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LOCAL IRREGULARITY CONJECTURE FOR 2-MULTIGRAPHS VERSUS CACTI

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Abstract. A multigraph is locally irregular if the degrees of the end-vertices of every multiedge are distinct. The locally irregular coloring is an edge coloring of a multigraph G such that every color induces a locally irregular submultigraph of G. A locally irregular colorable multigraph G is any multigraph which admits a locally irregular coloring. We denote by $\mathrm{lir}(G)$ the locally irregular chromatic index of a multigraph G, which is the smallest number of colors required in the locally irregular coloring of the locally irregular colorable multigraph G. In case of graphs the definitions are similar. The Local Irregularity Conjecture for 2-multigraphs claims that for every connected graph G, which is not isomorphic to K_2, multigraph ${}^2G$ obtained from G by doubling each edge satisfies $\mathrm{lir}({}^2G) \leq 2$. We show this conjecture for cacti. This class of graphs is important for the Local Irregularity Conjecture for 2-multigraphs and the Local Irregularity Conjecture which claims that every locally irregular colorable graph G satisfies $\mathrm{lir}(G) \leq 3$. At the beginning it has been observed that all not locally irregular colorable graphs are cacti. Recently it has been proved that there is only one cactus which requires 4 colors for a locally irregular coloring and therefore the Local Irregularity Conjecture was disproved.

Keywords: locally irregular coloring, decomposable, cactus graphs, 2-multigraphs.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

All graphs and multigraphs considered in this paper are finite. The main interest of this paper are edge colorings of a multigraph. Let $G = (V, E)$ be a graph. We say that a graph is *locally irregular* if the degrees of the two end-vertices of every edge are distinct. A *locally irregular coloring* of a graph G is an edge coloring of G such that every color induces a locally irregular subgraph of G. We denote by $\mathrm{lir}(G)$ the *locally irregular chromatic index* of a graph G which is the smallest number k such that there exists a locally irregular coloring of G with k colors. This problem is closely related to the well known 1-2-3 Conjecture proposed by Karoński, Łuczak and Thomason in [6].

Note that not every graph has a locally irregular coloring. We define the family $\mathfrak{T}$ recursively in the following way:

- the triangle K_3 belongs to $\mathfrak{T}$,
- if G is a graph from $\mathfrak{T}$, then any graph G' obtained from G by identifying a vertex $v \in V(G)$ of degree 2, which belongs to a triangle in G, with an end vertex of a path of even length or with an end vertex of a path of odd length such that the other end vertex of that path is identified with a vertex of a new triangle.

The family $\mathfrak{T}'$ consists of the family $\mathfrak{T}$, all odd length paths and all odd length cycles. In [2] Baudon, Bensmail, Przybyło and Woźniak proved that only the graphs from the family $\mathfrak{T}'$ are not locally irregular colorable (do not admit locally irregular coloring). They also proposed the Local Irregularity Conjecture which says that every connected graph $G \notin \mathfrak{T}'$ satisfies $\mathrm{lir}(G) \leq 3$.

However, Sedlar and Škrekovski in [12] showed that the bow-tie graph B which is presented in Figure 1 does not have locally irregular coloring with three colors. They also proved in [11] that every locally irregular colorable cactus $G \neq B$ satisfies $\mathrm{lir}(G) \leq 3$. By a *cactus* we mean a graph in which no two cycles intersect in more than one vertex.

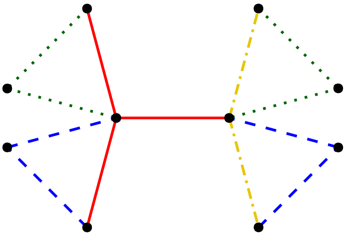

Fig. 1. The bow-tie graph B and its locally irregular coloring with four colors

Furthermore, they proposed the following new version of the Local Irregularity Conjecture.

Conjecture 1.1 ([11]). *Every connected graph $G \notin \mathfrak{T}'$ except for the bow-tie graph B satisfies* $\mathrm{lir}(G) \leq 3$.

This conjecture was proved for some graph classes for example trees [1], graphs with minimum degree at least 10^{10} [10], r-regular graphs where $r \geq 10^7$ [2], decomposable split graphs [7] and decomposable claw-free graphs with maximum degree 3 [8]. For planar graphs it is known that $\mathrm{lir}(G) \leq 15$ [3]. For general connected graphs Bensmail, Merker and Thomassen [4] proved that $\mathrm{lir}(G) \leq 328$ if $G \notin \mathfrak{T}'$. Later the constant upper bound was lowered to $\mathrm{lir}(G) \leq 220$ by Lužar, Przybyło and Soták [9].

Before we present the Local Irregularity Conjecture for 2-multigraphs we present more definitions. By 2-*multigraph*, denoted by ${}^2G$, we mean the multigraph obtained from a graph G by doubling each edge. We call an *edge multiplicity* the number of single edges forming a multiedge e in a multigraph G, and we denote it by $\mu_G(e)$. Multigraph

H is a *submultigraph* of a multigraph G if H is a subgraph of G and for each multiedge e of H, $\mu_H(e) \le \mu_G(e)$ holds. Analogically, multigraph H is an *induced submultigraph* of G if H is an induced subgraph of G and for each multiedge e of H, $\mu_H(e) = \mu_G(e)$ holds. We denote by $\hat{d}(v)$ degree of the vertex v in a multigraph (the number of single edges incident with the vertex v). The *locally irregular coloring* of a multigraph G is an edge coloring of G such that every color induces a locally irregular submultigraph of G. We say that multigraphs $\hat{G}_1$ and $\hat{G}_2$ create a *decomposition* of a multigraph $\hat{G}$ if for each edge e of G, $\mu_{\hat{G}_1}(e) + \mu_{\hat{G}_2}(e) = \mu_{\hat{G}}(e)$ holds. We say that a multigraph is *locally irregular colorable* if it satisfies the locally irregular coloring. By the *locally irregular chromatic index* of a locally irregular colorable multigraph G, denoted by $\mathrm{lir}(G)$, we mean the smallest number of colors required in a locally irregular coloring of G. We will say that a multiedge is colored red-blue if one element of the multiedge is red and the second is blue. We proposed in [5] the following conjecture for 2-multigraphs.

Conjecture 1.2 (Local Irregularity Conjecture for 2-multigraphs [5])**.** *For every connected graph G which is not isomorphic to K_2 we have* $\mathrm{lir}(^2G) \le 2$.

We also proved that for general connected graphs $\mathrm{lir}(^2G) \le 76$ if G is not isomorphic to K_2. Moreover, we proved the following theorem concerning the above conjecture.

Theorem 1.3 ([5])**.** *The Local Irregularity Conjecture for 2-multigraphs holds for: paths, cycles, wheels W_n, complete graphs K_n, for $n \ge 3$, bipartite graphs and complete k-partite graphs, for $k \ge 3$.*

Now, we recall the locally irregular coloring of multipaths and multicycles from the proof of above theorem given by us in [5] which will be useful throughout the proof of our main result. We will denote by P_n a path with n vertices.

For a multipath ^2P_n of even length we color the first two multiedges blue, the next two multiedges red, and we repeat this color sequence until the end of the multipath. We do not consider the multipath of length one. For a longer multipath of odd length we color the first multiedge blue, the second red-blue, the third red, and then we color the remaining multiedges in the same way as multipath of even length.

We color multicycles of length from three to seven as in Figure 2. The coloring of longer multicycle we obtain by replacing two red multiedges with a multipath of length $4k + 2$ colored: the first two multiedges red, the next two multiedges blue, the subsequent two multiedges red, and so on in the appropriate colored multicycle of length from four to seven.

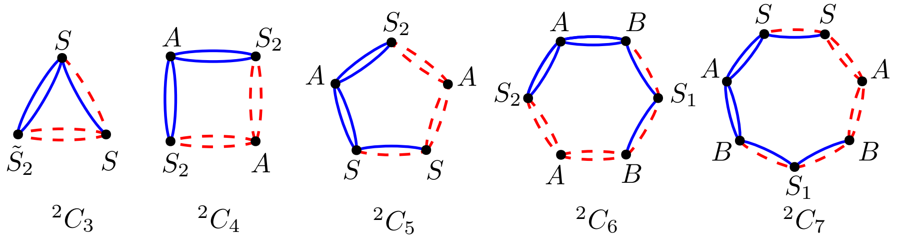

Fig. 2. A locally irregular coloring of the multicycles. The meaning of vertex labels will be explained in Theorem 2.3

In this paper we will show that Conjecture 1.2 holds for cacti. The mentioned results for graphs so far indicate that cacti are important class of graphs for the problem of locally irregular coloring, because graphs from the family $\mathfrak{T}$ which is not locally irregular colorable are cacti and the only known counterexample for the Local Irregularity Conjecture is also a cactus. This motivated us to check the Local Irregularity Conjecture for 2-multigraphs for cacti.

2. MAIN RESULT

We need to introduce the notation and important lemma for trees. In the proof of our main result we will often use locally irregular coloring of rooted multitrees presented in the proof of this lemma. First, we will denote by $T(r)$ a tree rooted at a vertex r. By a *leaf edge* we mean an edge that contain a vertex which is a leaf in a tree T. A *shrub* is any tree rooted at a leaf. The only edge in a shrub G incident to the root we will call the *root edge* of G. We will call *branch* rooted at vertex x, denoted by $B(x)$, the following subgraph in rooted tree T. If x has only one son x' in T a branch $B(x)$ includes: edge xx', edges from x' to all its sons $x_1, \ldots, x_k$ and if x_j has only one neighbor x'_j which is a leaf in T we also include to the branch $B(x)$ leaf edge $x_jx'_j$ for $j = 1, \ldots, k$. If x has more than one son in T a branch $B(x)$ includes: edges from x to all its sons $x_1, \ldots, x_k$ and if x_j has only one neighbor x'_j which is a leaf in T we also include to the branch $B(x)$ leaf edge $x_jx'_j$ for $j = 1, \ldots, k$.

Remark 2.1. There are only four branches: K_2 rooted at an initial vertex, path P_4 rooted at an internal vertex, path P_4 rooted at an initial vertex and path P_5 rooted at a central vertex, which are not locally irregular.

We will call such branches *exceptional*. We will use corresponding notation for 2-multigraphs.

Lemma 2.2. *For each tree T rooted at the vertex r there exists a decomposition of ${}^2T$ into branches generated by a red-blue almost locally irregular coloring in this sense the only possible conflict may occur between r and one or two of its neighbours. Moreover, all branches contain at least two multiedges, except for the situation when T is isomorphic to K_2, and all leaf edges are monochromatic in these branches.*

Proof. The lemma holds obviously for $T = K_2$. Assume that $T \neq K_2$. First, we decompose ${}^2T(r)$ into branches in the following way. Let ${}^2B(r)$ be the first branch from our decomposition. Let x_i for $i = 1, \ldots, p$ be a leaf in ${}^2B(r)$, which is not a leaf in T. The next branches of the decomposition are ${}^2B(x_1), \ldots, {}^2B(x_p)$. We start coloring from ${}^2B(r)$, next we color branches with root at a vertex from ${}^2B(r)$ and so on. Branch ${}^2B(r)$ is colored by one color except for the situation when ${}^2B(r) = {}^2T(r)$ and ${}^2B(r)$ is a multipath ${}^2P_4$ rooted at one of its ends. In this situation we color root multiedge blue and the rest of this branch red. With such coloring there is a conflict between the root r and its son. Note that if ${}^2B(r)$ is one of the exceptional branches that is colored one color, there is a conflict between r and one or two of his sons. Let ${}^2B(x)$ where $x \neq r$ be a branch from the above decomposition. If ${}^2B(x)$ is not an exceptional branch, we color the whole branch ${}^2B(x)$ using only one color different from the color of multiedge from the vertex x to its father. Note that this multiedge is monochromatic. We can easily see that in such coloring the multiedges to leaves in branch are still monochromatic.

If ${}^2B(x)$ is an exceptional branch, we denote by x_0 the father of x. Assume that multiedge x_0x is colored red. We use the coloring of exceptional branch ${}^2B(x)$ presented in Figure 3.

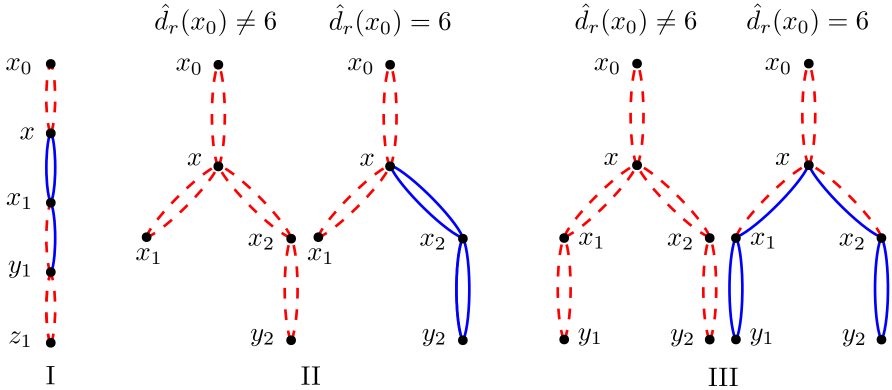

Fig. 3. The coloring of branch ${}^2B(x)$ which is an exception

Note that multiedge x_0x has always only one color. □

Remark that in the coloring of rooted multitree presented in the proof of above lemma, the root and its sons are always even where by *even* vertex we mean a vertex which has even degrees in both colors and similarly by *odd* vertex we mean a vertex which has odd degrees in both colors. Note that all vertices are either odd or even because in 2-multigraph every vertex has even sum of red and blue degrees.

In multicactus we call *cyclic* all vertices on multicycles and the remaining vertices are *woody*. Now we are ready to prove our main result for cacti.

Theorem 2.3. *For every cactus G, the multigraph ${}^2G$ satisfies* $\mathrm{lir}({}^2G) \leq 2$.

Proof. We give a construction of locally irregular coloring of ${}^2G$. We will use the method of *vertex labeling* which will help us to do this. We treat multicactus as a multigraph obtained from a multicycle by *adding* to vertices multicycles and multitrees where by adding or *joining* a multicycle to a vertex x we mean identifying the vertex x with one of the vertices on the multicycle, extending locally irregular coloring and vertex labeling to this multicycle. Similarly, by adding or joining rooted multitree ${}^2T(r)$ to a vertex x we mean identifying the vertex x with the root of multitree ${}^2T(r)$, extending locally irregular coloring to ${}^2T(r)$, and possibly changing the label of x.

Now we briefly present the outline of our construction. First, we color locally irregularly the longest multicycle in ${}^2G$ and label its vertices. (Details and possible exceptions are described below.)

Let x be the vertex in the already-colored part to which we want to join still uncolored elements such as multicycles or multitree.

Next, we select one vertex on the colored multicycle and attach to it all the multicycles and multitree to be attached to it, and extend both coloring and labeling of the cyclic vertices to the attached elements. As a reminder, woody vertices do not get labels. Then we take one by one remaining vertices on this colored multicycle, and we treat them analogously.

If x is a cyclic vertex, then in general how the attached elements are colored depends on the label of the vertex x and also on the element we are attaching.

If x is not a cyclic vertex, then it must be a leaf. We proceed as before. Note that there must be a multicycle among the elements we add, so x will get some label.

All the necessary details of each part of the construction will be given below.

Remark 2.4. It may happen that the vertex x changes its label during this procedure. In some situations, instead of a multicycle, we are forced to consider a slightly modified structure, namely the so-called multicycle with spikes (see below). Finally, in some situations, instead of adding new elements to a fixed vertex, we have to consider two adjacent vertices simultaneously in the addition process.

Initial part of the locally irregular coloring of ^{2}G. We color in a standard way, except for the situation described below, the longest multicycle in ^2G using the same method as in the proof of Theorem 1.3. This coloring will be called *standard*. During the rest of the proof we will often use this standard coloring of multicycle. We also label all vertices on this colored multicycle as in Figure 2. We use labels A, B, S_1, S_2, $\tilde{S}_2$ and S with the following meaning:

- A – an even vertex which has red degree or blue degree greater than two, always remains labelled A and whose neighbours have labels from the set $\{B, S_2, S\}$.
- B – an odd vertex which always remains labelled B and whose neighbours have labels from the set $\{A, S_1\}$.
- S_1 – an even vertex x with: $\hat{d}(x) = 4$, $\hat{d}_r(x) = \hat{d}_b(x) = 2$, two incident multiedges colored red-blue which gets label A if we join something to it and whose neighbours are odd.
- S_2 – an even vertex x on multicycle of length greater than three with: $\hat{d}(x) = 4$, $\hat{d}_r(x) = \hat{d}_b(x) = 2$, two incident multiedges colored first blue second red which gets label B if we join something to it and whose neighbours have label A.
- $\tilde{S}_2$ – even vertex x on $^2C_3$ with: $\hat{d}(x) = 4$, $\hat{d}_r(x) = \hat{d}_b(x) = 2$, two incident multiedges colored first blue second red which gets label A if we join something to it and whose neighbours have label S.
- We label by S two odd adjacent vertices. Label S occurs only in a pair of vertices, and we call such pair a *special pair* S. The neighbours of this pair are even.

Now we present the idea behind vertex labelling. Let us take, for example, a multicycle of length divisible by four. On this multicycle in the standard coloring occur only vertices labelled A and S_2. Obviously neighboring vertices on this multicycle have different red and blue degrees. If we will add further elements, color and label them, according to the rules above, then vertices labelled A remain labelled A and vertices labelled S_2 get label B or remain labelled S_2 (if we do not add something to them). Either way, we are sure that for neighboring vertices on the multicycle the degrees in both colors are different (without having to calculate them exactly).

Unfortunately, not all multicycles have a length divisible by four. Already on a multicycle ${}^2C_6$ appears a vertex with the label S_1 which should turn into A, if we add something to this vertex, but this is impossible when we want to add a path of length two (see below). Note, moreover, that for odd-length multicycles, at least two neighboring vertices must be of the same parity, which makes their distinguishing not automatic. These problems make it necessary to consider many cases.

Note that multitree ${}^2P_3$ rooted at its end creates particular problems. We will call such multipath a *spike*. Note that a spike creates particular problems only when an individual spike is joined to a cyclic vertex x. Therefore, by adding or joining a spike we always mean adding an individual spike and nothing more. Let us observe that it cannot be added to the vertex labelled S_1 and also to the vertex labelled $\tilde{S}_2$. We can avoid this problem in two ways.

First, trying to change colors in standard coloring of multicycle so that the vertex to which we add spike has another label. Sometimes it is impossible, for example in the situation when multicycle ${}^2C_3$ has a spike added to each vertex. This particular case is presented in Figure 4.

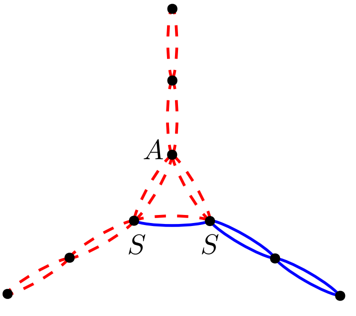

Fig. 4. The coloring of ${}^2C_3$ with spikes

Secondary, in the situation when the initial multicycle has length $4k+2$ or $4k+3$ for $k \geq 1$ and the next part of the construction requires adding a spike (to the vertex with label S_1) we will consider the multicycle with spike instead of multicycle. Note that in the initial multicycle which has length $4k+2$ or $4k+3$ for $k \geq 1$ we have only one vertex labelled S_1. The coloring of this multicycle with spike is presented in Figure 5. Note that in this figure we presented coloring only for multicycles of length $n=6$ and $n=7$, but we can easily extend this coloring for longer multicycles with spike.

We may assume that in the next part of the construction we will never add a spike to the vertex labeled $\tilde{S}_2$ on ${}^2C_3$. If it is not true we can easily recolor standard ${}^2C_3$. Note that this problem does not apply to the vertex labelled S_2.

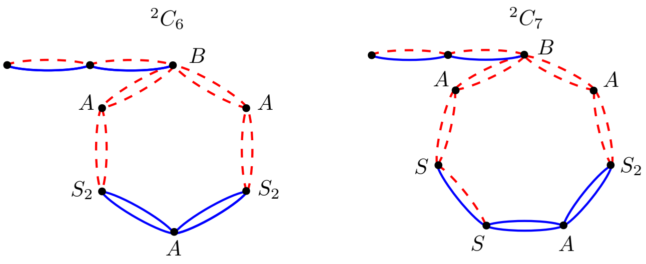

Fig. 5. The initial coloring of $^2C_6$ with spike and $^2C_7$ with spike

Remark 2.5. In the next part of the construction new labels for particular pairs of vertices will appear.

Joining all multitrees and multicycles to vertices on chosen colored multicycle in ^{2}G. We choose one colored multicycle in ^2G. First, we join all multitrees and multicycles to vertices labeled A, B, S_1, S_2 and $\tilde{S}_2$ on chosen multicycle. Then we consider pair of vertices labelled S or with another labels which appear later.

Let $\tilde{G}$ be already colored part of the multigraph ^2G. This means in particular that each cyclic vertex in $\tilde{G}$ has its own label. Let x be a cyclic vertex in $\tilde{G}$. We describe how to add subsequent elements to x. The method of coloring and labelling them depends mostly on the label of x. We establish a rule that if we start joining elements to x we join all of them and in the remaining part of the construction we shall not add anything to x. We assume that multicycles which we join are colored standardly and labelled as in Figure 2.

Joining to the vertex labelled A. Recall that we do not change the parity and the label of the vertex labelled A. Therefore, adding multicycles is very simple in this case. By a *dominating* color at a vertex x we mean color in which the vertex x has greater degree. We join to the vertex x labelled A all multicycles:

– $^2C_3$ using their vertices labelled $\tilde{S}_2$,
– $^2C_{4k}$ and $^2C_{4k+1}$ using their vertices labelled A so that we increase the degree of x at dominating color,
– $^2C_{4k+2}$ and $^2C_{4k+3}$ using their vertices labelled S_1.

After joining multicycles, we join rooted multitree. So we join a rooted multitree which is not a multishrub to the vertex x labelled A starting from multiedges incident to its root colored with the dominating color at x using Lemma 2.2. If the rooted multitree joined to the vertex x labelled A is isomorphic to $^2K_2$ or $^2P_3$ we color it with the dominating color at x. Moreover, if the rooted multitree joined to the vertex x labelled A is isomorphic to $^2P_5$ we color it: the multiedge xr' with the dominating color at x, the second multiedge with the other color, the third multiedge red-blue and the last multiedge with the dominating color at x. If the rooted multitree joined to the vertex x labelled A is a multishrub and is not isomorphic to $^2K_2$, $^2P_3$ or $^2P_5$, we color its root multiedge xr' with the dominating color at x and the rest of this multishrub rooted at the vertex r' we color starting from the different color then we used for xr'

from Lemma 2.2. If $\hat{d}(r') = 6$ in ${}^2G$ and we have pendant multipath ${}^2P_3$ from the vertex r', we recolor this multipath red-blue. More precisely we recolor each multiedge of this multipath using both colors red and blue.

Joining to the vertex labelled B. Recall that a vertex labelled B is odd and remains odd (with label B). Therefore, adding multitrees is very simple in this case because from Lemma 2.2 we always have $\hat{d}(x) \neq \hat{d}(r')$ for each r' which is a neighbour of x in added multitree. When we add each multicycle, we identify a vertex y on this multicycle with the vertex x. We need to make sure that neighbours of y on the multicycle are even. For multicycle ${}^2C_{4k}$ or ${}^2C_{4k+1}$ we use standard coloring of a multicycle and the vertex y with label S_2. For multicycle ${}^2C_{4k+2}$ or ${}^2C_{4k+3}$ except for ${}^2C_3$ we denote by y the vertex with label B and greater degree at dominating color at x. Multiedges incident to a vertex labelled S_1 of a joined multicycle are recolored in a dominating color at y. Thus, the vertex x remains odd. For a multicycle ${}^2C_3$ we use standard coloring of a multicycle and the vertex y with label $\tilde{S}_2$. Then we recolor red multiedge colored red-blue in the standard coloring of joined ${}^2C_3$. Then we label the neighbors of y on ${}^2C_3$ by $\mathbf{T}$. Using bold we would like to stress the fact that given pair of vertices appears first time. These two adjacent vertices on ${}^2C_3$ create new special pair T where by a *special pair* T we mean a pair of even adjacent vertices on ${}^2C_3$ which has neighbour labelled B and the form: multiedge colored red, first vertex from pair T, multiedge colored red, second vertex from pair T and multiedge colored blue (see Figure 8).

Joining to the vertex labelled $\mathbf{S_1}$ or $\mathbf{\tilde{S}_2}$. We will never join a spike to the vertex labelled S_1 or $\tilde{S}_2$. Note that if we had to join a spike to the vertex labelled S_1 or $\tilde{S}_2$ then instead of the standard coloring and labeling for the multicycle we would use coloring and labeling for the multicycle with spike. We join all multicycles to the vertex x labelled S_1 or $\tilde{S}_2$ using the same method as when we join to the vertex labelled A. Then we join multitree to the vertex x using the method of joining to the vertex labelled A. Recall that after joining all the elements the vertex x gets label A.

Joining to the vertex labelled $\mathbf{S_2}$. Recall that in this case we have to change the parity and label of the vertex x. If we add multicycles then we use first added multicycle to change parity of the vertex x, and then we add remaining elements using the same method as when we add to the vertex labelled B. Now we describe how to add first multicycle. Let y be the vertex labelled B on ${}^2C_{4k+2}$ or ${}^2C_{4k+3}$ except for ${}^2C_3$ such that its neighbour y_1 is labelled S_1. We join this multicycle to x using the vertex y. If we have a spike at vertex y_1, we replace this multicycle by a multicycle with spike. The coloring of this multicycle with spike is presented in Figure 6.

A multicycle ${}^2C_{4k}$ or ${}^2C_{4k+1}$ in the standard coloring either has no odd vertex or has an odd vertex with an odd neighbour. Therefore, this coloring cannot be used in this case. Thus, before we join multicycle ${}^2C_{4k}$ to the vertex x labelled S_2 we recolor two multiedges incident to one of the vertices labelled A with red-blue. Thus, this vertex y_1 gets label S_1 and its neighbours get label B. Then we join this recolored multicycle ${}^2C_{4k}$ to the vertex x using similar method as when we join ${}^2C_{4k+2}$ or ${}^2C_{4k+3}$ to the vertex labelled S_2. Note that we do not have any pair of adjacent vertices with the same label on joined multicycle ${}^2C_{4k}$.

Before we join multicycle ${}^2C_{4k+1}$ except for ${}^2C_5$ to x, we recolor two multiedges incident to one of the vertices labelled A which has no neighbour from the special pair S red-blue. Thus, this vertex y_1 gets label S_1 and its neighbours get label B. Then we join this recolored multicycle ${}^2C_{4k+1}$ to x using the same method as when we join ${}^2C_{4k}$ to the vertex labelled S_2.

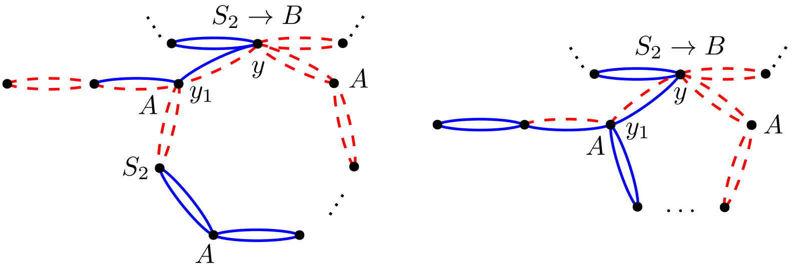

Fig. 6. The coloring of a multicycle ${}^2C_{4k+2}$ or ${}^2C_{4k+3}$ with spike joined to the vertex labelled S_2 (left) and ${}^2C_{4k}$ or ${}^2C_{4k+1}$ with spike joined to the vertex labelled S_2 (right)

The coloring of multicycle ${}^2C_5$, ${}^2C_5$ with spike and ${}^2C_3$ joined to the vertex labelled S_2 is presented in Figure 7. Note that we used first time labels $\mathbf{S'}$ and $\mathbf{T'}$. Labels S' and T' appear only in pairs, on adjacent vertices. We call them, respectively, the *special pair S' and T'*.

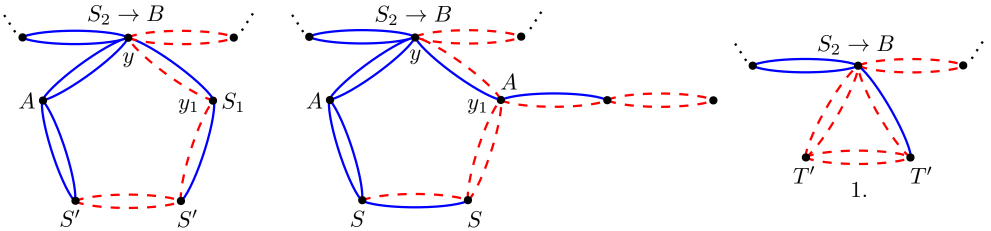

Fig. 7. The coloring of multicycle ${}^2C_5$, ${}^2C_5$ with spike and ${}^2C_3$ joined to the vertex labelled S_2 on multicycle

We use the label S' in two situations:

1. for both vertices from a pair of adjacent vertices on ${}^2C_5$ which would become the special pair S if we joined something to both vertices from this pair, because now this pair has: two neighbours labelled A, S_1 and the form: multiedge colored blue, first vertex from pair S', multiedge colored red, second vertex from pair S' and multiedge colored red-blue (or symmetrically, by *symmetrical* coloring we mean swapping all colors);
2. for both vertices from a pair of adjacent vertices on ${}^2C_3$ which would become the special pair S if we joined something to both vertices from this pair, because now

this pair has: neighbour labelled A and the form as in the first situation; the second situation appears later.

We use the label T' for both vertices from a pair of adjacent vertices on ${}^2C_3$ which would become the special pair T if we joined something to both vertices from this pair, because now it has one neighbor labelled B and the form:

1. multiedge colored red, first vertex from pair T', multiedge colored red, second vertex from pair T' and multiedge colored red-blue (or symmetrically),
2. multiedge colored red, first vertex from pair T', multiedge colored blue, second vertex from pair T' and multiedge colored red-blue (or symmetrically). This form appears later (see Figure 8).

We always use presented coloring of ${}^2C_5$ with spike when we join ${}^2C_5$ to the vertex labelled S_2 and at least one of the neighbours of y on joined ${}^2C_5$ has a spike, because we would like to avoid potential problems with special pair S'.

Now, we introduce the method of joining only the multitree ${}^2T$ rooted at r to the vertex x labelled S_2. We consider two cases. Assume that red is the dominating color at the vertex x.

Case 1. $\hat{d}(r) = 2$ in ${}^2T$. We color root multiedge rx' red-blue. If $\hat{d}(x') > 4$, we color multitree rooted at vertex x' using Lemma 2.2 so that blue is the dominating color at the vertex x'. If $\hat{d}(x') = 4$, we color multiedge $x'x_1$ red-blue and then we color multitree rooted at the vertex x_1 using Lemma 2.2 so that blue is the dominating color at the vertex x_1.

Case 2. $\hat{d}(r) > 2$ in ${}^2T$. We treat this rooted multitree as the set of multishrubs with common root. We choose one of those multishrubs and color it in the same way as in the first case. Then we color the remaining part of multitree using the same method as when we join it to the vertex labelled B so that red is the dominating color at x.

Now, we present in details method of joining elements to pairs of vertices labelled S, T, T' and S' on the multicycle. Note that in any colored multicycle there is at most one pair of vertices labelled S or S' or T or T'. When we consider each of these pairs, we join at the same time all multitrees and all multicycles to both vertices creating this pair. Recall that pair of vertices labelled T or T' will appear only on the multicycle ${}^2C_3$. In Figure 8 we recall the form of each pair, but it may happen that we will have pairs with symmetrical coloring, and then we treat those pairs analogously.

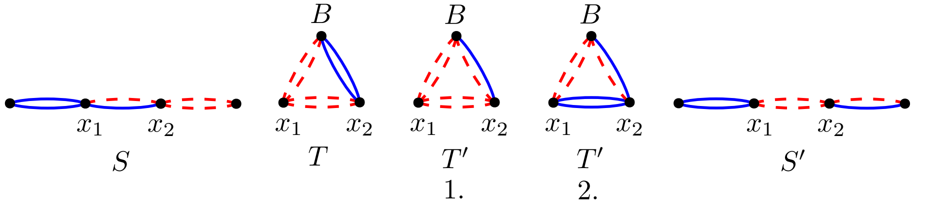

Fig. 8. The form of pair vertices labelled S, T, T' and S'

Note that in each pair of vertices both vertices creating this pair have the same label and this label create the name of pair.

Pair S. We consider two cases.

Case 1. We join only one multicycle ${}^2C_3$ or ${}^2C_{4k}$ or ${}^2C_{4k+1}$ to the special pair S. When we join ${}^2C_3$ we use one of the following colorings presented in Figure 9. Note that we create the special pair T' in the second form on this multicycle ${}^2C_3$.

When we join multicycle ${}^2C_{4k}$ or ${}^2C_{4k+1}$ to the vertex x_1 we first recolor multiedge x_1x_2 blue, and then we join this multicycle using the method of joining to the vertex labelled S_2. When we join multicycle ${}^2C_{4k}$ or ${}^2C_{4k+1}$ to the vertex x_2 we first recolor multiedge x_1x_2 red, and then we join this multicycle using the method of joining to the vertex labelled S_2.

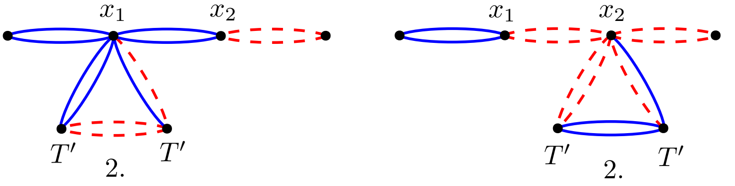

Fig. 9. Colorings of multicycle ${}^2C_3$ joined to the special pair S

Case 2. Otherwise, we join rooted multitree and all kinds of multicycles to the vertex x_1 using the method of joining to the vertex labelled B so that blue is the dominating color at x_1. Then we join rooted multitree and all kinds of multicycles to the vertex x_2 using the method of joining to the vertex labelled B so that red is the dominating color at x_2. Note that red multiedges at x_1 come from multicycles ${}^2C_{4k}$, ${}^2C_{4k+1}$ and ${}^2C_3$, similarly, blue multiedges at the vertex x_2 come from multicycles ${}^2C_{4k}$, ${}^2C_{4k+1}$ and ${}^2C_3$.

Obviously, we can have conflict between vertices x_1 and x_2. First, we consider conflict when one vertex from pair S suppose x_1 admits $\hat{d}(x_1) = 4$. It means that $\hat{d}_b(x_1) = \hat{d}_b(x_2) = 3$. Therefore, it suffices that we recolor symmetrically one of the multicycles ${}^2C_{4k+3}$, ${}^2C_{4k+2}$ or rooted multitree joined to the vertex x_2 to solve this conflict. If we have similar conflict and x_2 admits $\hat{d}(x_2) = 4$ we solve it analogously. Assume that $\hat{d}(x_1) > 4$, $\hat{d}(x_2) > 4$ and we have conflict between x_1 and x_2. Thus, we have definitely joined at least two multicycles from the set of all multicycles ${}^2C_{4k+1}$, ${}^2C_{4k}$ and ${}^2C_3$ to the vertex x_1 or x_2. To solve this conflict we recolor using the coloring from the method of joining to the vertex labelled S_2 and so that we increase the degree in the dominating color exactly two multicycles from the set of all multicycles ${}^2C_{4k+1}$, ${}^2C_{4k}$ and ${}^2C_3$ joined to one vertex from pair S. Thus, we always solve this conflict, because using above procedure we increase the degree in the dominating color by two and decrease the degree in the other color by two in one vertex from pair S. Thus, we are done with special pair S.

Remark 2.6. During the procedure of adding multicycles, described above, new labels were assigned to the special pairs T, T' and S'. Therefore, below we have to describe how to add further elements to these pairs.

The pair S', appearing only on the $^2C_5$ and $^2C_3$ multicycles, is unique in this sense, we should start the process of adding elements to these multicycles from this pair to avoid potential problems with neighbors of pair S' on $^2C_5$.

Pair T. When we join to only one vertex from the special pair T we choose the vertex x_1, and we use the method of joining to the vertex labeled A. When we join $^2K_2$ to one vertex and something to the second vertex from par T we join blue $^2K_2$ to x_2 and remaining elements to x_1 using the method of joining to the vertex labeled A. The particular coloring of special pair T with added a spike to each vertex from this pair is presented in Figure 10.

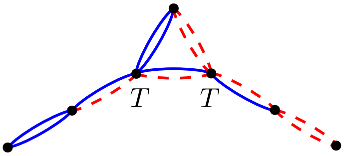

Fig. 10. The particular coloring of special pair T with spikes

We consider the situation when we will join something (except for a spike) to one vertex from pair T and a spike to the second vertex from pair T. We join a spike colored red to the vertex x_1, and then we join elements to the vertex x_2 starting from multicycles and then possibly multitree using the same method as when we join to the vertex labelled A and so that blue is the dominating color at x_2. If we have a conflict ($\hat{d}_r(x_1) = \hat{d}_r(x_2) = 6$), we recolor a spike joined to the vertex x_1 blue.

In the remaining situation we join starting from multicycles then we join multitrees to vertices x_1 and x_2 using the same method as when we join to the vertex labelled A so that red is the dominating color at x_1 and blue at x_2. If we have conflict between x_1 and x_2 without recoloring, we move all joins from x_1 to x_2 and from x_2 to x_1. Thus, we are done with special pair T.

Pair T'. First, we present the general method of joining multitree ^2T rooted at r to the vertex x_2. We consider two cases.

Case 1. $\hat{d}(r) = 2$ in ^2T. We color the root multiedge rx' red-blue. Then we color the multitree rooted at the vertex x' using Lemma 2.2.

Case 2. $\hat{d}(r) \geq 4$ in ^2T. We treat the multitree rooted at r as a set of multishrubs with common root. We choose one multishrub rooted at r and color it in the same way as in Case 1. Then we join the remaining part of ^2T to the vertex x_2 using the method of joining to the vertex labelled A starting from the root multiedge in the dominating color at x_2 if $\hat{d}(r) = 4$ in ^2T and starting from the other color of multiedges incident to r in the remaining part of ^2T if $\hat{d}(r) > 4$ in ^2T.

Now, we present in detail method of joining elements to the special pair T' in the first form. When we join to only one vertex from pair T', we choose the vertex x_1, and we use the method of joining to the vertex labelled A.

When we join only a multitree to each vertex from special pair T', we join the first multitree to x_1 using the same method as when we join to the vertex labelled A and the second multitree to the vertex x_2 using the general method of joining multitree rooted at r to the vertex x_2 so that blue is the dominating color at x_2 if we join multitree with root of degree equal at least six. If we join multitree with root of degree equal to two and multitree with root of degree equal to four we join multitree with root of degree equal to two to the vertex x_2 and multitree with root of degree equal to four to the vertex x_1 to avoid conflict.

In the remaining case we first join exactly one multicycle to the vertex x_2 using the coloring presented in Figure 11.

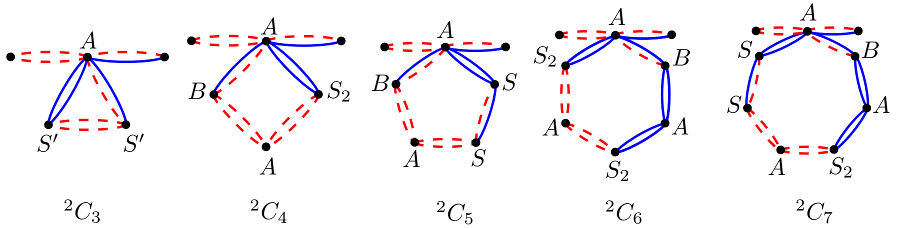

Fig. 11. The coloring of multicycle joined at first to the vertex x_2 from special pair T'

The coloring of longer multicycle we obtain by replacing two red multiedges with a multipath of length $4k+2$ colored: the first two multiedges red, the next two multiedges blue, the subsequent two multiedges red, and so on in the appropriate colored multicycle of length from four to seven. Then we continue joining multicycles and possibly multitree to the vertex x_2 using the same method as when we join to the vertex labelled A so that blue is the dominating color at x_2. Next, we join elements to the vertex x_1 using the same method as when we join to the vertex labelled A so that red is the dominating color at x_1. At the end if we have conflict between vertices x_1 and x_2, namely $\hat{d}_r(x_1) = \hat{d}_r(x_2)$, we recolor symmetrically (we swap color of all multiedges) joined at first multicycle to the vertex x_2.

Now, we present the method of joining elements to the special pair T' in the second form. When we join to only one vertex from pair T', we choose the vertex x_2 and consider two situations. In the first situation we join only multitree using the general method of joining multitree to the vertex x_2. In the second situation when we join multicycle or multicycles and possibly multitree we use the method of joining to the vertex x_2 form special pair T' in the first form. When we join something to both vertices from special pair T' in the second form we start with recoloring multiedge x_1x_2 red. Thus, pair T' get the first form, and we use the above method of joining to pair T' in the first form. Thus, we are done with special pair T'.

Pair S'. This pair occurs only on multicycle $^2C_3$ and $^2C_5$. Additionally, if the pair S' occurs on a multicycle $^2C_5$ joined to the colored part of multicactus at vertex x, then neighbors of the vertex x on $^2C_5$ have no spike and the distance between x and x_1 is the same as the distance between x and x_2. By the above properties of special pair S' and the fact that we start considering a colored $^2C_5$ which contain a pair S' with this pair we can choose where we join (to x_1 or to x_2) the first and the second part of elements to this special pair S'.

When we join to only one vertex from special pair S', we choose vertex x_2, and we use the method of joining to the vertex labelled B, so that red is the dominating color at x_2. Then we can also join rooted multitree to the vertex x_1 using the same method as when we join to the vertex labelled S_2 so that blue is the dominating color at x_1.

We consider other case when we join at least one multicycle to each vertex from special pair S'. First, we join all from the first part of joins to this pair S' to the vertex x_2 using the method of joining to the vertex labelled B, so that red is the dominating color at x_2. Then we join exactly one multicycle to x_1 using the method of joining to the vertex labelled S_2, so that $\hat{d}_r(x_1) = 3$ and $\hat{d}_b(x_2) = 5$. Next, we join multicycles and possibly multitree to the vertex x_1 using the same method as when we join to the vertex labelled B so that blue is the dominating color at x_1. At the end if we have conflict between vertices x_1 and x_2, namely $\hat{d}_r(x_1) = \hat{d}_r(x_2)$, we recolor symmetrically joined at first multicycle to the vertex x_1. Thus, we are done with special pair S'.

Joining multicycles and additional multitree to a leaf in the colored part of ^{2}G. Note that we never join only a single multitree to a leaf in ^2G. We denote by x chosen leaf in colored part of ^2G and by x_0 the only neighbour of x in colored part of ^2G. We present the method of joining the longest multicycle to the leaf. After joining this multicycle vertex x gets label A or B and has different parity than x_0. Therefore, we can easily join remaining elements to x. We will consider four main cases in view of the color of multiedge xx_0 and parity of x_0 degrees in both colors.

Case 1. Multiedge xx_0 is monochromatic, without loss of generality red, and x_0 is even. First, we describe the method of joining $^2C_3$ to the vertex x. When we join $^2C_3$ and something else to x we use the first coloring of $^2C_3$ presented in Figure 12. Note that if we join also the only multitree to x we should tend to make blue the dominating color at x to avoid potential conflict. In other case when we join only $^2C_3$ to x we use the second coloring of $^2C_3$ presented in Figure 12.

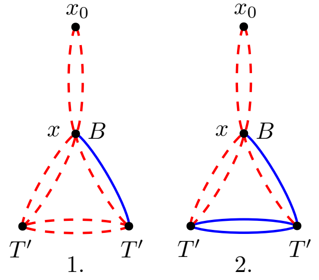

Fig. 12. Colorings of multicycle $^2C_3$ joined to the vertex x in Case 1

We join multicycle of length greater than three to the vertex x using the method of joining to the vertex labelled S_2. Thus, in all this case the vertex x with joined multicycle has label B.

Case 2. Multiedge xx_0 is monochromatic, without loss of generality red, and x_0 is odd. We join multicycle to the vertex x using the method of joining to the vertex labelled A so that red is the dominating color at x. Thus, the vertex x with joined multicycle has label A.

Case 3. Multiedge xx_0 is colored red-blue and x_0 is even. We join multicycle to the vertex x using the method of joining to the vertex labelled B so that red is the dominating color at x. Thus, the vertex x with joined multicycle has label B.

Case 4. Multiedge xx_0 is colored red-blue and x_0 is odd. We join a multicycle to the vertex x using the coloring presented in Figure 13.

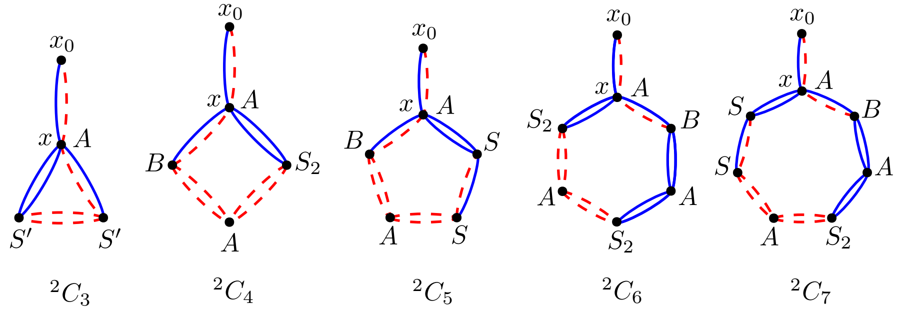

Fig. 13. The coloring of multicycle joined to the vertex x in Case 4

The coloring of longer multicycle we obtain by replacing two red multiedges with a multipath of length $4k+2$ colored: the first two multiedges red, the next two multiedges blue, the subsequent two multiedges red, and so on in the appropriate colored multicycle of length from four to seven. Thus, the vertex x with joined multicycle has label A. So we are done. □

As an immediate consequence of the above theorem and Theorem 1.3 we get the following result.

Corollary 2.7. *The Local Irregularity Conjecture for* 2*-multigraphs holds for graphs from the family* $\mathfrak{T}'$.

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