ON THE PATH PARTITION OF GRAPHS

Mekkia Kouider and Mohamed Zamime

Communicated by Gyula O.H. Katona

Abstract. Let G be a graph of order n. The maximum and minimum degree of G are denoted by Δ and δ, respectively. The path partition number $\mu(G)$ of a graph G is the minimum number of paths needed to partition the vertices of G. Magnant, Wang and Yuan conjectured that

$$\mu(G) \leq \max\left\{\frac{n}{\delta+1}, \frac{(\Delta-\delta)\, n}{(\Delta+\delta)}\right\}.$$

In this work, we give a positive answer to this conjecture, for $\Delta \geq 2\delta$.

Keywords: path, partition.

Mathematics Subject Classification: 05C20.

1. INTRODUCTION

Throughout the paper, all graphs are finite, simple and undirected. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. We denote by n the order of G. The *neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V : uv \in E\}$. The *degree* of v, denoted by $d(v)$, is the size of its neighborhood. The *minimum degree* of the graph G is denoted by $\delta(G)$, and the *maximum degree* by $\Delta(G)$.

Let A and B be two subsets of $V(G)$. Let $\varepsilon(A, B)$ be the number of edges with one end vertex in the set A the other one in the set B.

In this work, we deal with the partition problem. The cover problem and the partition problem constitute a large and important class of well studied problems in the fields of graph theory. A *cycle cover* of a graph (resp. a *path cover*) is a set $\mathcal{C}$ of cycles (resp. paths) of the graph such that each vertex belongs to at least one cycle (resp. one path) of $\mathcal{C}$. Many results on these concepts, have been given in the literature. For example, Kouider [5, 6], and Kouider and Lonc [7] studied the problem of covering a graph by a minimum number of cycles. More details and references can be found in the survey of Manuel [12].

Among the many variations of the partition problem, we mention the *path partition* that has been studied intensively for about sixty years. A family $\mathcal{P}$ of paths is called a *path partition* of a graph G if its members cover the vertices of the graph and are vertex disjoint. Its cardinality $|\mathcal{P}|$ is the number of paths of $\mathcal{P}$. The *path partition number* of G is

$$\mu(G) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a path partition of } G\}.$$

The concept of path partition number was introduced by Ore [13] in 1961. Several works have been done in this topic. See for example [1, 2, 4, 8, 9].

In 1996, Reed proved in [14] the following result.

Theorem 1.1 ([14]). *Let G be a connected cubic graph on n vertices. Then*

$$\mu(G) \leq \left\lceil \frac{n}{9} \right\rceil.$$

Furthermore, for 2-connected graphs, a better bound is established by Yu [15].

Theorem 1.2. *Let G be a 2-connected cubic graph on n vertices. Then*

$$\mu(G) \leq \left\lceil \frac{n}{10} \right\rceil.$$

For regular graphs, in 2009, Magnant and Martin [10] conjectured the following.

Conjecture 1.3 ([10]). Let G be a d-regular graph on n vertices. Then

$$\mu(G) \leq \frac{n}{d+1}.$$

They verified this last conjecture for the case $d \leq 5$ (see [10]). In 2018, Han obtained an asymptotic answer.

Theorem 1.4 ([4]). *For every c, $0 < c < 1$ and $\alpha > 0$, there exists n_0 such that if $n \geq n_0$, $d \geq cn$ and G is a d-regular graph on n vertices, then $n/(d+1)$ vertex-disjoint paths cover all vertices of G except αn.*

Gruskys and Letzter [3] improved this result by allowing to take $\alpha = 0$.

In 2016, Magnant, Wang and Yuan [11] extended Conjecture 1.3 to general graphs as follows.

Conjecture 1.5 ([11]). Let G be a graph on n vertices. Then

$$\mu(G) \leq \max\left\{\frac{n}{\delta+1}, \frac{(\Delta-\delta)\, n}{(\Delta+\delta)}\right\}.$$

If true, the last conjecture would be sharp. For $\delta + 2 \leq \Delta$, the bound is achieved by the collection of disjoint copies of $K_{\delta,\Delta}$. For $\delta = \Delta$, it is achieved by the collection of disjoint copies of complete graphs $K_{\delta+1}$. This conjecture is proved in [11] for the case $\delta = 1$ and $\delta = 2$.

In this work, we prove Conjecture 1.5 for all graphs with maximum degree Δ at least 2δ.

Theorem 1.6. *Let G be a graph of order n of minimum degree δ, ($\delta \geq 2$), and maximum degree Δ with $\Delta \geq 2\delta$. Then*

$$\mu(G) \leq \frac{(\Delta - \delta)\, n}{(\Delta + \delta)}.$$

We remark that $\frac{n}{\delta+1} \leq \frac{(\Delta-\delta)n}{(\Delta+\delta)}$ if and only if $\delta + 2 \leq \Delta$. So for $\delta \geq 2$ and $\Delta \geq 2\delta$, the inequality of the theorem is equivalent to

$$\mu(G) \leq \max\left\{\frac{n}{\delta+1}, \frac{(\Delta - \delta)\, n}{(\Delta + \delta)}\right\}$$

which is the inequality of Conjecture 1.5.

2. PRELIMINARIES

Let us introduce the following notations and definitions. Let $\mathcal{P}$ be a minimum path partition of $V(G)$. So, $|\mathcal{P}| = \mu(G)$. Let p_i be the number of paths of order $i \in \{1, 2\}$ in $\mathcal{P}$.

We may suppose that $p_1 + p_2 \neq 0$, otherwise we have $\mu(G) \leq \frac{n}{3}$. As $\Delta \geq 2\delta$, we get $\mu(G) \leq \frac{(\Delta-\delta)n}{(\Delta+\delta)}$ and the problem is resolved.

Let V_1 be the set of isolated vertices of $\mathcal{P}$ and V_2 be the set of end vertices of the isolated edges of $\mathcal{P}$. We denote by R any path in $\mathcal{P}$, and we write $R = R[a,b] = [a, \ldots, b]$ if a and b are the end vertices of R. We set $End(R) = \{a, b\}$. Let $Int\,(R)$ be the set of internal vertices of R. Let $\mathcal{A} \subseteq \mathcal{P}$. We denote by $Int(\mathcal{A})$ (resp. $End(\mathcal{A})$) the set of internal (resp. end) vertices of the paths of $\mathcal{A}$. For i fixed, we denote by R_i any path of order i. We set $\mathcal{R}_i$ the set of paths of order i. By *abcd* or $[a, b, c, d]$ we denote a path with 4 vertices. For i odd, $i \geq 3$, let us set $C_i = \bigcup_{R \in \mathcal{R}_i} c\,(R)$, where $c(R)$ denotes the central vertex of the path R.

Example 2.1. Let us illustrate the above notations relative to a partition in Figure 1. We consider

$$\begin{aligned}
&\mathcal{R}_1 = \{x_1\}, \quad \mathcal{R}_2 = \{[x_2, x_3],\ [x_4, x_5],\ [x_6, x_7]\},\\
&\mathcal{R}_3 = \{[x_8, \ldots, x_9]\},\\
&\mathcal{R}_4 = \{[x_{10}, \ldots, x_{11}], [x_{12}, \ldots, x_{13}]\},\\
&\mathcal{R}_5 = \{[x_{14}, \ldots, x_{15}], [x_{16}, \ldots, x_{17}], [x_{18}, \ldots, x_{19}], [x_{20}, \ldots, x_{21}]\},\\
&End(\mathcal{R}_3) = \{x_8, x_9\}, \quad End(\mathcal{R}_4) = \{x_{10}, x_{11}, x_{12}, x_{13}\},\\
&End(\mathcal{R}_5) = \{x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}\},\\
&C_3 = \{w_1\}, \quad C_5 = \{w_5, w_6, w_7, w_8\}.
\end{aligned}$$

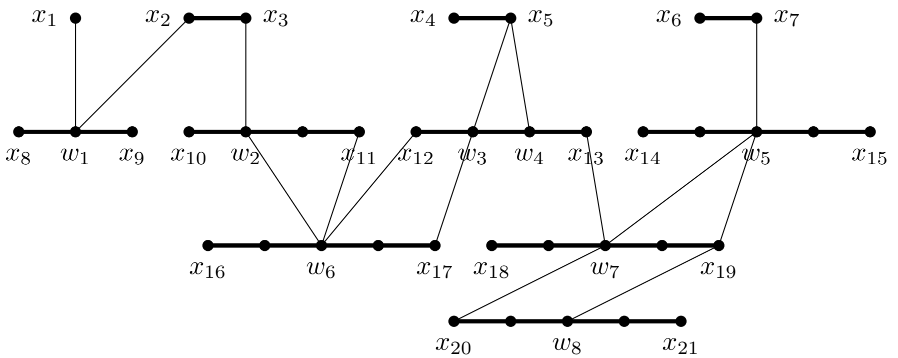

Fig. 1. Illustration of the definitions

For $x \in End(R)$, $N_{ext}(x)$ is the set of non path neighbors (neighbors of x outside its own path R) and $N_{ext}(X') = \bigcup_{x \in X'} N_{ext}(x)$ with $X' \subset End(\mathcal{P})$. Now using $N_{ext}(V_1 \cup V_2)$, we define a subset X of $End(\mathcal{P})$, and we denote $N_{ext}(X)$ by W. Let $X_1 = V_1 \cup V_2$, $W_1 = N_{ext}(x_1)$ and for $t \geq 1$, X_t being defined, let

$$X_{t+1} = X_t \cup \left(\bigcup_{N_{ext}(X_t) \cap Int(R) \neq \emptyset, R \in \mathcal{R}} End(R) \right).$$

Let $s \geq 1$ the first integer such that $X_s = X_{s+1}$. Let us set $X = X_s$, $W = N_{ext}(x)$ and for $t \in \{1, \ldots, s\}$, let $W_{t+1} = N_{ext}(X_{t+1}) \setminus N_{ext}(X_t)$. Then $W = \bigcup_{i=1}^{i=s} W_i$. Here is an example of that construction.

Example 2.2. For the partition in Figure 1, we have

$$\begin{aligned}
X_1 &= \{x_1, x_2, \ldots, x_7\}, \\
X_2 &= X_1 \cup \{x_8, x_9, \ldots, x_{15}\}, \\
X_3 &= X_2 \cup \{x_{16}, x_{17}, x_{18}, x_{19}\}, \\
X_4 &= X_3 \cup \{x_{20}, x_{21}\} = X, \\
W_1 &= \{w_1, w_2, \ldots, w_5\}, \quad W_2 = \{w_6, w_7\}, \quad W_3 = \{w_8\}, \\
W &= \{w_1, w_2, \ldots, w_8\}.
\end{aligned}$$

Let $X_0 = \emptyset$. Pick $w_r \in W_r$ for some r. By definition of w_r, there exists a sequence

$$\alpha(w_r) = x_1 w_1, x_2 w_2, \ldots, x_r w_r,$$

where for each $t \in \{1, \ldots, r\}$, $x_t \in X_t - X_{t-1}$, $w_t \in W_t$ and $x_t w_t$ is an edge joining two paths of the partition. In addition, for each $t \in \{1, \ldots, r-1\}$, w_t and x_{t+1} are in the same path of the partition.

The sequence $\alpha(w_r)$ *has a good order* if the vertex w_r belongs to a path R with end vertices, say x_{r+1} and x'_{r+1}, in $X_{r+1} - X_r$. *The vertex* w_r *is then said to be of good order*. Using a sequence $\alpha(w_r)$ with *good order*, we can define two new partitions as follows.

For each $i \in \{1, \ldots, r+1\}$, we orient the paths of $\mathcal{P}$ such that each x_i is the terminal extremity. We denote by w_t^+ and w_t^- the successor and the predecessor of w_t, respectively.

(1) $\mathcal{P}_1(w_r)$ is obtained from $\mathcal{P}$ by deleting the edges $w_t w_t^+$, $1 \le t \le r$ and adding the edges $x_t w_t$ for $1 \le t \le r$;

(2) $\mathcal{P}_2(w_r)$ is obtained from $\mathcal{P}$ by deleting the edges $w_t w_t^+$, $1 \le t \le r-1$ and the edge $w_r w_r^-$ and adding the edges $x_t w_t$ for $1 \le t \le r$.

If we consider the sets of edges of these partitions we note that

$$E(\mathcal{P}_2) = (E(\mathcal{P}_1) - w_r w_r^-) \cup w_r w_r^+ .$$

Furthermore, $|\mathcal{P}_2| = |\mathcal{P}_1| = \mu(G)$.

For example, the sequence $\alpha(w_2)$ in the graph of Figure 2, defines two partitions. We have

$$\mathcal{P}_1(w_2) = \left\{x'_1 x_1 w_1 w_1^-, w_1^+ x_2 w_2 w_2^- x'_3, x_3 w_2^+, x'_4 w_3^- w_3 w_3^+ x_4\right\}$$

and

$$\mathcal{P}_2(w_2) = \left\{x'_1 x_1 w_1 w_1^-, w_1^+ x_2 w_2 w_2^+ x_3, w_2^- x'_3, x'_4 w_3^- w_3 w_3^+ x_4\right\} .$$

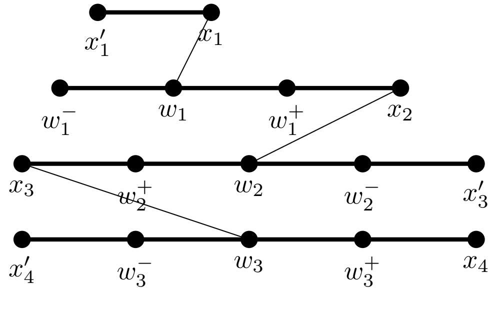

Fig. 2. Graph with $\mu(G) = 4$

We denote by $R_i[x', x]$ any path of order i oriented from x' to x. So x' is the initial end of R_i and x is its terminal end.

Observation 2.3. *If* $w \in W_t$*, then for some* i*,* w *belongs to some path* $R_i[x', x]$*. The path* $R_{i_1}[w^+, x]$ *is in* $\mathcal{P}_1(w)$ *and the path* $R_{i_2}[x', w^-]$ *is in* $\mathcal{P}_2(w)$*. Note that the subpath* $R_{i_1}[w^+, x]$ *(resp.* $R_{i_2}[x', w^-]$*) is of order* i_1 *(resp.* i_2*) such that* $i_1 + i_2 + 1 = i$*.*

3. PROOF OF THEOREM 1.6

We choose a minimum path-partition $\mathcal{P}_0$ such that:

(1) p_1 is minimum,
(2) if (1) is satisfied, then p_2 is minimum.

Let $\mathcal{P}'$ be the set of paths with end vertices in X. Let $p = |\mathcal{P}'|$. Let p_i be the number of paths of order i in $\mathcal{P}'$.

Let us outline the sketch of the proof.

In view to bound $\mu(G)$ we want to bound p_1 and p_2. We consider the set X generated by $V_1 \cup V_2$, and therefore the two sets $W = N_{ext}(X)$ and $\varepsilon(X, W)$. Note that the cardinality of X is $2p - p_1$, The proof of our theorem is done through the following steps. We want to bound in two manners the number of edges $\varepsilon(X, W)$. The upper bound will use W and Δ, the lower bound will use X and δ.

In the first part of the proof, we show some claims relative to the set W and one relative to the lower bound of $\varepsilon(x, W)$ for $x \in X$.

In the second part of the proof, we calculate the bounds of $\varepsilon(X, W)$. We get finally an upper bound for $p_1 + 2p_2$ in function of p, δ and Δ, and, then an upper bound for $\mu(G)$.

3.1. CLAIMS

Claim 3.1.

(1) *For each* $v \in V_1$, $N(v) \subset C_3$.
(2) *For each* $a \in V_2$, $N(a) \subset C_3 \cup Int(\mathcal{R}_4) \cup C_5$.

So, $\begin{cases} |N(a) \cap R| \leq 1, & \textit{for every path } R \textit{ of order } 3 \textit{ or } 5, \\ |N(a) \cap R| \leq 2, & \textit{for every path } R \textit{ of order } 4. \end{cases}$

Proof. (1) Let v be a vertex of V_1. By the minimality of $\mathcal{P}_0$, v is not adjacent to an end vertex of another path in $\mathcal{P}_0$. Let w be a neighbor of v in a path oriented from x' to x. We have two partitions. In $\mathcal{P}_0$, we replace the path v and the path $R_i[x', x]$ either by the pair of paths $vR_{i_1}[w, x]$, $R_{i_2}[x', w^-]$ or by the pair of paths $R_{i'_1}[x', w]\,v$, $R_{i'_2}[w^+, x]$. By the minimality of p_1, w is both predecessor of x and successor of x'. So the order of $R_i[x', x]$ is 3, and w is the center of $R_i[x', x]$. Thus, $N(v) \subset C_3$.

(2) Let w be a neighbor of a in $R_i[x', x]$. As precedently, we get two partitions and by definition of $\mathcal{P}_0$, each of $R_i[x', w^-]$ and $R_i[w^+, x]$ should be of order at most two. So $N(a) \subset C_3 \cup C_5 \cup Int(\mathcal{R}_4)$, completing the proof of Claim 3.1. □

Claim 3.2. *Let* W_a *be the set of vertices of good order in* W. *Then:*

(1) $W_a \subset C_3 \cup Int(\mathcal{R}_4) \cup C_5$,
(2) $W = W_a$.

Proof. (1) Suppose that there exists $w \in W_a$ such that w is in the path $R_i\,[x', x]$. By Observation 2.3, we have $i - 1 = i_1 + i_2$. If $i_1 \geq 3$ (resp. $i_2 \geq 3$), then $\mathcal{P}_1\,(w)$ (resp. $\mathcal{P}_2\,(w)$) contains $p_1 - 1$ paths of order 1 or p_1 paths of order 1 and $p_2 - 1$ paths of order 2. A contradiction with the definition of $\mathcal{P}_0$. Thus, $W_a \subset C_3 \cup Int\,(\mathcal{R}_4) \cup C_5$.

(2) Suppose that $W \neq W_a$. In $W_b - W_a$ there exists necessarily a vertex $w = w_r$ with sequence $\alpha\,(w) = x_1 w_1, x_2 w_2, \ldots, x_r w_r$ with $x_t \in X_t - X_{t-1}$ and x_r is an end vertex of some path $R = [x'_r, \ldots, x_r]$. By the definition of w, the vertex w belongs to a path $R' = [x'_j, \ldots, x_j]$ with $x_j \in X_j$, $j \leq r$. By the definition of X_j, the path R' contains one element of W_a, say w_a. By the definition of W_a, w_a is adjacent to a vertex x'' of X_{j-1}, end vertex of a path R''. Since $W_a \subset C_3 \cup Int\,(\mathcal{R}_4) \cup C_5$, then $|R'| = 4$ or 5 and $w = w_a^-$ or $w = w_a^+$. The end vertex $x_r \in X_r$ and the end vertex $x'' \in X_{j-1}$ are adjacent respectively to w and w_a, successive vertices of the same path R'.

We get a partition with $p - 1$ paths. We replace the three paths R, R' and R'' by the two paths composed by $R \cup R' \cup R'' \cup \{x_r w, x'' w_a\} - \{w w_a\}$ (see Figure 3), a contradiction with the minimality of $\mathcal{P}_0$. Thus, $W_b = \emptyset$ and so, $W = W_a$. This completes the proof of Claim 3.2.

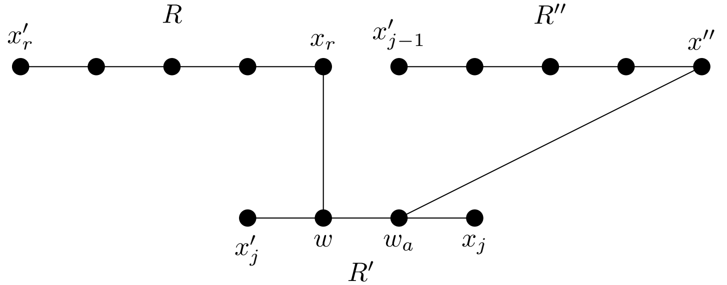

Fig. 3. Paths R, R', R''

□

Claim 3.3. *For each path R of order 4 in $\mathcal{P}_0$, we have $|W \cap V(R)| \leq 2$. Furthermore, if $|W \cap V(R)| = 2$, then there is a unique $x \in X$ such that $W \cap V(R) \neq \emptyset$ and thus $\varepsilon(X, R) = 4$.*

Proof. By the minimality of $\mathcal{P}_0$ for each path $R \in \mathcal{R}_4$, we have $|W \cap V(R)| \leq 2$. Now, assume that there exists a path $R \in \mathcal{R}_4$ such that $W \cap V\,(R) = \{w_1, w_2\}$ where $w_i \in N_{ext}\,(x_i)$, $i = 1$, 2. By taking off the edge $w_1 w_2$ and adding the edges $x_1 w_1$ and $x_2 w_2$ we obtain a partition with $p - 1$ paths, a contradiction. So $x_1 = x_2$. Let $R'(x_1)$ be the path of extremity x_1 in the partition. We may suppose $R = [x'_0, w_1, w_2, x_0]$. If x_0 is neighbor of w_1, then we replace the two paths R and $R'(x_1)$ by the path $R'(x_1) \cup [w_2, x_0, w_1, x'_0]$. We get a partition with $p - 1$ paths, a contradiction. So, there is no edge $x_0 w_1$. Similarly, there is no edge $x'_0 w_2$. It follows that $\varepsilon(X, R) = 4$, completing the proof of Claim 3.3. □

For $i \in \{1, 2\}$, let $\mathcal{R}_4^i$ be the set of paths of order 4, which contain exactly i elements of W.

For the lower bound we shall need the following claim.

Claim 3.4.

(1) *If* $x \in X \cap V(\mathcal{R}_1 \cup \mathcal{R}_3)$*, then* x *has* $d(x)$ *neighbors in* W*.*
(2) *If* x_1*,* $x_2 \in X \cap V(\mathcal{R}_4)$*, then the set* $\{x_1, x_2\}$ *has* $d(x_1) + d(x_2) - 1$ *neighbors in* W *if* x_1*,* x_2 *belong to* $V\left(\mathcal{R}_4^1\right)$*, and it has* $d(x_1) + d(x_2)$ *neighbors if* x_1*,* x_2 *belong to* $V\left(\mathcal{R}_4^2\right)$*.*
(3) *If* $x \in X \cap V(\mathcal{R}_2 \cup \mathcal{R}_5)$*, then* x *has* $d(x) - 1$ *neighbors in* W*.*

Proof. (1) If $x \in V_1$, then by Claim 3.1, $N(x) \subset W$. If $x \in X \cap V(\mathcal{R}_3)$, then let $R = xyx'$. By the minimality of μ, x and x' are not adjacent. It follows that $N(x) \subset W$.

(2) Let $R' = x_1yzx_2$. By the minimality of p, there is no edge x_1x_2.

First assume that $W \cap R' = \{y\}$. By definition of W, y is neighbor of an end vertex, say x of path $R = [x', \ldots, x]$. If $x_1z \in E(G)$, then we replace the two paths R and R' by exactly one path $x_2zx_1yx \ldots x'$, a contradiction with the minimality of the path partition. It follows that $N(x_1) \subset W$. Note that $(N(x_2) - \{z\}) \subset W$.

Now assume that $W \cap V(R') = \{y, z\}$, then $N(x_1) \subset W$ and $N(x_2) \subset W$.

(3) Clearly if $x \in V_2$, then x has $d(x) - 1$ neighbors in W. Let $R'' = x_1yztx_2$. By Claim 3.2, we have $W \cap V(R'') = \{z\}$. It follows that $N(x_1) - \{y\} \subset W$ and $N(x_2) - \{t\} \subset W$, completing the proof of Claim 3.4. □

Now we bound first $|\mathcal{P}'|$ then $\mu(G)$.

3.2. CALCULATIONS OF THE BOUNDS

(1) *The bound of* $p_1 + 2p_2$*.*

Let $k = \frac{\Delta}{\delta}$. We shall prove the following inequality.

Claim 3.5.

$$p_1 + 2p_2 \leq (p_3 + p_4 + p_5)(k - 2) + \frac{2}{\delta}p_2,$$

where p_i *is the number of paths of order* i *in* $\mathcal{P}'$*.*

Proof. Put $p'_4 = \left|\mathcal{R}_4^1\right|$ and $p''_4 = \left|\mathcal{R}_4^2\right|$. Let $\varepsilon(X, W)$ be the number of edges between X and W. Let w be any vertex of W. Observe that for $w \in Int\,(\mathcal{R}_5)$, w has at most $\Delta - 2$ neighbors in X. For $w \in Int\left(\mathcal{R}_4^1\right)$, then w has at least one neighbor which does not belong to X and so, w has at most $\Delta - 1$ neighbors in X. By Claim 3.3, if $w \in Int\left(\mathcal{R}_4^2\right)$, then w has exactly two neighbors in X. If $w \in W \cap Int(\mathcal{R}_3)$ then w has at most Δ neighbors in X. By Claims 3.1 and 3.2, for each $i \geq 6$, $W \cap Int\,(\mathcal{R}_i) = \emptyset$ and $X \cap End\,(\mathcal{R}_i) = \emptyset$. It follows that

$$\varepsilon(X, W) \leq p_3\Delta + p'_4\,(\Delta - 1) + 4p''_4 + p_5(\Delta - 2).$$

On the other hand, by Claim 3.4,

$$\begin{aligned}\varepsilon(X,W) \geq & \sum_{x\in End(\mathcal{R}_1)} d(x) + \sum_{x\in End(\mathcal{R}_2)} (d(x)-1) \\ & + \sum_{x\in End(\mathcal{R}_3)\cap X} d(x) \\ & + \sum_{x_1,x_2\in End(\mathcal{R}_4^1)\cap X} (d(x_1)+d(x_2)-1) \\ & + \sum_{x_1,x_2\in End(\mathcal{R}_4^2)\cap X} (d(x_1)+d(x_2)) \\ & + \sum_{x\in End(\mathcal{R}_5)\cap X} (d(x)-1).\end{aligned}$$

As for each $x \in X$ we have $\delta \leq d(x)$, it follows that

$$\begin{aligned}& p_1\delta + 2p_2(\delta-1) + 2p_3\delta + p_4'(2\delta-1) + p_4''2\delta + 2p_5(\delta-1) \\ & \leq p_3\Delta + p_4'(\Delta-1) + 4p_4'' + p_5(\Delta-2).\end{aligned}$$

As $k \geq 2$ and $\delta \geq 2$, then $\Delta \geq 4$ and we replace $4p_4''$ by $\Delta p_4''$ in the last inequality. Then

$$\begin{aligned}& p_1\delta + 2p_2(\delta-1) + 2p_3\delta + p_4'(2\delta-1) + p_4''2\delta + 2p_5(\delta-1) \\ & \leq p_3\Delta + p_4'(\Delta-1) + p_4''\Delta + p_5(\Delta-2).\end{aligned}$$

As $p_4 = p_4' + p_4''$, we get

$$p_1\delta + 2p_2(\delta-1) \leq p_3(\Delta-2\delta) + p_4(\Delta-2\delta) + p_5(\Delta-2\delta).$$

Since $\Delta = k\delta$, then

$$p_1 + 2p_2 \leq (p_3+p_4+p_5)(k-2) + \frac{2}{\delta}p_2.$$

This completes the proof of Claim 3.5. □

(2) *Calculation of the bound of* $|\mathcal{P}'| = p$.

By Claim 3.5, there exists $r \leq k-2$, such that

$$p_1 + 2p_2 = r(p_3+p_4+p_5) + \frac{2}{\delta}p_2. \tag{3.1}$$

Let us call n_1 the order of $V(\mathcal{P}')$. Recall that $\mathcal{P}'$ is the set of paths with end vertices in X, $p = |\mathcal{P}'|$ and p_i is the number of paths of order i in $\mathcal{P}'$. By Claims 3.1 and 3.2,

$\mathcal{P}'$ contains no path of order at least 6. It follows that

$$p = p_1 + p_2 + p_3 + p_4 + p_5$$

and

$$n_1 = p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5.$$

Using equality (3.1), we obtain

$$p = (r+1)(p_3 + p_4 + p_5) + \left(\frac{2}{\delta} - 1\right) p_2.$$

As $\delta \geq 2$, we get $p \leq (r+1)(p_3 + p_4 + p_5)$. Again by equality (3.1), we have

$$n_1 = (r+3)p_3 + (r+4)p_4 + (r+5)p_5 + \frac{2}{\delta} p_2.$$

This yields

$$n_1 \geq (r+3)(p_3 + p_4 + p_5).$$

Since $r \leq (k-2)$, we get the inequality $p \leq \frac{k-1}{k+1} n_1$.

(3) *The bound of* $\mu(G)$.

Let $G_2 = G - V(\mathcal{P}')$. Let $n_2 = n - n_1$. Clearly $\mu(G) \leq p + \mu(G_2)$. We know that $p \leq \frac{k-1}{k+1} n_1$. Recall that each path of $\mathcal{P}_0$ contained in G_2 has order at least 3. It follows that $\mu(G_2) \leq \frac{n_2}{3}$. Since $k \geq 2$, we have $\frac{1}{3} \leq \frac{k-1}{k+1}$ and so $\mu(G_2) \leq \frac{k-1}{k+1} n_2$. Thus, $\mu(G) \leq \frac{k-1}{k+1} n$. This finishes the proof of the theorem.

Acknowledgements

The authors are grateful to the anonymous referee whose careful reading gave them the opportunity to improve the quality of the paper.

REFERENCES

[1] M. Chen, J. Li, L. Wang, L. Zhang, *On partitioning simple bipartite graphs in vertex-disjoint paths*, Southeast Asian Bull. Math. **31** (2007), no. 2, 225–230.

[2] H. Enomoto, K. Ota, *Partitions of a graph into paths with prescribed endvertices and lengths*, J. Graph Theory **34** (2000), no. 2, 163–169.

[3] V. Gruslys, S. Letzter, *Cycle partitions of regular graphs*, arXiv:1808.00851.

[4] J. Han, *On vertex-disjoint paths in regular graphs*, Electron. J. Combin. **25** (2018), no. 2, #P2.12.

[5] M. Kouider, *Neighborhood and covering vertices by cycles*, Combinatorica **20** (2000), no. 2, 219–226.

[6] M. Kouider, *Covering vertices by cycles*, J. Graph Theory **18** (1994), no. 8, 757–776.

[7] M. Kouider, Z. Lonc, *Covering cycles and k-term degree sums*, Combinatorica **16** (1996), 407–412.

[8] J. Li, G. Steiner, *Partitioning a bipartite graph into vertex-disjoint paths*, Ars Combin. **81** (2006), 161–173.

[9] Ch. Lu, Q. Zhou, *Path covering number and L(2, 1)-labeling number of graphs*, Discrete Appl. Math. **161** (2013), 2062–2074.

[10] C. Magnant, D.M. Martin, *A note on the path cover number of regular graphs*, Australas. J. Combin. **43** (2009), 211–217.

[11] C. Magnant, H. Wang, *Path partition of almost regular graphs*, Australas. J. Combin. **64** (2016), 334–340.

[12] P. Manuel, *Revisiting path-type covering and partitioning problems*, (2018), hal-01849313.

[13] O. Ore, *Arc coverings of graphs*, Ann. Mat. Pura Appl. **55** (1961), no. 4, 315–321.

[14] B. Reed, *Paths, stars and the number three*, Combin. Probab. Comput. **5** (1996), no. 3, 277–295.

[15] G. Yu, *Covering* 2*-connected* 3*-regular graphs by disjoint paths*, J. Graph Theory **18** (2018), no. 3, 385–401.

Mekkia Kouider (corresponding author)
km@lri.fr

University Paris Sud
France

Mohamed Zamime
zamimemohamed@yahoo.com

University Yahia Fares of Medea
Department of Technology
Mathematics Laboratory and its Applications
Medea, Algeria

Received: January 17, 2023.
Revised: June 24, 2023.
Accepted: July 7, 2023.