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WEAK SIGNED ROMAN k-DOMINATION IN DIGRAPHS

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Abstract. Let $k \geq 1$ be an integer, and let D be a finite and simple digraph with vertex set $V(D)$. A weak signed Roman k-dominating function (WSRkDF) on a digraph D is a function $f\colon V(D) \to \{-1, 1, 2\}$ satisfying the condition that $\sum_{x \in N^-[v]} f(x) \geq k$ for each $v \in V(D)$, where $N^-[v]$ consists of v and all vertices of D from which arcs go into v. The weight of a WSRkDF f is $w(f) = \sum_{v \in V(D)} f(v)$. The weak signed Roman k-domination number $\gamma^k_{wsR}(D)$ is the minimum weight of a WSRkDF on D. In this paper we initiate the study of the weak signed Roman k-domination number of digraphs, and we present different bounds on $\gamma^k_{wsR}(D)$. In addition, we determine the weak signed Roman k-domination number of some classes of digraphs. Some of our results are extensions of well-known properties of the weak signed Roman domination number $\gamma_{wsR}(D) = \gamma^1_{wsR}(D)$ and the signed Roman k-domination number $\gamma^k_{sR}(D)$.

Keywords: digraph, weak signed Roman k-dominating function, weak signed Roman k-domination number, signed Roman k-dominating function, signed Roman k-domination number.

Mathematics Subject Classification: 05C20, 05C69.

1. TERMINOLOGY AND INTRODUCTION

In this paper we continue the study of signed Roman dominating functions in graphs and digraphs (see for example the survey article [2]). Let $k \geq 1$ be an integer, G a simple graph with vertex set $V(G)$, and $N[v] = N_G[v]$ the closed neighborhood of the vertex v. A *weak signed Roman k-dominating function* (WSRkDF) on a graph G is defined in [10] as a function $f\colon V(G) \to \{-1, 1, 2\}$ such that $\sum_{x \in N_G[v]} f(x) \geq k$ for every $v \in V(G)$. A weak signed Roman k-dominating function f on a graph G is called a *signed Roman k-dominating function* (SRkDF) on G if every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$ (see [6]). The weight of a WSRkDF or an SRkDF f on a graph G is $w(f) = \sum_{v \in V(G)} f(v)$. The *weak signed Roman k-domination number* $\gamma^k_{wsR}(G)$ or *signed Roman k-domination number* $\gamma^k_{sR}(G)$ of G is the minimum weight of a WSRkDF or an SRkDF on G, respectively. The special case $\gamma_{sR}(G) = \gamma^1_{sR}(G)$ was investigated by Ahangar, Henning, Löwenstein, Zhao and Samodivkin [1].

Let now D be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. The sets $N^+_D(v) = N^+(v) = \{x \,|\, (v,x) \in A(D)\}$ and $N^-_D(v) = N^-(v) = \{x \,|\, (x,v) \in A(D)\}$ are called the *out-neighborhood* and *in-neighborhood* of the vertex v. Likewise, $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$. We write $d^+_D(v) = d^+(v) = |N^+(v)|$ for the *out-degree* of a vertex v and $d^-_D(v) = d^-(v) = |N^-(v)|$ for its *in-degree*. The *minimum* and *maximum in-degree* are $\delta^- = \delta^-(D)$ and $\Delta^- = \Delta^-(D)$ and the *minimum* and *maximum out-degree* are $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x,y) \in A(D)$, the vertex y is an *out-neighbor* of x and x is an *in-neighbor* of y, and we also say that x *dominates* y or y *is dominated by* x. For a real-valued function $f\colon V(D) \to \mathbf{R}$, the weight of f is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. Consult [4] and [5] for notation and terminology which are not defined here.

For an integer $p \geq 1$, we define a set $S \subseteq V(D)$ to be a *p-dominating set* of D if for all $v \notin S$, v is dominated by p vertices in S. The *p-domination number* $\gamma_p(D)$ of a digraph D is the minimum cardinality of a p-dominating set of D.

A *weak signed Roman k-dominating function* (abbreviated WSRkDF) on D is defined as a function $f\colon V(D) \to \{-1, 1, 2\}$ such that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq k$ for every $v \in V(D)$. A weak signed Roman k-dominating function f on D is called a *signed Roman k-dominating function* on D if every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$ (see [8]). The weight of a WSRkDF or an SRkDF f on a digraph D is $w(f) = \sum_{v \in V(D)} f(v)$. The *weak signed Roman k-domination number* $\gamma^k_{wsR}(D)$ or *signed Roman k-domination number* $\gamma^k_{sR}(D)$ of D is the minimum weight of a WSRkDF or an SRkDF on D, respectively. A $\gamma^k_{wsR}(D)$-function or a $\gamma^k_{sR}(D)$-function is a weak signed Roman k-dominating function or a signed Roman k-dominating function on D of weight $\gamma^k_{wsR}(D)$ or $\gamma^k_{sR}(D)$, respectively. For a WSRkDF or an SRkDF f on D, let $V_i = V_i(f) = \{v \in V(D) : f(v) = i\}$. A weak signed Roman k-dominating function or a signed Roman k-dominating function $f\colon V(D) \to \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(D)$. The special cases $k = 1$ were introduced and investigated by Sheikholeslami and Volkmann [7] and Volkmann [11].

The weak signed Roman k-domination number exists when $\delta^- \geq \frac{k}{2} - 1$. The definitions lead to $\gamma^k_{wsR}(D) \leq \gamma^k_{sR}(D)$. Therefore each lower bound on $\gamma^k_{wsR}(D)$ is also a lower bound on $\gamma^k_{sR}(D)$.

Our purpose in this work is to initiate the study of the weak signed Roman k-domination number in digraphs. We present basic properties and sharp bounds on $\gamma^k_{wsR}(D)$. In particular we show that many lower bounds on $\gamma^k_{sR}(D)$ are also valid for $\gamma^k_{wsR}(D)$. In addition, we determine the weak signed Roman k-domination number of some classes of digraphs. Some of our results are extensions of known properties of the signed Roman domination number $\gamma_{sR}(D) = \gamma^1_{sR}(D)$ by Sheikholeslami and

Volkmann [7] and the signed Roman k-domination number $\gamma^k_{sR}(G)$ of graphs G, given by Henning and Volkmann in [6].

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1.1. *If $D(G)$ is the associated digraph of a graph G, then we have $\gamma^k_{wsR}(D(G)) = \gamma^k_{wsR}(G)$.*

Let K_n and K^*_n be the complete graph and complete digraph of order n, respectively. In [9] and [10], the author determines the weak signed Roman k-domination number of complete graphs K_n for $n \geq k \geq 1$.

Proposition 1.2 ([9, 10])**.** *If $n \geq k \geq 1$, then $\gamma^k_{wsR}(K_n) = k$.*

Using Observation 1.1 and Proposition 1.2, we obtain the weak signed Roman k-domination number of complete digraphs.

Corollary 1.3. *If $n \geq k \geq 1$, then $\gamma^k_{wsR}(K^*_n) = k$.*

Proposition 1.4 ([10])**.** *Let $k \geq 1$ be an integer, and let $K_{p,p}$ be the complete bipartite graph of order $2p$. If $p \geq k+3$, then $\gamma^k_{wsR}(K_{p,p}) = 2k+2$. If $k+1 \leq p \leq k+2$, then $\gamma^k_{wsR}(K_{p,p}) = p+k-1$. If $k \geq 2$, then $\gamma^k_{wsR}(K_{k,k}) = 2k$ and $\gamma_{wsR}(K_{1,1}) = 1$. If $k \geq 2$, then $\gamma^k_{wsR}(K_{k-1,k-1}) = 2k-2$.*

Using Observation 1.1 and Proposition 1.4 , we obtain the weak signed Roman k-domination number of complete bipartite digraphs $K^*_{p,p}$.

Corollary 1.5. *If $p \geq k+3$, then $\gamma^k_{wsR}(K^*_{p,p}) = 2k+2$. If $k+1 \leq p \leq k+2$, then $\gamma^k_{wsR}(K^*_{p,p}) = p+k-1$. If $k \geq 2$, then $\gamma^k_{wsR}(K^*_{k,k}) = 2k$ and $\gamma_{wsR}(K^*_{1,1}) = 1$. If $k \geq 2$, then $\gamma^k_{wsR}(K^*_{k-1,k-1}) = 2k-2$.*

2. PRELIMINARY RESULTS

In this section we present basic properties of the weak signed Roman k-dominating functions and the weak signed Roman k-domination numbers of digraphs.

Lemma 2.1. *If $f = (V_{-1}, V_1, V_2)$ is a WSRkDF on a digraph D of order n and minimum in-degree $\delta^-(D) \geq \frac{k}{2} - 1$, then*

(a) $|V_{-1}| + |V_1| + |V_2| = n$,
(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$,
(c) $V_1 \cup V_2$ *is a* $\lceil \frac{k+1}{2} \rceil$*-dominating set of* D.

Proof. Since (a) and (b) are immediate, we only prove (c). If $|V_{-1}| = 0$, then $V_1 \cup V_2 = V(D)$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set of D. Let now $|V_{-1}| \geq 1$, and let $v \in V_{-1}$

an arbitrary vertex. Assume that v has j in-neighbors in V_1 and q in-neighbors in V_2. The condition $f(N^-[v]) \geq k$ leads to $j + 2q - 1 \geq k$ and so $q \geq \frac{k+1-j}{2}$. This implies

$$j + q \geq j + \frac{k+1-j}{2} = \frac{k+j+1}{2} \geq \frac{k+1}{2}.$$

Therefore v has at least $j+q \geq \lceil \frac{k+1}{2} \rceil$ in-neighbors in $V_1 \cup V_2$. Since v was an arbitrary vertex in V_{-1}, we deduce that $V_1 \cup V_2$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set of D. □

Corollary 2.2. *If D is a digraph of order n and minimum in-degree $\delta^-(D) \geq \frac{k}{2} - 1$, then $\gamma^k_{wsR}(D) \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) - n$.*

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma^k_{wsR}(D)$-function. Then it follows from Lemma 2.1 that

$$\begin{aligned}\gamma^k_{wsR}(D) &= |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \\ &\geq 2|V_1 \cup V_2| - n \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) - n.\end{aligned}$$

□

The digraph without arcs and the digraph qK_2^* show that Corollary 2.2 is sharp for $k = 1$ and $k = 2$. For the case $\Delta^-(D) \geq \frac{k+1}{2}$, we can improve Corollary 2.2 slightly.

Theorem 2.3. *If D is a digraph of order n with $\delta^-(D) \geq \frac{k}{2} - 1$ and $\Delta^-(D) \geq \frac{k+1}{2}$, then*

$$\gamma^k_{wsR}(D) \geq \min\left\{2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) + 2 - n,\, 2\gamma_k(D) + 1 - n,\, 2\gamma_{k+1}(D) - n\right\}.$$

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma^k_{wsR}(D)$-function. If $|V_2| \geq 2$, then it follows from Lemma 2.1 that

$$\gamma^k_{wsR}(D) = 2|V_1| + 3|V_2| - n = 2|V_1 \cup V_2| + |V_2| - n \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) + 2 - n.$$

If $|V_2| = 1$ and $v \in V_{-1}$ is an arbitrary vertex, then we deduce from the condition $f(N^-[v]) \geq k$ that v has at least k in-neighbors in $V_1 \cup V_2$. Hence $V_1 \cup V_2$ is a k-dominating set and thus

$$\gamma^k_{wsR}(D) = 2|V_1 \cup V_2| + |V_2| - n \geq 2\gamma_k(D) + 1 - n.$$

Let now $|V_2| = 0$. If $|V_{-1}| = 0$, then $V_1 = V(D)$ and therefore $\gamma^k_{wsR}(D) = |V_1| = n$. If v is a vertex with $d^-(v) = \Delta^-(D)$, then the condition $\Delta^-(D) \geq \frac{k+1}{2}$ implies that $V(D) \setminus \{v\}$ is a $\lceil \frac{k+1}{2} \rceil$-dominating set of D. Thus, $\gamma_{\lceil \frac{k+1}{2} \rceil}(D) \leq n - 1$, and we obtain

$$\gamma^k_{wsR}(D) = n = 2(n-1) + 2 - n \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(D) + 2 - n.$$

Finally, let $|V_2| = 0$ and $|V_{-1}| \geq 1$. If $v \in V_{-1}$ is an arbitrary vertex, then we deduce from the condition $f(N^-[v]) \geq k$ that v has at least $k+1$ in-neighbors in V_1. Hence V_1 is a $(k+1)$-dominating set and thus

$$\gamma^k_{wsR}(D) = 2|V_1| - n \geq 2\gamma_{k+1}(D) - n,$$

and the proof is complete. □

The proof of the next proposition is identically with the proof of Proposition 7 in [8] and is therefore omitted.

Proposition 2.4. *Assume that $f = (V_{-1}, V_1, V_2)$ is a WSRkDF on a digraph D of order n with $\delta^-(D) \geq \frac{k}{2} - 1$. If $\Delta^+(D) = \Delta^+$ and $\delta^+(D) = \delta^+$, then*

(i) $(2\Delta^+ + 2 - k)|V_2| + (\Delta^+ + 1 - k)|V_1| \geq (\delta^+ + k + 1)|V_{-1}|$,
(ii) $(2\Delta^+ + \delta^+ + 3)|V_2| + (\Delta^+ + \delta^+ + 2)|V_1| \geq (\delta^+ + k + 1)n$,
(iii) $(\Delta^+ + \delta^+ + 2)\omega(f) \geq (\delta^+ - \Delta^+ + 2k)n + (\delta^+ - \Delta^+)|V_2|$,
(iv) $\omega(f) \geq (\delta^+ - 2\Delta^+ + 2k - 1)n/(2\Delta^+ + \delta^+ + 3) + |V_2|$.

3. BOUNDS ON THE WEAK SIGNED ROMAN k-DOMINATION NUMBER

We start with a general upper bound, and we characterize all extremal digraphs.

Theorem 3.1. *Let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{wsR}(D) \leq 2n$ with equality if and only if k is even, $\delta^-(D) = \frac{k}{2} - 1$, and each vertex of D is of minimum in-degree or has an out-neighbor of minimum in-degree.*

Proof. Define the function $g : V(D) \to \{-1, 1, 2\}$ by $g(x) = 2$ for each vertex $x \in V(D)$. Since $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$, the function g is a WSRkDF on D of weight $2n$ and thus $\gamma^k_{wsR}(D) \leq 2n$.

Now let k be even, $\delta^-(D) = \frac{k}{2} - 1$, and assume that each vertex of D is of minimum in-degree or has an out-neighbor of minimum in-degree. Let f be a $\gamma^k_{wsR}(D)$-function, and let $x \in V(D)$ be an arbitrary vertex. If $d^-(x) = \frac{k}{2} - 1$, then $f(N^-[x]) \geq k$ implies $f(x) = 2$. If x is not of minimum in-degree, then x has an out-neighbor w of minimum in-degree. Now the condition $f(N^-[w]) \geq k$ leads to $f(x) = 2$. Thus f is of weight $2n$, and we obtain $\gamma^k_{wsR}(D) = 2n$ in this case.

Conversely, assume that $\gamma^k_{wsR}(D) = 2n$. If $k = 2p + 1$ is odd, then $\delta^-(D) \geq p$. Define the function $h : V(D) \to \{-1, 1, 2\}$ by $h(w) = 1$ for an arbitrary vertex w and $h(x) = 2$ for each vertex $x \in V(D) \setminus \{w\}$. Then

$$h(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 1 + 2\delta^-(D) \geq 1 + 2p = k$$

for each $v \in V(D)$. Thus the function h is a WSRkDF on D of weight $2n - 1$, a contradiction to the assumption $\gamma^k_{wsR}(D) = 2n$.

Let now k be even and assume that there exists a vertex w such that $d^-(w) \geq \frac{k}{2}$ and $d^-(x) \geq \frac{k}{2}$ for each out-neighbor of w. Define the function $h_1 : V(D) \to \{-1, 1, 2\}$ by $h_1(w) = 1$ and $h_1(x) = 2$ for each vertex $x \in V(D) \setminus \{w\}$. Then $h_1(N^-[v]) \geq k + 1$ for each vertex $v \in N^-[w]$ and $h_1(N^-[x]) \geq k$ for each vertex $x \notin N^-[w]$. Hence the function h_1 is a WSRkDF on D of weight $2n - 1$, and we obtain the contradiction $\gamma^k_{wsR}(D) \leq 2n - 1$. This completes the proof. □

The proof of Theorem 3.1 also leads to the next result.

Theorem 3.2. *Let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{sR}(D) \leq 2n$ with equality if and only if k is even, $\delta^-(D) = \frac{k}{2} - 1$, and each vertex of D is of minimum in-degree or has an out-neighbor of minimum in-degree.*

Proposition 3.3. *If D is a digraph of order n with minimum in-degree $\delta^- \geq k-1$, then $\gamma^k_{wsR}(D) \leq \gamma^k_{sR}(D) \leq n$.*

Proof. Define the function $f\colon V(D) \to \{-1,1,2\}$ by $f(x)=1$ for each vertex $x \in V(D)$. Since $\delta^- \geq k-1$, the function f is an SRkDF on D of weight n and thus $\gamma^k_{wsR}(D) \leq \gamma^k_{sR}(D) \leq n$. □

A digraph D is *r-regular* if $\Delta^+(D) = \Delta^-(D) = \delta^+(D) = \delta^-(D) = r$. As an application of Proposition 2.4 (iii), we obtain a lower bound on the weak signed Roman k-domination number for r-regular digraphs.

Corollary 3.4. *If D is an r-regular digraph of order n with $r \geq \frac{k}{2} - 1$, then $\gamma^k_{sR}(D) \geq \gamma^k_{wsR}(D) \geq kn/(r+1)$.*

The special case $k=1$ of Corollary 3.4 can be found in [11]. Using Corollary 3.4 and Observation 1.1, we obtain the next known result.

Corollary 3.5 ([10])**.** *If G is an r-regular graph of order n with $r \geq \frac{k}{2} - 1$, then $\gamma^k_{wsR}(G) \geq kn/(r+1)$.*

Example 3.6. If H is a $(k-1)$-regular digraph of order n, then it follows from Corollary 3.4 that $\gamma^k_{sR}(H) \geq \gamma^k_{wsR}(H) \geq n$ and so $\gamma^k_{sR}(H) = \gamma^k_{wsR}(H) = n$, according to Proposition 3.3.

Example 3.6 demonstrates that Proposition 3.3 and Corollary 3.4 are both sharp. If $k \geq 2$, then Corollary 1.5 implies that $\gamma^k_{wsR}(K^*_{k,k}) = 2k$. This is a further example showing the sharpness of Proposition 3.3.

Theorem 3.7. *If D is a digraph of order n with $\delta^-(D) \geq \frac{k}{2} - 1$, then*

$$\gamma^k_{wsR}(D) \geq k + 1 + \Delta^-(D) - n.$$

Proof. Let $w \in V(D)$ be a vertex of maximum in-degree, and let f be a $\gamma^k_{wsR}(D)$-function. Then the definitions imply

$$\begin{aligned} \gamma^k_{wsR}(D) &= \sum_{x \in V(D)} f(x) = \sum_{x \in N^-[w]} f(x) + \sum_{x \in V(D) - N^-[w]} f(x) \\ &\geq k + \sum_{x \in V(D) - N^-[w]} f(x) \geq k - (n - (\Delta^-(D) + 1)) \\ &= k + 1 + \Delta^-(D) - n, \end{aligned}$$

and the proof of the desired lower bound is complete. □

If $n \geq k \geq 1$, then it follows from Corollary 1.3 that $\gamma^k_{wsR}(K^*_n) = k$. Therefore, the bound given in Theorem 3.7 is sharp.

A digraph D is *out-regular* or *r-out-regular* if $\Delta^+(D) = \delta^+(D) = r$. If D is not out-regular, then the next lower bound on the weak signed Roman k-domination number holds.

Corollary 3.8. *Let D be a digraph of order n, minimum in-degree $\delta^- \geq \frac{k}{2} - 1$, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^+ < \Delta^+$, then*

$$\gamma^k_{wsR}(D) \geq \left(\frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+ + 3}\right) n.$$

Proof. Multiplying both sides of the inequality in Proposition 2.4 (iv) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Proposition 2.4 (iii), we obtain the desired lower bound. □

Corollary 3.9 ([8])**.** *Let D be a digraph of order n, minimum in-degree $\delta^- \geq \frac{k}{2} - 1$, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^+ < \Delta^+$, then*

$$\gamma^k_{sR}(D) \geq \left(\frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+ + 3}\right) n.$$

Since the bound given in Corollary 3.9 is sharp (see [8]), the bound given in Corollary 3.8 is sharp too.

Since $\Delta^+(D(G)) = \Delta(G)$ and $\delta^+(D(G)) = \delta(G)$, Corollary 3.8 and Observation 1.1 lead to the next known result.

Corollary 3.10 ([6, 10])**.** *Let G be a graph of order n, minimum degree $\delta \geq \frac{k}{2} - 1$ and maximum degree Δ. If $\delta < \Delta$, then*

$$\gamma^k_{sR}(G) \geq \gamma^k_{wsR}(G) \geq \left(\frac{2\delta + 3k - 2\Delta}{2\Delta + \delta + 3}\right) n.$$

The special case $k = 1$ of Corollary 3.10 can be found in [1, 9].

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v the arc uv belongs to $\overline{D}$ if and only if uv does not belong to D. Using Corollary 3.5 one can prove the following Nordhaus–Gaddum type inequality analogously to Theorem 17 in [8].

Theorem 3.11. *If D is an r-regular digraph of order n such that $r \geq \frac{k}{2} - 1$ and $n - r - 1 \geq \frac{k}{2} - 1$, then*

$$\gamma^k_{wsR}(D) + \gamma^k_{wsR}(\overline{D}) \geq \frac{4kn}{n+1}.$$

If n is even, then $\gamma^k_{wsR}(D) + \gamma^k_{wsR}(\overline{D}) \geq \frac{4k(n+1)}{n+2}$.

Example 3.12. Let $k \geq 1$ be an integer, and let H and $\overline{H}$ be $(k-1)$-regular digraphs of order $n = 2k - 1$. In view of Example 3.6, we have $\gamma^k_{wsR}(H) + \gamma^k_{wsR}(\overline{H}) = 2n$. This leads to

$$\gamma^k_{wsR}(H) + \gamma^k_{wsR}(\overline{H}) = 2n = \frac{4kn}{n+1}.$$

Example 3.12 shows that the Nordhaus-Gaddum bound in Theorem 3.11 is sharp.

4. SPECIAL FAMILIES OF DIGRAPHS

Example 4.1. If $k \geq 1$ and $n \geq \frac{k}{2}$ are integers, then $\gamma^k_{wsR}(K^*_n) = k$.

Proof. If $n \geq k$, then Corollary 1.3 leads to the desired result. Let now $k > n \geq \frac{n}{2}$. Corollary 3.5 implies $\gamma^k_{wsR}(K^*_n) \geq k$. For the converse inequality, let the function $f\colon V(K^*_n) \to \{-1, 1, 2\}$ assign to $k - n$ vertices the value 2 and to the remaining $2n - k$ vertices the value 1. Then f is a WSRkDF on K^*_n of weight $\omega(f) = k$ and so $\gamma_{wsR}(K^*_n) \leq k$. This leads to $\gamma^k_{wsR}(K^*_n) = k$ also in this case. □

Let C_n be an oriented cycle of order n. In [7] and [11] it was shown that $\gamma_{sR}(C_n) = \gamma_{wsR}(C_n) = \frac{n}{2}$ when n is even and $\gamma_{sR}(C_n) = \gamma_{wsR}(C_n) = \frac{n+3}{2}$ when n is odd. Now we determine $\gamma^k_{wsR}(C_n)$ and $\gamma^k_{sR}(C_n)$ for $2 \leq k \leq 4$.

Theorems 3.1 and 3.2 immediately lead to $\gamma^4_{sR}(C_n) = \gamma^4_{wsR}(C_n) = 2n$. In addition, according to Example 3.6, we have $\gamma^2_{sR}(C_n) = \gamma^2_{wsR}(C_n) = n$.

Example 4.2. For $n \geq 2$, we have $\gamma^3_{wsR}(C_n) = \gamma^3_{sR}(C_n) = \lceil \frac{3n}{2} \rceil$.

Proof. Corollary 3.5 implies $\gamma^3_{sR}(C_n) \geq \gamma^3_{wsR}(C_n) \geq \lceil \frac{3n}{2} \rceil$. For the converse inequality we distinguish two cases.

Case 1. Assume that $n = 2t$ is even for an integer $t \geq 1$. Let $C_{2t} = v_0v_1 \ldots v_{2t-1}v_0$. Define $f\colon V(C_{2t}) \to \{-1, 1, 2\}$ by $f(v_{2i}) = 1$ and $f(v_{2i+1}) = 2$ for $0 \leq i \leq t - 1$. Then $f(N^-[v_j]) = 3$ for each $0 \leq j \leq 2t - 1$, and therefore f is an SR3DF on C_{2t} of weight $\omega(f) = 3t$. Thus $\gamma^3_{wsR}(C_n) \leq \gamma^3_{sR}(C_n) \leq 3t$. Consequently, $\gamma^3_{wsR}(C_n) = \gamma^3_{sR}(C_n) = 3t = \lceil \frac{3n}{2} \rceil$ in this case.

Case 2. Assume now that $n = 2t + 1$ is odd for an integer $t \geq 1$. Let $C_{2t+1} = v_0v_1 \ldots v_{2t}v_0$. Define $f\colon V(C_{2t}) \to \{-1, 1, 2\}$ by $f(v_{2i}) = 1$, $f(v_{2i+1}) = 2$ for $0 \leq i \leq t - 1$ and $f(v_{2t}) = 2$. Then $f(N^-[v_j]) \geq 3$ for each $0 \leq j \leq 2t$, and therefore f is an SR3DF on C_{2t+1} of weight $\omega(f) = 3t + 2$. Thus $\gamma^3_{wsR}(C_n) \leq \gamma^3_{sR}(C_n) \leq 3t + 2$. Consequently, $\gamma^3_{wsR}(C_n) = \gamma^3_{sR}(C_n) = 3t + 2 = \lceil \frac{3n}{2} \rceil$ in the second case. □

A digraph is *connected* if its underlying graph is connected. A *rooted tree* is a connected digraph with a vertex r of in-degree 0, called the *root*, such that every vertex different from the root has in-degree 1.

Proposition 4.3. *If T is a rooted tree of order $n \geq 1$, then $\gamma^2_{wsR}(T) = \gamma^2_{sR}(T) = n+1$.*

Proof. Let f be a $\gamma^2_{wsR}(T)$-function, and let r be the root of T. Since $d^-(r) = 0$ and $d^-(x) = 1$ for $x \in V(T) \setminus \{r\}$, we note that $f(r) = 2$ and $f(x) \geq 1$ for $x \in V(T) \setminus \{r\}$. Thus $\gamma^2_{sR}(T) \geq \gamma^2_{wsR}(T) \geq n+1$. On the other hand, the function $g : V(T) \to \{-1, 1, 2\}$ defined by $g(r) = 2$ and $f(x) = 1$ for $x \in V(T) \setminus \{r\}$, is an SR2DF on T of weight $\omega(g) = n+1$. Hence $\gamma^2_{wsR}(T) \leq \gamma^2_{sR}(T) \leq n+1$ and thus $\gamma^2_{wsR}(T) = \gamma^2_{sR}(T) = n+1$. □

Corollary 4.4. *If P_n is an oriented path of order $n \geq 1$, then $\gamma^2_{wsR}(P_n) = \gamma^2_{sR}(P_n) = n + 1$.*

5. FURTHER LOWER BOUNDS

Let S_1 be an orientation of the star $K_{1,n-1}$ such that the center w has out-degree $n-1$. In addition, let S_2 consists of S_1 together with an arc vw for an arbitrary leaf v of $K_{1,n-1}$.

Theorem 5.1. *Let D be a digraph of order $n \geq 2$. Then $\gamma_{wsR}(D) \geq 3-n$, with equality if and only if $D \in \{S_1, S_2\}$.*

Proof. If $\Delta^-(D) \geq 1$, then Theorem 3.7 implies $\gamma_{wsR}(D) \geq 3-n$. Clearly, this remains valid for $\Delta^-(D) = 0$, and the lower bound is proved.

If $D \in \{S_1, S_2\}$, then define $g : V(D) \to \{-1, 1, 2\}$ by $g(w) = 2$ and $g(x) = -1$ for $x \in V(D) \setminus \{w\}$. Then g is a weak signed Roman dominating function on D of weight $3-n$ and thus $\gamma_{wsR}(D) = 3-n$.

Assume now that $\gamma_{wsR}(D) = 3-n$, and let f be a $\gamma_{wsR}(D)$-function. This implies that D has exactly one vertex w with $f(w) = 2$ and $n-1$ vertices $y_1, y_2, \ldots, y_{n-1}$ such that $f(y_i) = -1$ for $1 \leq i \leq n-1$. By the definition, w dominates y_i for $1 \leq i \leq n-1$. If there exists an arc $y_i y_j$ for $i \neq j$, then $f(N^-[y_j]) \leq 0$, a contradiction. If y_i and y_j dominate w for $i \neq j$, then $f(N^-[w]) \leq 0$, a contradiction. Thus, $D \in \{S_1, S_2\}$, and the proof is complete. □

Theorem 5.2. *Let D be a digraph of order $n \geq 2$. Then $\gamma^2_{wsR}(D) \geq 4-n$, with equality if and only if $D = K_2^*$.*

Proof. If $\Delta^-(D) = 0$, then $\gamma^2_{wsR}(D) = 2n > 4-n$. If $\Delta^-(D) \geq 1$, then Theorem 3.7 implies $\gamma_{wsR}(D) \geq 4-n$, and the lower bound is proved. If $D = K_2^*$, then it follows from Example 4.1 that $\gamma^2_{swR}(D) = 2 = 4-n$.

Assume now that $\gamma^2_{wsR}(D) = 4-n$, and let f be a $\gamma^2_{wsR}(D)$-function. This implies that D has exactly two vertices u and v with $f(u) = f(v) = 1$ and $n-2$ vertices $x_1, x_2, \ldots, x_{n-2}$ such that $f(x_i) = -1$ for $1 \leq i \leq n-2$. It follows that $n = 2$, u dominates v and v dominates u and thus $D = K_2^*$. □

Theorem 5.3. *Let $k \geq 3$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$. Then*

$$\gamma^k_{wsR}(D) \geq k + \left\lceil \frac{k}{2} \right\rceil - n,$$

with equality if and only if $D = K^*_{\lceil \frac{k}{2} \rceil}$.

Proof. Since $\Delta^-(D) \geq \delta^-(D) \geq \lceil \frac{k}{2} \rceil - 1$, it follows from Theorem 3.7 that

$$\gamma^k_{wsR}(D) \geq k + 1 + \Delta^-(D) - n \geq k + 1 + \left\lceil \frac{k}{2} \right\rceil - 1 - n = k + \left\lceil \frac{k}{2} \right\rceil - n,$$

and the desired lower bound is proved. If $D = K^*_{\lceil \frac{k}{2} \rceil}$, then Example 4.1 shows that

$$\gamma^k_{swR}(D) = k = k + \left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil.$$

Conversely, assume that $\gamma^k_{wsR}(D) = k + \lceil \frac{k}{2} \rceil - n$, and let f be $\gamma^k_{wsR}(D)$-function. If $\Delta^-(D) \geq \lceil \frac{k}{2} \rceil$, then Theorem 3.7 implies $\gamma^k_{wsR}(D) \geq k + \lceil \frac{k}{2} \rceil + 1 - n$, a contradiction. Thus, $\Delta^-(D) = \delta^-(D) = \lceil \frac{k}{2} \rceil - 1$. If there exists a vertex w with $f(w) = -1$, then we obtain the contradiction

$$k \leq f(N^-[w]) \leq -1 + 2\Delta^-(D) = -1 + 2\left(\left\lceil \frac{k}{2} \right\rceil - 1\right) \leq k - 2.$$

So $f(x) \geq 1$ for each $x \in V(D)$. Next we distinguish two cases.

Case 1. Assume that k is even. If there exixsts a vertex w with $f(w) = 1$, then we arrive at the contradiction

$$k \leq f(N^-[w]) \leq 1 + 2\Delta^-(D) = 1 + 2\left(\frac{k}{2} - 1\right) = k - 1.$$

Therefore $f(x) = 2$ for all $x \in V(D)$. We deduce that $\omega(f) = 2n = k + \frac{k}{2} - n$ and thus $n = \frac{k}{2}$. Consequently, $D = K^*_{\lceil \frac{k}{2} \rceil}$ in this case.

Case 2. Assume that k is odd. If there exists a vertex w with $f(w) = 1$, then w has exactly $\frac{k-1}{2}$ in-neighbors of weight 2. Suppose that D has $t \geq 0$ further vertices of weight 1 and $s \geq 0$ further vertices of weight 2. Then $n = 1 + \frac{k-1}{2} + s + t$ and hence

$$2n = 2s + 2t + k + 1. \tag{5.1}$$

On the other hand we observe that $\omega(f) = 2n - (t + 1) = k + \frac{k+1}{2} - n$ and thus

$$6n = 3k + 2t + 3. \tag{5.2}$$

Combining (5.1) and (5.2), we find that $6s + 4t = 0$ and therefore $s = t = 0$. It follows that $n = \frac{k+1}{2}$ and so $D = K^*_{\lceil \frac{k}{2} \rceil}$.

Finally, assume that $f(x) = 2$ for each $x \in V(D)$. Then $\omega(f) = 2n = k + \frac{k+1}{2} - n$, and we obtain the contradiction $6n = 3k + 1$. □

Let $\{u, v, x_1, x_2, \ldots, x_{n-2}\}$ be the vertex set of the digraph B of order $n \geq 2$ such that u and v dominate x_i for $1 \leq i \leq n - 2$. In addition, let $B_1 = B \cup \{vu\}$, $B_2 = B_1 \cup \{uv\}$, $B_3 = B_1 \cup \{x_1u\}$, $B_4 = B_2 \cup \{x_1u\}$, $B_5 = B_2 \cup \{x_1v, x_1u\}$ and $B_6 = B_2 \cup \{x_1u, x_2v\}$.

Theorem 5.4. *Let D be a digraph of order $n \geq 2$. If $D \notin \{S_1, S_2\}$, then $\gamma_{wsR}(D) \geq 4 - n$, with equality if and only if*

$$D \in \{B, B_1, B_2, B_3, B_4, B_5, B_6\}.$$

Proof. Theorem 5.1 implies $\gamma_{wsR}(D) \geq 4 - n$. If

$$D \in \{B, B_1, B_2, B_3, B_4, B_5, B_6\},$$

then define the function $g : V(D) \to \{-1, 1, 2\}$ by $g(u) = g(v) = 1$ and $g(x_i) = -1$ for $1 \leq i \leq n - 2$. Then g is a weak signed Roman dominating function on D of weight $4 - n$ and thus $\gamma_{wsR}(D) = 4 - n$.

Assume now that $\gamma_{wsR}(D) = 4 - n$, and let f be a $\gamma_{wsR}(D)$-function. This implies that D has exactly two vertices u and v with $f(u) = f(v) = 1$ and $n-2$ vertices $x_1, x_2, \ldots, x_{n-2}$ such that $f(x_i) = -1$ for $1 \le i \le n-2$. By the definition, u and v dominate x_i for $1 \le i \le n-2$. If there exists an arc $x_i x_j$ for $i \ne j$, then $f(N^-[x_j]) \le 0$, a contradiction. If x_i and x_j dominate u or v for $i \ne j$, then $f(N^-[u]) \le 0$ or $f(N^-[v]) \le 0$, a contradiction. If x_1 dominates u, then v dominates u and $D = B_3$ or $D = B_4$. If x_1 dominates u and v, then v dominates u and u dominates v and $D = B_5$. If x_1 dominates u and x_2 dominates v, the $D = B_6$. Finally, if there is no arc from x_i to $\{u, v\}$, then $D \in \{B, B_1, B_2\}$. □

Let $\{u, v, x_1, x_2, \ldots, x_{n-2}\}$ be the vertex set of the digraph L of order $n \ge 2$ such that u and v dominate x_i for $1 \le i \le n-2$ and u dominates v. In addition, let $L_1 = L \cup \{vu\}$, $L_2 = L_1 \cup \{x_1 u\}$, $L_3 = L \cup \{x_1 v\}$, $L_4 = L_1 \cup \{x_1 u, x_1 v\}$, and $L_5 = L_1 \cup \{x_1 u, x_2 v\}$. Using Theorem 5.2 instead of Theorem 5.1, one can prove the next result analogously to Theorem 5.4.

Theorem 5.5. *Let D be a digraph of order $n \ge 3$. Then $\gamma^2_{wsR}(D) \ge 5 - n$, with equality if and only if $D \in \{L, L_1, L_2, L_3, L_4, L_5\}$.*

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