{n-2}\Big)^{-1}$,

— for $n = 2$,

$$g(t,w) = \left(\frac{b}{a}\right)^{2t} a^2 f\left(\left(\frac{b}{a}\right)^{t} a, w\right),$$

$$k_0(s) = 2\pi\left(\frac{b}{a}\right)^{s} aK\left(a, \left(\frac{b}{a}\right)^{s} a\right),$$

$$k_1(s) = 2\pi\left(\frac{b}{a}\right)^{s} aK\left(b, \left(\frac{b}{a}\right)^{s} a\right).$$

Since each mapping which transforms problem 1 to problem 2, is diffeomorphism, we get the following result.

Proposition 2.1. *The boundary value problem* (1.1) *has a positive radial solution if and only if the problem* (2.4) *has a positive solution.*

Therefore, firstly we shall investigate the problem (2.4). The method we use it standard for boundary value problems. Namely, we reformulate the problem (2.4) as an equivalent fixed point problem. We will employ the following fixed point theorem.

Theorem 2.2 ([11])**.** *Let K be a cone in a Banach space X, i.e. K is nonempty convex and closed set such that:*

(C1) $\lambda K \subset K$ *for* $\lambda \geq 0$,
(C2) $K \cap (-K) = \{0\}$, *where* 0 *denotes the zero in* X.

Let Ω_1 and Ω_2 be two bounded open neighbourhoods of zero such that $\overline{\Omega}_1 \subset \Omega_2$ and $T\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ a completely continuous mapping. Suppose that one of the following two conditions is satisfied:

(CE) $\|T(x)\| \leq \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|T(x)\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_2$,

and

(CC) $\|T(x)\| \geq \|x\|$ *for* $x \in K \cap \partial\Omega_1$ *and* $\|T(x)\| \leq \|x\|$ *for* $x \in K \cap \partial\Omega_2$.

Then operator T *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. EXISTENCE OF SOLUTIONS FOR AUXILIARY PROBLEM

Denote by $C([0,1])$ the Banach space of all continuous functions $w\colon [0,1] \to \mathbb{R}$ with the norm

$$\|w\| = \sup_{t\in[0,1]} |w(t)|.$$

Moreover, $C^1([0,1])$ is the space of all continuous whose first derivatives belong to $C([0,1])$ and $AC([0,1])$ is the space of all absolutely continuous functions defined on $[0,1]$.

We say that $w\colon [0,1] \to \mathbb{R}$ is *Carathéodory's solution* to the problem (2.4) if $w \in C^1([0,1])$, $w' \in AC([0,1])$, $w''(t) = -g(t, w(t))$ for almost every t and the boundary conditions of (2.4) hold.

For any function $w \in C([0,1])$, we shall use the following notation

$$H_0(w) := \int_0^1 k_0(s)h(w(s))\,\mathrm{d}s, \quad H_1(w) := \int_0^1 k_1(s)h(w(s))\,\mathrm{d}s. \tag{3.1}$$

Then by direct computation we obtain the following

Lemma 3.1. *A function* $w\colon [0,1] \to \mathbb{R}$ *is Carathéodory's solution of the problem* (2.4) *if and only if* w *satisfies the following integral equation*

$$w(t) = \int_0^1 G(t,s)g(s,w(s))\,\mathrm{d}s + \big(H_1(w) - H_0(w)\big) \cdot t + H_0(w),$$

where $G\colon [0,1] \times [0,1] \to \mathbb{R}$ *is the Green function corresponding to the Dirichlet boundary value problem, i.e.*

$$G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Consider the operator $T\colon C([0,1]) \to C([0,1])$, defined by

$$(Tw)(t) = \int_0^1 G(t,s)g(s,w(s))\,\mathrm{d}s + (1-t) \cdot H_0(w) + t \cdot H_1(w),$$

for $w \in C([0,1])$.

We assume the following assumption holds: for arbitrary $M > 0$, there is a function $h_M \in L^1(0,1)$ such that $g(t,w) \le h_M(w)$ for almost every $t \in [0,1]$ and $w \in [0,M]$. Then the function g is said L^1*–Carathéodory's function.*

Then by using the classical Arzelá–Ascoli Theorem ([16, 17]), we get

Lemma 3.2. *The operator T is completely continuous.*

Now, we shall prove that the problem (2.4) has at least one positive solution.

The following assumptions will be needed:

(i) $\lim\limits_{w\to 0^+} \sup\limits_{t\in[0,1]\setminus S_1} \frac{g(t,w)}{w} = 0$,

(ii) $\lim\limits_{w\to 0^+} \frac{h(w)}{w} = 0$,

(iii) $\lim\limits_{w\to +\infty} \inf\limits_{t\in[c,d]\setminus S_2} \frac{g(t,w)}{w} = +\infty$,

(iv) $\lim\limits_{w\to +\infty} \frac{h(w)}{w} = +\infty$,

(v) $\int\limits_c^d k_i(s)\,\mathrm{d}s > 0$ for $i = 0, 1$,

where $[c,d] \subset (0,1)$ is some interval and $S_1, S_2 \subset \mathbb{R}$ are some sets with zero Lebesgue measure.

Note that by the assumption (i) we get $g(t,0) = 0$ for $t \in [0,1] \setminus S_1$. On the other hand, the assumption (ii) yields $h(0) = 0$. Then the zero function is a solution of problem (2.4). Applying Theorem 2.2 with condition (CE) we shall show the existence of nontrivial solution to problem (2.4)

Theorem 3.3. *Assume that the conditions* (i), (ii) *hold. If we suppose that the condition* (iii) *is satisfied or both conditions* (iv) *and* (v) *are satisfied, then the problem* (2.4) *has at least one positive solution.*

Proof. In the Banach space $C([0,1])$ we define a set

$$K := \left\{ w \in C([0,1]) : w(t) \ge 0 \,\text{for}\; t \in [0,1],\; \inf_{t\in[c,d]} w(t) \ge \min\{c, 1-d\}\, \|w\| \right\}.$$

It is easy to see that K is a cone in the space $C([0,1])$.

In the first part of the proof we shall show that the operator T maps all nonnegative functions into the cone K. It is clear that $Tw \ge 0$ for $w \ge 0$.

Observe that

$$G(t,s) \le s(1-s), \tag{3.2}$$

for any pair $(t,s) \in [0,1] \times [0,1]$, hence

$$\|Tw\| \le \int\limits_0^1 s(1-s)g(s,w(s))\,\mathrm{d}s + \max\{H_0(w), H_1(w)\}.$$

On the other hand, if $t \in [c,d]$ and $s \in [0,1]$ we have

$$\begin{aligned} G(t,s) &= \begin{cases} s(1-t), & s \le t, \\ t(1-s), & t \ge s, \end{cases} \\ &\ge \begin{cases} s(1-d), & s \le t, \\ c(1-s), & t \ge s, \end{cases} \\ &\ge \min\{c, 1-d\} s(1-s). \end{aligned} \tag{3.3}$$

Thus, we obtain

$$\begin{aligned} (Tw)(t) &\ge \int_0^1 \min\{c,1-d\} s(1-s) g(s, w(s))\, \mathrm{d}s \\ &\quad + \min\{c,1-d\} \max\{H_0(w), H_1(w)\} \\ &= \min\{c,1-d\} \Bigg(\int_0^1 s(1-s) g(s,w(s))\, \mathrm{d}s + \max\{H_0(w), H_1(w)\} \Bigg) \\ &\ge \min\{c,1-d\} \|Tw\|, \end{aligned}$$

for $t \in [c,d]$. Hence,

$$\inf_{t \in [c,d]} (Tw)(t) \ge \min\{c,1-d\} \|Tw\|,$$

which implies that $T(K) \subset K$.

In the next step, we shall introduce two bounded open neighbourhoods of zero in $C([0,1])$ such that $\overline{\Omega}_1 \subset \Omega_2$, for which the condition (CE) of Theorem 2.2 will be satisfied.

By assumption (i), there exists a constant $M_1 > 0$ such that

$$g(s,w) \le 3 \cdot w \quad \text{for } 0 < w < M_1 \text{ and } s \in [0,1] \setminus S_1.$$

Now, we can choose $\varepsilon > 0$ such that

$$\varepsilon \cdot \int_0^1 k_i(s)\, \mathrm{d}s \le \frac{1}{2} \quad \text{for } i = 0, 1.$$

Next, by assumption (ii), there is a constant $M_2 > 0$ such that

$$h(w) \le \varepsilon \cdot w \quad \text{for } 0 < w < M_2. \tag{3.4}$$

Let $R_1 := \min\{M_1, M_2\}$ and $\Omega_1 := B_{C([0,1])}(0, R_1)$, where $B_{C([0,1])}(w_0, R)$ is the open ball in $C([0,1])$ of radius $R > 0$ with center at $w_0 \in C([0,1])$. Assume that

$w \in K \cap \partial\Omega_1$. Then $w(s) \leq M_1$ and $w(s) \leq M_2$ for $s \in [0,1]$. Hence, by (3.2) and (3.4) for any $t \in [0,1]$ we obtain

$$\begin{aligned}
(Tw)(t) &= \int_0^1 G(t,s)g(s,w(s))\,\mathrm{d}s + \big(H_1(w) - H_0(w)\big)t + H_0(w) \\
&\leq \int_0^1 s(1-s)g(s,w(s))\,\mathrm{d}s + \max\{H_0(w), H_1(w)\} \\
&= \int_0^1 s(1-s)g(s,w(s))\,\mathrm{d}s + \int_0^1 k_i(s)h(w(s))\,\mathrm{d}s \\
&\leq \int_0^1 s(1-s)3w(s)\,\mathrm{d}s + \int_0^1 k_i(s)\varepsilon w(s)\,\mathrm{d}s \\
&\leq \frac{1}{2}\|w\| + \frac{1}{2}\|w\| = \|w\|.
\end{aligned}$$

Therefore $\|Tw\| \leq \|w\|$ for $w \in K \cap \partial\Omega_1$.

First, we consider the case in which the assumption (ii) holds. We can choose a number $L > 0$ such that

$$L(\min\{c, 1-d\})^2 \int_c^d s(1-s)\,\mathrm{d}s = 1.$$

By the condition (iii), there is a constant $M_3 > \min\{c, 1-d\}R_1$ such that

$$g(s,w) \geq L \cdot w \quad \text{for } w \geq M_3 \text{ and } s \in [0,1] \setminus S_2. \tag{3.5}$$

Define $R_2 := \frac{1}{\min\{c,1-d\}} M_3$ and $\Omega_2 := B_{C([0,1])}(0, R_2)$. Then, we observe that $\overline{\Omega}_1 \subset \Omega_2$.

Let $w \in K$ and $\|w\| = R_2$. Since the function w belongs to the cone K, we have

$$w(t) \geq \min\{c, 1-d\}\|w\| = M_3 \quad \text{for } t \in [c,d].$$

According to the above inequality, by (3.3) and (3.5) taking $t_0 \in [c,d]$, we get

$$\begin{aligned}
(Tw)(t_0) &= \int_0^1 G(t_0,s)g(s,w(s))\,\mathrm{d}s + (H_1(w) - H_0(w))\,t_0 + H_0(w) \\
&\geq \int_0^1 G(t_0,s)g(s,w(s))\,\mathrm{d}s \\
&\geq \int_c^d G(t_0,s)g(s,w(s))\,\mathrm{d}s \\
&\geq \int_c^d L\min\{c,1-d\}s(1-s)w(s)\,\mathrm{d}s \\
&\geq L(\min\{c,1-d\})^2 \int_c^d s(1-s)\,\mathrm{d}s\,\|w\| = \|w\|\,,
\end{aligned}$$

thus $\|Tw\| \geq \|w\|$ for $w \in K \cap \partial\Omega_2$.

Applying the condition (CE) of Theorem 2.2 to T, we obtain that T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e. BVP (2.4) has a positive solution.

In the last part of the proof, we assume that the conditions (iv) and (v) are satisfied (instead of (iii)). We do not change the definition of the set Ω_1, for which the first inequality of the condition (CE) is fulfilled. However, we introduce the new definition of Ω_2 and we will show that $\|Tw\| \geq \|w\|$ for $w \in K \cap \partial\Omega_2$.

We set

$$\eta := \frac{1}{\min\{c,1-d\}\min\left\{\int_c^d k_i(s)\,\mathrm{d}s : i = 0,1\right\}}. \tag{3.6}$$

By the assumption (iv), there is a constant $M_4 > \min\{c,1-d\}R_1$ such that $h(w) \geq \eta w$ for $w \geq M_4$. Let $R_2 := \frac{1}{\min\{c,1-d\}}M_4$ and $\Omega_2 := B_{C([0,1])}(0,R_2)$. Then $\overline{\Omega}_1 \subset \Omega_2$. Take $w \in K$ such that $\|w\| = R_2$. Hence $w(t) \geq M_4$ for $t \in [c,d]$. One can note that

$$(H_1(w) - H_0(w))\,t + H_0(w) \geq \min\{H_0(w), H_1(w)\}. \tag{3.7}$$

for any $w \in C([0,1])$. Positivity of g, (3.6) and (3.7) yields

$$\begin{aligned}(Tw)(t) &\geq \int_0^1 G(t,s)g(s,w(s))\,\mathrm{d}s + (H_1(w)-H_0(w))\,t_0 + H_0(w)\\ &\geq H_j(w) = \int_0^1 k_j(s)h(w(s))\,\mathrm{d}s\\ &\geq \int_c^d k_j(s)h(w(s))\,\mathrm{d}s\\ &\geq \int_c^d k_j(s)\eta w(s)\,\mathrm{d}s\\ &\geq \|w\|\cdot\eta\min\{c,1-d\}\int_c^d k_j(s)\,\mathrm{d}s = \|w\|\,,\end{aligned}$$

where

$$H_j(w) = \min\{H_0(w), H_1(w)\}.$$

Therefore $\|Tw\| \geq \|w\|$ for $w \in K \cap \partial\Omega_2$ and, by Theorem 2.2, BVP (2.4) has at least one positive solution. □

Now, we are ready formulate the main result for elliptic problem (1.1), but at first we introduce the following assumptions:

(i') $\lim\limits_{u\to 0^+} \sup\limits_{r\in[a,b]\setminus P_1} \frac{f(r,u)}{u} = 0$, where P_1 is some set with zero Lebesgue measure;

(iii') $\lim\limits_{u\to +\infty} \inf\limits_{r\in[r_1,r_2]\setminus P_2} \frac{f(r,u)}{u} = +\infty$, where $[r_1,r_2] \subset (a,b)$ is some interval and P_2 is some set with zero Lebesgue measure;

(v') $\int\limits_\Omega K(r,|y|)h\big(u(y)\big)\,\mathrm{d}y > 0$ for $r = a,b$.

Theorem 3.4. *Assume that $f\colon [a,b]\times\mathbb{R}_+ \to \mathbb{R}_+$ is L^1–Carathéodory's function and the conditions* (i'), (ii) *hold. If the condition* (iii') *is fulfilled or both the assumptions* (iv) *and* (v) *are satisfied the problem* (1.1) *has at least one positive radial solution.*

Proof. Since the assumptions (i'), (iii'), (v') provide respectively the assumptions (i), (iii), (v) we can use Theorem 3.3 for the auxiliary problem (2.4) after appropriate change of variable. Therefore, by Proposition 2.1 we get the problem (1.1) has at least one positive radial solution. □

4. EXAMPLES

To illustrate how our main result can be used in practise we present some examples.

Example 4.1. Consider the problem

$$-\Delta u = u^p g(|x|), \quad u|_{\partial\Omega} = \int\limits_{\Omega} |xy| u^q(y)\,\mathrm{d}y,$$

where $\Omega = B(0,b) \setminus \overline{B}(0,a) \subset \mathbb{R}^n$, $0 < a < b$, $p, q > 1$ and $g\colon [a,b] \to \mathbb{R}_+$ is continuous. Then via Theorem 3.4 the above problem admits a positive radial solution.

Example 4.2. Denote by $L^\infty\big([a,b],\mathbb{R}_+\big)$ the set of all measurable nonnegative functions defined on $[a,b]$ which are bounded almost everywhere and by $C\big(\mathbb{R}_+,\mathbb{R}_+\big)$ set of all continuous nonnegative functions on $\mathbb{R}_+$. Let us consider the problem (1.1) with $f(r,u) = \sum_{i=1}^m A_i(r)B_i(u)$, where $A_i \in L^\infty\big([a,b],\mathbb{R}_+\big)$, $B_i \in C\big(\mathbb{R}_+,\mathbb{R}_+\big)$, $\lim_{u\to 0^+} \frac{B_i(u)}{u} = 0$ and $\lim_{u\to+\infty} \frac{B_i(u)}{u} = +\infty$, $i = 1,\dots,m$. Moreover, we assume that the conditions (ii) from Section 3 is satisfied. Since $A_i \in L^\infty\big([a,b],\mathbb{R}_+\big)$, there are constants $M_i > 0$ and a sets $P_i \in [a,b]$ with measure zero such that $A_i(r) \le M_i$ for $r \in [a,b] \setminus P_i$. Let us define set $P := \bigcup_{i=1}^m P_i$, which has measure zero. Then f is L^1-Carathéodory's function. Moreover, since $\lim\limits_{u\to 0^+} \frac{B_i(u)}{u} = 0$ and $\lim\limits_{u\to+\infty} \frac{B_i(u)}{u} = +\infty$, the conditions (i') and (iii') hold with $P_1 = P_2 = P$. Hence, by Theorem 3.4, the problem (1.1) has at least one positive solution.

REFERENCES

[1] C. Bandle, C.V. Coffman, M. Marcus, *Nonlinear elliptic problems in annular domains*, J. Differential Equations **69** (1987), no. 3, 322–345.

[2] A. Boucherif, *Second-order boundary value problems with integral boundary conditions*, Nonlinear Anal. **70** (2009), 364–371.

[3] F. Cianciaruso, G. Infante, P. Pietramala, *Multiple positive radial solutions for Neumann elliptic systems with gradient dependence*, Math. Methods Appl. Sci. **41** (2018), 6358–6367.

[4] W.A. Day, *Extensions of a property of the heat equation to linear thermoelasticity and other theories*, Quart. Appl. Math. **40** (1982), no. 3, 319–330.

[5] W.A. Day, *Heat Conduction within Linear Thermoelastacity*, Springer Tracts in Natural Philosophy, vol. 30, Springer-Verlag, New York, 1982.

[6] R. Enguica, L. Sanchez, *Radial solutions for a nonlocal boundary value problem*, Bound. Value Probl. **2006** (2006), Article no. 32950.

[7] P. Fijałkowski, B. Przeradzki, *On a boundary value problem for a nonlocal elliptic equation*, J. Appl. Anal. **9** (2003), no. 2, 201–209.

[8] P. Fijałkowski, B. Przeradzki, *On a radial positive solution to a nonlocal elliptic equation*, Topol. Methods Nonlinear Anal. **21** (2003), no. 2, 293–300.

[9] A. Friedman, *Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions*, Quart. Appl. Math. **44** (1986), no. 3, 401–407.

[10] X. Garaizar, *Existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Differential Equations **70** (1987), no. 1, 69–92.

[11] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Inc., Boston, 1988.

[12] G. Infante, *Nonzero positive solutions of a multi-parameter elliptic system with functional BCs*, Topol. Methods Nonlinear Anal. **52** (2018), 665–675.

[13] G. Infante, *Nonzero positive solutions of nonlocal elliptic systems with functional BCs*, J. Elliptic Parabol. Equ. **5** (2019), 493–505.

[14] G.L. Karakostas, P.Ch. Tsamatos, *On a nonlocal boundary value problem at resonance*, J. Math. Anal. Appl. **259** (2001), no. 1, 209–218.

[15] G.L. Karakostas, P.Ch. Tsamatos, *Sufficient conditions for the existence of nonnegative solutions of a nonlocal boundary value problem*, Appl. Math. Lett. **15** (2002), 401–407.

[16] M. Krukowski, *Arzelà-Ascoli's theorem in uniform spaces*, Discrete Contin. Dyn. Syst. Ser. B **23** (2018), no. 1, 283–294.

[17] M. Krukowski, *Arzelà-Ascoli's theorem via the Wallman compactification*, Quaest. Math. **41** (2018), no. 3, 349–357.

[18] S. Lin, *On the existence of positive radial solutions for nonlinear elliptic equations in annular domains*, J. Differential Equations **81** (1989), no. 2, 221–233.

[19] O. Nica, *Initial-value problems for first-order differential systems with general nonlocal conditions*, Electron. J. Differential Equations (2012), no. 74, 1–15.

[20] O. Nica, *Nonlocal initial value problems for first order differential systems*, Fixed Point Theory **13** (2012), no. 2, 603–612.

[21] C.V. Pao, *Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions*, J. Comput. Appl. Math. **88** (1998), no. 1, 225–238.

[22] R. Stańczy, *Decaying solutions for semilinear elliptic equations in exterior domains*, Topol. Methods Nonlinear Anal. **14** (1999), no. 2, 363–370.

[23] R. Stańczy, *Bounded solutions for nonlinear elliptic equations in unbounded domains*, J. Appl. Anal. **6** (2000), no. 1, 129–138.

[24] R. Stańczy, *Positive solutions for superlinear elliptic equations*, J. Math. Anal. Appl. **283** (2003), no. 1, 159–166.

[25] K. Szymańska-Dębowska, *On two systems of non-resonant nonlocal boundary value problems*, An. Stiint. Univ. Ovidius Constanta Ser. Mat. **21** (2013), no. 3, 257–267.

[26] K. Szymańska-Dębowska, J. Mawhin, *Convex sets and second order systems with nonlocal boundary conditions at resonance*, Proc. Amer. Math. Soc. **145** (2017), no. 5, 2023–2032.

[27] K. Szymańska-Dębowska, J. Mawhin, *Convex sets, fixed points and first order systems with nonlocal boundary conditions at resonance*, J. Nonlinear Convex Anal. **18** (2017), no. 1, 149–160.

[28] K. Szymańska-Dębowska, J. Mawhin, *Couples of lower and upper slopes and resonant second order ordinary differential equations with nonlocal boundary conditions*, Math. Bohem. **141** (2016), no. 2, 239–259.

[29] J.R.L. Webb, *Remarks on nonlocal boundary value problems at resonance*, Appl. Math. Comput. **216** (2010), no. 2, 497–500.

[30] W.M. Whyburn, *Differential equations with general boundary conditions*, Bull. Amer. Math. Soc. **48** (1942), no. 10, 692–704.

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