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ON THE CONCEPT OF GENERALIZATION OF $\mathcal{I}$-DENSITY POINTS

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Abstract. This paper deals with essential generalization of $\mathcal{I}$-density points and $\mathcal{I}$-density topology. In particular, there is an example showing that this generalization of $\mathcal{I}$-density point yields the stronger concept of density point than the notion of $\mathcal{I}(\mathcal{J})$-density. Some properties of topologies generated by operators related to this essential generalization of density points are provided.

Keywords: density topology, generalization of density topology.

Mathematics Subject Classification: 54A05, 54A10.

1. INTRODUCTION

A classical notion of Lebesgue density point describes a local behavior of measurable subsets of $\mathbb{R}$ around a given point. Thanks of the Lebesgue density theorem we know that almost all points of a measurable set are its density points. This yields a curious global description of measurable sets. Starting from 1982 the ideas of Wilczyński (see [5, 8] and see also [2]) brought an analogous theory of density points in the Baire category case. It opened extensive studies in this area by several authors. More subtle and general notion of category density points have been proposed. Our paper presents a new general approach in this direction. However, there were some generalizations of the density point with respect to category presented in [3, 6, 7], we hope to obtain essential applications connected with respective-type topology and the corresponding class of real-valued functions, which could play a role in the contemporary real analysis.

Throughout the paper we will use the standard notation: $\mathbb{R}$ will be the set of real numbers, $\mathbb{N}$ the set of natural numbers and $\mathcal{T}_{nat}$ the natural topology on $\mathbb{R}$. By $\lambda(A)$ we shall denote the Lebesgue measure of a measurable set A and by $|I|$ the length of an interval I. The symbol $\mathcal{S}$ will stand for the σ–algebra of sets having the Baire property in $\mathbb{R}$ and $\mathcal{I}$ for the σ-ideal of first category sets in $\mathbb{R}$. For $z, \alpha \in \mathbb{R}$, $A \subset \mathbb{R}$ we define $A + z = \{a + z \colon a \in A\}$, $\alpha A = \{\alpha a \colon a \in A\}$. If A, B are sets, then by $A \triangle B$ we denote the symmetric difference of sets A and B.

Let $Y \in \mathcal{S}$ be a bounded set of the second category and $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ be the sequences of the real numbers converging to zero such that $a_n \neq 0$ for every $n \in \mathbb{N}$. Set $Y_n = a_nY + b_n$ for every $n \in \mathbb{N}$ and $\mathcal{Y} = \{Y_n\}_{n\in\mathbb{N}}$.

We shall say that x_0 is an $\mathcal{I}(\mathcal{Y})$-*density point* of a set $A \in \mathcal{S}$ if

$$\chi_{(A-x_0-b_n)\frac{1}{a_n}\cap Y}(x) \xrightarrow[n\to\infty]{\mathcal{I}} \chi_Y(x),$$

which means that

$$\underset{\{n_k\}_{k\in\mathbb{N}}\nearrow}{\forall} \quad \underset{\{n_{k_m}\}_{m\in\mathbb{N}}}{\exists} \quad \underset{\Theta\in\mathcal{I}}{\exists} \quad \underset{x\notin\Theta}{\forall} \quad \chi_{\left(A-x_0-b_{n_{k_m}}\right)\frac{1}{a_{n_{k_m}}}\cap Y}(x) \underset{m\to\infty}{\longrightarrow} \chi_Y(x).$$

The above condition is equivalent to

$$\underset{\{n_k\}_{k\in\mathbb{N}}\nearrow}{\forall} \quad \underset{\{n_{k_m}\}_{m\in\mathbb{N}}}{\exists} \quad \limsup_{m\to\infty} \left(\left(Y_{n_{k_m}} \setminus A\right) - x_0 - b_{n_{k_m}}\right) \frac{1}{a_{n_{k_m}}} \in \mathcal{I}.$$

The concept of an $\mathcal{I}(\mathcal{Y})$-density is generalization of so called $\mathcal{I}(\mathcal{J})$-density ([6]), $\mathcal{I}\langle s\rangle$-density ([3]) and finally $\mathcal{I}$-density ([8]).

Let us fix a bounded set $Y \in \mathcal{S}$ of the second category and the sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ converging to zero such that $a_n \neq 0$ for every $n \in \mathbb{N}$. Let $Y_n = a_nY + b_n$ for every $n \in \mathbb{N}$ and $\mathcal{Y} = \{Y_n\}_{n\in\mathbb{N}}$.

Let us define for every $A \in \mathcal{S}$

$$\Phi_{\mathcal{I}(\mathcal{Y})}(A) = \{x \in \mathbb{R} : x \text{ is an } \mathcal{I}(\mathcal{Y})\text{-density point of } A\}.$$

Theorem 1.1. *For every sets A, $B \in \mathcal{S}$ we have:*

(1) $\Phi_{\mathcal{I}(\mathcal{Y})}(\emptyset) = \emptyset, \quad \Phi_{\mathcal{I}(\mathcal{Y})}(\mathbb{R}) = \mathbb{R}$,
(2) $\Phi_{\mathcal{I}(\mathcal{Y})}(A \cap B) = \Phi_{\mathcal{I}(\mathcal{Y})}(A) \cap \Phi_{\mathcal{I}(\mathcal{Y})}(B)$,
(3) $A \triangle B \in \mathcal{I} \Rightarrow \Phi_{\mathcal{I}(\mathcal{Y})}(A) = \Phi_{\mathcal{I}(\mathcal{Y})}(B)$,
(4) $A \triangle \Phi_{\mathcal{I}(\mathcal{Y})}(A) \in \mathcal{I}$.

Properties (1)–(3) are the consequence of definition of $\mathcal{I}(\mathcal{Y})$-density. Before we prove the condition (4) we have the following property.

Property 1.2. *If $V \in \mathcal{T}_{nat}$, then*

$$V \subset \Phi_{\mathcal{I}(\mathcal{Y})}(V) \subset \overline{V}.$$

Proof. Let us assume that $V \neq \emptyset$ and $x_0 \in V$. Then there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset V$. Moreover, there exists $n_0 \in \mathbb{N}$ such that $Y_n \subset (-\delta, \delta)$ for $n > n_0$. We conclude that

$$\chi_{(V-x_0-b_n)\frac{1}{a_n}\cap Y}(x) \xrightarrow[n\to\infty]{\mathcal{I}} \chi_Y(x).$$

It implies that $x_0 \in \Phi_{\mathcal{I}(\mathcal{Y})}(V)$.

Now let us assume that $x_0 \in \Phi_{\mathcal{I}(\mathcal{Y})}(V) \setminus \overline{V}$. Then $x_0 \in \mathbb{R} \setminus \overline{V} \subset \Phi_{\mathcal{I}(\mathcal{Y})}(\mathbb{R} \setminus \overline{V})$. Hence,

$$x_0 \in \Phi_{\mathcal{I}(\mathcal{Y})}(V) \cap \Phi_{\mathcal{I}(\mathcal{Y})}(\mathbb{R} \setminus \overline{V}) = \Phi_{\mathcal{I}(\mathcal{Y})}(\emptyset) = \emptyset.$$

This contradiction ends the proof. $\square$

Now we prove condition (4). Let $A \in \mathcal{S}$. Then $A = V \triangle C$, where $V \in \mathcal{T}_{nat}$ and $C \in \mathcal{I}$. By condition (2) and Property 1.2 we get that $\Phi_{\mathcal{I}(\mathcal{Y})}(A) = \Phi_{\mathcal{I}(\mathcal{Y})}(V) \supset V$. Hence, $A \setminus \Phi_{\mathcal{I}(\mathcal{Y})}(A) \subset A \setminus V \in \mathcal{I}$. At the same time $\Phi_{\mathcal{I}(\mathcal{Y})}(A) \setminus A \subset \overline{V} \setminus A \in \mathcal{I}$, so that $A \triangle \Phi_{\mathcal{I}(\mathcal{Y})}(A) \in \mathcal{I}$.

Corollary 1.3. *In the virtue of Theorem 1.1 the operator $\Phi_{\mathcal{I}(\mathcal{Y})}$ is the lower density operator on the space $(\mathbb{R}, \mathcal{S}, \mathcal{I})$.*

Since the pair $(\mathcal{S}, \mathcal{I})$ has the hull property, we have the following theorem.

Theorem 1.4 ([4])**.** *The family*

$$\mathcal{T}_{\mathcal{I}(\mathcal{Y})} = \{A \in \mathcal{S} : A \subset \Phi_{\mathcal{I}(\mathcal{Y})}(A)\}$$

is a topology on $\mathbb{R}$ essentially stronger than $\mathcal{T}_{nat}$.

According to properties of topologies generated by lower density operator, we have the following theorem.

Theorem 1.5 ([4])**.**

(a) *$A \in \mathcal{I}$ if and only if A is $\mathcal{T}_{\mathcal{I}(\mathcal{Y})}$-closed and $\mathcal{T}_{\mathcal{I}(\mathcal{Y})}$-nowhere dense,*
(b) *$\mathcal{I} = \mathcal{M}(\mathcal{T}_{\mathcal{I}(\mathcal{Y})})$, where $\mathcal{M}(\mathcal{T}_{\mathcal{I}(\mathcal{Y})})$ is the family of meager sets with respect to $\mathcal{T}_{\mathcal{I}(\mathcal{Y})}$,*
(c) *$\mathcal{B}or(\mathcal{T}_{\mathcal{I}(\mathcal{Y})}) = \mathcal{B}(\mathcal{T}_{\mathcal{I}(\mathcal{Y})}) = \mathcal{S}$, where $\mathcal{B}or(\mathcal{T}_{\mathcal{I}(\mathcal{Y})})$ is the family of Borel sets and $\mathcal{B}(\mathcal{T}_{\mathcal{I}(\mathcal{Y})})$ is the family of sets having the Baire property with respect to $\mathcal{T}_{\mathcal{I}(\mathcal{Y})}$,*
(d) *$\langle \mathbb{R}, \mathcal{T}_{\mathcal{I}(\mathcal{Y})} \rangle$ is a Baire space,*
(e) *$\mathcal{T}_{\mathcal{I}(\mathcal{Y})} = \{\Phi_{\mathcal{I}(\mathcal{Y})}(A) \setminus B \colon A \in \mathcal{S}, B \in \mathcal{I}\}$,*
(f) *$A \in \mathcal{I}$ if and only if A is $\mathcal{T}_{\mathcal{I}(\mathcal{Y})}$-closed and $\mathcal{T}_{\mathcal{I}(\mathcal{Y})}$-discrete,*
(g) *A is $\mathcal{T}_{\mathcal{I}(\mathcal{Y})}$-compact if and only if A is finite,*
(h) *$\langle \mathbb{R}, \mathcal{T}_{\mathcal{I}(\mathcal{Y})} \rangle$ is neither a first countable, nor a second countable, nor a separable space,*
(i) *$\langle \mathbb{R}, \mathcal{T}_{\mathcal{I}(\mathcal{Y})} \rangle$ is not a Lindelöf space.*

Let $\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$ be a sequence of closed intervals tending to zero, it means that $\lim_{n\to\infty} \operatorname{diam}(\{0\} \cup J_n) = 0$. According to paper [6], we shall say that x_0 is an $\mathcal{I}(\mathcal{J})$-density point of a set $A \in \mathcal{S}$ if the sequence $\{\chi_{\frac{2}{|J_n|}(A - x_0 - s(J_n)) \cap [-1,1]}(x)\}_{n\in\mathbb{N}}$ converges to the characteristic function $\chi_{[-1,1]}$ with respect to the σ-ideal of the first category sets, where $s(J_n)$ is a center of interval J_n, for $n \in \mathbb{N}$.

It is easy to observe that for any sequence of closed intervals $\mathcal{J} = \{[c_n, d_n]\}_{n\in\mathbb{N}}$ we can find sequence $\mathcal{Y}$ generating the same density points i.e. $\Phi_{\mathcal{I}(\mathcal{Y})} = \Phi_{\mathcal{I}(\mathcal{J})}$. It is sufficient to consider $Y = [-1, 1]$, $a_n = \frac{d_n - c_n}{2}$ and $b_n = \frac{c_n + d_n}{2}$ for $n \in \mathbb{N}$.

2. MAIN RESULT

In the following theorem it is shown that the concept of $\mathcal{I}(\mathcal{Y})$-density is essential extension of the concept of $\mathcal{I}(\mathcal{J})$-density.

Theorem 2.1. *There exists a bounded set $Y \in \mathcal{S}$ of the second category and sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ of the real numbers converging to zero and $a_n \neq 0$ for every $n \in \mathbb{N}$ such that operators $\Phi_{\mathcal{I}(\mathcal{Y})} \neq \Phi_{\mathcal{I}(\mathcal{J})}$ for every sequence of closed intervals $\mathcal{J}$ tending to zero.*

Proof. Let $K_l = \left[\frac{2}{3^l}, \frac{1}{3^{l-1}}\right]$ for $l \in \mathbb{N}$. Define $Y = \bigcup_{l\in\mathbb{N}} K_l$ and $a_n = 3^{-n}$, $b_n = 2 \cdot 3^{-n}$ for $n \in \mathbb{N}$. Obviously $Y \subset [0,1]$ and

$$Y_n = a_n Y + b_n \subset K_n \quad \text{for every } n \in \mathbb{N}.$$

Suppose that there exists a sequence $\mathcal{J} = \{J_m\}_{m\in\mathbb{N}}$ of closed intervals tending to zero such that $\Phi_{\mathcal{I}(\mathcal{J})} = \Phi_{\mathcal{I}(\mathcal{Y})}$. Since $0 \in \Phi_{\mathcal{I}(\mathcal{Y})}([0,1))$, we have $0 \in \Phi_{\mathcal{I}(\mathcal{J})}([0,1))$. By Conclusion 6 in [7] we can assume that $J_m \cap (-\infty, 0) = \emptyset$.

Observe that for almost every $m \in \mathbb{N}$ interval J_m intersects at most one interval K_l for $l \in \mathbb{N}$. It means that

$$\exists_{m_0\in\mathbb{N}} \forall_{m>m_0} \ \operatorname{card}(N(m)) \leq 1,$$

where

$$N(m) = \{l \in \mathbb{N} \colon \lambda(J_m \cap K_l) > 0\}.$$

On the contrary, suppose that

$$\forall_{j\in\mathbb{N}} \exists_{m_j>j} \ \operatorname{card}(N(m_j)) > 1.$$

Put $l_j = \min N(m_j)$. Then $\lambda(J_{m_j} \cap K_{l_j}) > 0$ and $\lambda(J_{m_j} \cap K_{l_j+1}) > 0$. Hence,

$$J_{m_j} \setminus Y \supset \left(\frac{1}{3^{l_j}}, \frac{2}{3^{l_j}}\right).$$

Since $\lambda(J_{m_j} \cap K_{l_j-1}) = 0$, we have $\lambda(J_{m_j}) \leq \frac{2}{3^{l_j-1}}$. Put

$$c_j = \left(\frac{1}{3^{l_j}} - s(J_{m_j})\right) \frac{2}{\lambda(J_{m_j})},$$
$$d_j = \left(\frac{2}{3^{l_j}} - s(J_{m_j})\right) \frac{2}{\lambda(J_{m_j})}$$

for $j \in \mathbb{N}$. Then

$$d_j - c_j = \frac{2}{3^{l_j}\lambda(J_{m_j})} \geq \frac{2 \cdot 3^{l_j-1}}{3^{l_j} \cdot 2} = \frac{1}{3}.$$

The sequences $\{c_j\}_{j\in\mathbb{N}}$ and $\{d_j\}_{j\in\mathbb{N}}$ are bounded. Hence, there exist subsequences $\{c_{j_k}\}_{k\in\mathbb{N}}$ and $\{d_{j_k}\}_{k\in\mathbb{N}}$ tending to c_0 and d_0, respectively. Clearly $d_0 - c_0 \geq \frac{1}{3}$ and

$$((J_{m_{j_k}} \setminus Y) - s(J_{m_{j_k}})\frac{2}{\lambda(J_{m_{j_k}})} \supset \left(\left(\frac{1}{3^{n_{j_k}}}, \frac{2}{3^{n_{j_k}}}\right) - s(J_{m_{j_k}}\right) \frac{2}{\lambda(J_{m_{j_k}})} \supset (c_{j_k}, d_{j_k}).$$

It implies that

$$\limsup_{k\to\infty}((J_{m_{j_k}} \setminus Y) - s(J_{m_{j_k}})\frac{2}{\lambda(J_{m_{j_k}})} \supset (c_0, d_0) \notin \mathcal{I}.$$

It means that $0 \notin \Phi_{\mathcal{I}(\mathcal{J})}(Y)$ but $0 \in \Phi_{\mathcal{I}(\mathcal{Y})}(Y)$. This contradiction proves that for almost every $m \in \mathbb{N}$ interval J_m intersects at most one interval K_l.

Analogously we can prove that for almost every $m \in \mathbb{N}$ interval J_m intersects at most one interval $a_n K_l + b_n$ for n, $l \in \mathbb{N}$.

Therefore,

$$\hat{N}(m) = \{(n,l) \in \mathbb{N}^2 \colon \lambda(J_m \cap (a_n K_l + b_n)) > 0\}$$

has at most one element for $m \in \mathbb{N}$.

If there exists an increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ such that $\hat{N}(m_j) = \emptyset$ for $j \in \mathbb{N}$, then putting $A = [-1,1] \setminus \bigcup_{j\in\mathbb{N}} J_{m_j}$ we obtain that $0 \in \Phi_{\mathcal{I}(\mathcal{Y})}(A) \setminus \Phi_{\mathcal{I}(\mathcal{J})}(A)$. Hence, we can assume that $\mathrm{card}(\hat{N}(m)) = 1$ for $m \in \mathbb{N}$. Let

$$L = \sup\{l \colon (n,l) \in \hat{N}(m), m \in \mathbb{N}\}.$$

We have two cases:

Case 1. $L < \infty$. Define $A = \bigcup_{m\in\mathbb{N}} J_m$. Then $0 \in \Phi_{\mathcal{I}(\mathcal{J})}(A)$. Moreover,

$$(A \cap Y_n) \subset \left(\bigcup_{l\le L} a_n K_l + b_n\right)$$

for $n \in \mathbb{N}$. Hence,

$$Y_n \setminus A \supset \left(\bigcup_{l>L} (a_n K_l + b_n)\right).$$

It implies that

$$\limsup_{n\to\infty}((Y_n \setminus A) - b_n)a_n^{-1} \supset \bigcup_{l>L} K_l \notin \mathcal{I}.$$

Therefore, $0 \notin \Phi_{\mathcal{I}(\mathcal{Y})}(A)$ but $0 \in \Phi_{\mathcal{I}(\mathcal{J})}(A)$.

Case 2. $L = \infty$. We choose sequence $\{m_j\}_{j\in\mathbb{N}}$ such that $n_j < n_{j+1}$ and $l_j < l_{j+1}$, where $\hat{N}(m_j) = \{(n_j, l_j)\}$. Define set $A = [-1,1] \setminus \bigcup_{j\in\mathbb{N}} J_{m_j}$. Then $0 \notin \Phi_{\mathcal{I}(\mathcal{J})}(A)$. We prove that $0 \in \Phi_{\mathcal{I}(\mathcal{Y})}(A)$. Fix $n_0 \in \mathbb{N}$ and observe that $Y_{n_0} \setminus A = \emptyset$ if $n_0 \neq n_j$ for every $j \in \mathbb{N}$. When $n_0 = n_j$ for some $j \in \mathbb{N}$, we have

$$Y_{n_0} \setminus A = Y_{n_0} \cap J_{m_j} \subset a_{n_j} K_{l_j} + b_{n_j}.$$

Thus,

$$((Y_{n_0} \setminus A) - b_{n_0})a_{n_0}^{-1} \subset K_{l_j}.$$

It follows that

$$\limsup_{n\to\infty}((Y_n \setminus A) - b_n)a_n^{-1} = \emptyset \in \mathcal{I}.$$

Therefore, $0 \in \Phi_{\mathcal{I}(\mathcal{Y})}(A)$.

Finally, we conclude that operators $\Phi_{\mathcal{I}(\mathcal{Y})}$ and $\Phi_{\mathcal{I}(\mathcal{J})}$ are not equal for any sequence $\mathcal{J}$ of closed intervals tending to zero. $\square$

As a consequence of definition of $\mathcal{I}(\mathcal{Y})$-density point, we have the following property.

Property 2.2. *Let Y^1, $Y^2 \in \mathcal{S} \setminus \mathcal{I}$ be bounded sets. Fix the sequences of the real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ converging to zero, $a_n \neq 0$ for every $n \in \mathbb{N}$. Define $Y^i_n = a_n Y^i + b_n$ for every $n \in \mathbb{N}$ and $i = 1, 2$. Then*

$$Y^1 \triangle Y^2 \in \mathcal{I} \Longrightarrow \Phi_{\mathcal{Y}^1} = \Phi_{\mathcal{Y}^2}.$$

The inverse property is not true. Let $Y^1 = [-1, 1]$, $Y^2 = [-\alpha, \alpha]$, where $0 < \alpha < 1$, and $a_n = \alpha^n$, $b_n = 0$ for $n \in \mathbb{N}$. Define $Y^i_n = a_n Y^i + b_n$ for every $n \in \mathbb{N}$ and $i = 1, 2$. Then $\Phi_{\mathcal{Y}^1} = \Phi_{\mathcal{Y}^2}$ however $Y^1 \triangle Y^2 \notin \mathcal{I}$.

Indeed, let $A \in \mathcal{S}$. Then

$$x_0 \in \Phi_{\mathcal{Y}^1}(A) \iff \chi_{(A-x_0)\frac{1}{\alpha^n}\cap[-1,1]}(x) \xrightarrow[n\to\infty]{\mathcal{I}} \chi_{[-1,1]}(x)$$

and

$$x_0 \in \Phi_{\mathcal{Y}^2}(A) \iff \chi_{(A-x_0)\frac{1}{\alpha^n}\cap[-\alpha,\alpha]}(x) \xrightarrow[n\to\infty]{\mathcal{I}} \chi_{[-\alpha,\alpha]}(x).$$

Observe that

$$x \in (A - x_0)\,\frac{1}{\alpha^n} \cap [-1, 1] \iff \alpha x \in (A - x_0)\,\frac{1}{\alpha^{n-1}} \cap [-\alpha, \alpha].$$

Therefore,

$$x_0 \in \Phi_{\mathcal{Y}^1}(A) \iff x_0 \in \Phi_{\mathcal{Y}^2}(A).$$

Even if we assume that Y^i are regular open sets such that $\inf Y^i = 0$ and $\sup Y^i = 1$, for $i = 1, 2$, the inverse property is not true.

Example 2.3. Let

$$Y_{(k,l)} = \left(\frac{3^k - 1}{3 \cdot 9^{l-1}}, \frac{3^{k+1} - 2}{9^l}\right)$$

and

$$V_{(k,l)} = \left(\frac{3^k - 1}{9^l}, \frac{3^{k+1} - 2}{3 \cdot 9^l}\right)$$

for k, $l \in \mathbb{N}$. Observe that intervals $Y_{(k,l)}$ (and $V_{(k,l)}$) are pairwise disjoint for k, $l \in \mathbb{N}$. Define

$$G^1 = \bigcup_{k\in\mathbb{N}} \bigcup_{2l>k} Y_{(k,l)}, \quad G^2 = \bigcup_{k\in\mathbb{N}} \bigcup_{2l\geq k} V_{(k,l)}$$

and $a_n = 3^{-n}$, $b_n = 0$ for $n \in \mathbb{N}$. Obviously G^1, G^2 are regular open subsets of $[0, 1]$ such that $\inf G^i = 0$ and $\sup G^i = 1$, $i \in \{1, 2\}$.

Moreover, $Y_{(k,l)} \cap V_{(p,q)} = \emptyset$ for $k,l,p,q \in \mathbb{N}$. We show that:

(i) $\sup Y_{(k,l)} < \inf V_{(p,q)}$ if $k-2l+2 \le p-2q$ or if $k-2l+1 = p-2q$ and $l \le q$,
(ii) $\sup V_{(p,q)} < \inf Y_{(k,l)}$ if $k-2l \ge p-2q$ or if $k-2l+1 = p-2q$ and $l > q$.

Indeed, if $k-2l+1 = p-2q$ and $l \le q$, then

$$\sup Y_{(k,l)} = 3^{k-2l+1} - 2\cdot 3^{-2l} < 3^{p-2q} - 3^{-2q} = \inf V_{(p,q)}.$$

If $k-2l+2 \le p-2q$, then

$$\sup Y_{(k,l)} < 3^{k-2l+1} \le 2\cdot 3^{p-2q-1} = 3^{p-2q} - 3^{p-2q-1} \le \inf V_{(p,q)}.$$

If $k-2l+1 = p-2q$ and $l > q$, then

$$\sup V_{(p,q)} = 3^{p-2q} - 2\cdot 3^{-2q-1} < 3^{k-2l+1} - 3^{-2l+1} = \inf Y_{(k,l)}.$$

If $k-2l \ge p-2q$, then

$$\sup V_{(p,q)} < 3^{p-2q} \le 2\cdot 3^{k-2l} = 3^{k-2l+1} - 3^{k-2l} \le \inf Y_{(k,l)}.$$

Therefore, $G^1 \cap G^2 = \emptyset$.

We show that $\Phi_{\mathcal{G}^1} = \Phi_{\mathcal{G}^2}$. First observe that $a_n G^1 \subset a_{n-1}G^2$ and $a_n G^2 \subset a_{n-1}G^1$. More precisely, $(a_n G^1 \cap a_{n-1}G^2) = a_n G^1$ and $(a_n G^2 \cap a_{n-1}G^1) = a_n G^2$.

Fix $A \in \mathcal{S}$ and $x_0 \in \Phi_{\mathcal{G}^1}(A)$. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers. Then there exists subsequence $\{n_{k_m}+1\}_{m\in\mathbb{N}}$ of sequence $\{n_k+1\}_{k\in\mathbb{N}}$ such that

$$\limsup_{m\to\infty} \left(\left(a_{n_{k_m}+1}G^1 \setminus A\right) - x_0\right) \frac{1}{a_{n_{k_m}+1}} \in \mathcal{I}.$$

Moreover,

$$(a_{n_{k_m}} G^2 \cap a_{n_{k_m}+1}G^1) = a_{n_{k_m}} G^2.$$

Hence,

$$\limsup_{m\to\infty} \left(\left(a_{n_{k_m}} G^2 \setminus A\right) - x_0\right) \frac{1}{a_{n_{k_m}}} \in \mathcal{I}.$$

It implies that $x_0 \in \Phi_{\mathcal{G}^2}(A)$.

Analogously, we show that $x_0 \in \Phi_{\mathcal{G}^2}(A)$ implies $x_0 \in \Phi_{\mathcal{G}^1}(A)$.

Theorem 2.4. *For every sequence $\mathcal{Y} = \{Y_n\}_{n\in\mathbb{N}}$ tending to zero there exists an interval set B consisting of closed intervals such that 0 is an $\mathcal{I}(\mathcal{Y})$-density point of B.*

Proof. Let $Y_n = a_n Y + b_n$ for $n \in \mathbb{N}$, where $Y \in \mathcal{S}$ be a bounded set of the second category. Then there exist an open set G and a set $P \in \mathcal{I}$ such that $Y = G \triangle P$. Define

$$Z_n = \{|y| \colon y \in a_n G + b_n\}$$

and $z_n = \sup Z_n$, $l_n = \lambda(Z_n)$ for $n \in \mathbb{N}$. The sequence $\{z_n\}_{n\in\mathbb{N}}$ is tending to zero, hence we can choose subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ such that $2^{k+1} z_{n_{k+1}} < l_{n_k}$ for $k \in \mathbb{N}$. Let

$$J(1) = \left\{n \in \mathbb{N} \colon \left(Z_n \cap \left(\frac{z_{n_1}}{2}, z_{n_1}\right)\right) \ne \emptyset\right\}.$$

Since the sequence $\mathcal{Y} = \{Y_n\}_{n\in\mathbb{N}}$ is tending to zero, the set $J(1)$ is finite. Obviously $n_1 \in J(1)$. Let

$$d_1 = \frac{1}{2}\min\left\{\lambda\left(Z_n \cap \left(\frac{z_{n_1}}{2}, z_{n_1}\right)\right) : n \in J(1)\right\}.$$

Assume that we have $J(i)$, d_i for $i = 1, \dots, k$. Put

$$J(k+1) = \left\{n \in \mathbb{N}\colon \left(Z_n \cap \left(\frac{z_{n_{k+1}}}{2}, z_{n_{k+1}}\right)\right) \neq \emptyset\right\}.$$

Since the sequence $\mathcal{Y} = \{Y_n\}_{n\in\mathbb{N}}$ is tending to zero, the set $J(k+1)$ is finite. By definition, we have $n_{k+1} \in J(k+1)$. Let

$$d_{k+1} = \frac{1}{2^{k+1}}\min\left\{d_k, \lambda\left(Z_n \cap \left(\frac{z_{n_{k+1}}}{2}, z_{n_{k+1}}\right)\right) : n \in J(k+1)\right\}.$$

Putting

$$B = \bigcup_{k\in\mathbb{N}} \left(z_{n_{k+1}}, z_{n_k} - d_k\right) \cup \left(-z_{n_k} + d_k, -z_{n_{k+1}}\right)$$

we obtain that B is an interval set. Moreover, if $Y_n \setminus B \neq \emptyset$, we have

$$\begin{aligned}(Y_n \setminus B) \subset \Big(&\left[-z_{n_{m(n)}}, -z_{n_{m(n)}} + d_{m(n)}\right] \\ &\cup \left[-z_{n_{m(n)+1}}, z_{n_{m(n)+1}}\right] \\ &\cup \left[z_{n_{m(n)}} - d_{m(n)}, z_{n_{m(n)}}\right] \Big),\end{aligned}$$

where $m(n) = \min\left\{k \in \mathbb{N}\colon Z_n \cap \left(\frac{z_{n_k}}{2}, z_{n_k}\right) \neq \emptyset\right\}$, for $n \in \mathbb{N}$. Notice that

$$d_{m(n)} \leq \frac{\lambda(Z_n)}{2^{m(n)}} = \frac{a_n\lambda(G)}{2^{m(n)}}$$

and

$$z_{n_{m(n)+1}} \leq \frac{l_{n_{m(n)}}}{2^{m(n)+1}} \leq \frac{a_n\lambda(G)}{2^{m(n)}}.$$

Hence, the set $Y_n \setminus B$ is contained in three intervals with length less than $\frac{a_n\lambda(G)}{2^{m(n)}}$.

Therefore, the set $((Y_n \setminus B) - b_n)\,\frac{1}{a_n}$ is contained in 3 intervals with length less than $\frac{\lambda(G)}{2^{m(n)}}$.

Hence, for every sequence $\{n_m\}_{m\in\mathbb{N}}$ the set

$$\limsup_{m\to\infty}\left(((Y_{n_m} \setminus B) - b_{n_m})\frac{1}{a_{n_m}}\right)$$

contains at most three points, and as a result it is the first category set. For that reason 0 is an $\mathcal{I}(\mathcal{Y})$-density point of B. □

Quite recently there was published paper [1] containing results that for every set $B \in \mathcal{S}$ the set of all $\mathcal{I}$-density points of B is $\mathcal{F}_{\delta\sigma}$ set. In the light of above results we are motivated to pose the following question.

Problem 2.5. Let $B \in \mathcal{S}$ and $Y \in \mathcal{S}$ be a bounded set of the second category. Is the set $\Phi_{\mathcal{I}(\mathcal{J})}(B)$ Borel? In the case of an affirmative answer, what is the Borel class of the set $\Phi_{\mathcal{I}(\mathcal{J})}(B)$?

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