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**EXISTENCE OF SOLUTIONS OF THE DIRICHLET
PROBLEM FOR AN INFINITE SYSTEM OF NONLINEAR
DIFFERENTIAL-FUNCTIONAL EQUATIONS
OF ELLIPTIC TYPE**

Abstract. The Dirichlet problem for an infinite weakly coupled system of semilinear differential-functional equations of elliptic type is considered. It is shown the existence of solutions to this problem. The result is based on Chaplygin's method of lower and upper functions.

Keywords: infinite systems, elliptic differential-functional equations, monotone iterative technique, Chaplygin's method, Dirichlet problem.

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1. INTRODUCTION

Let S be an infinite set. Let $G \subset \mathbb{R}^m$ be an open bounded domain with $C^{2+\alpha}$ boundary ($\alpha \in (0, 1)$). Let $\mathcal{B}(S)$ be the Banach space of all bounded functions $w: S \rightarrow \mathbb{R}$, $w(i) = w^i$ ($i \in S$) with the norm

$$\|w\|_{\mathcal{B}(S)} := \sup_{i \in S} |w^i|.$$

In $\mathcal{B}(S)$ there is a partly order $w \leq \tilde{w}$ defined as $w^i \leq \tilde{w}^i$ for every $i \in S$. Elements of $\mathcal{B}(S)$ will be denoted by $(w^i)_{i \in S}$, too.

Let $C(\bar{G})$ be the space of all continuous functions $v: \bar{G} \rightarrow \mathbb{R}$ with the norm

$$\|v\|_{C(\bar{G})} := \max_{x \in \bar{G}} |v(x)|.$$

In this space $v \leq \tilde{v}$ means that $v(x) \leq \tilde{v}(x)$ for every $x \in \bar{G}$. By $C^{l+\alpha}(\bar{G})$, where $l = 0, 1, 2, \dots$ and $\alpha \in (0, 1)$, we denote the space of all continuous functions in \bar{G} whose

derivatives of order less or equal l exist and are Hölder continuous with exponent α in G (see [5], pp. 52–53). By $H^{l,p}(G)$ we denote the Sobolev space of all functions whose weak derivatives of order l are included in $L^p(G)$ (see [1], pp. 44–46). A notation $g \in C^{l+\alpha}(\partial G)$ (resp. $g \in H^{l,p}(\partial G)$) means that there exists a function $\mathbf{g} \in C^{l+\alpha}(\bar{G})$ (resp. $\mathbf{g} \in H^{l,p}(G) \cap C(\bar{G})$) such that $\mathbf{g}(x) = g(x)$ for every $x \in \partial G$. In these spaces norms are defined as

$$|g|_{C^{l+\alpha}(\partial G)} := \inf\{|\mathbf{g}|_{C^{l+\alpha}(\bar{G})} : \mathbf{g} \in C^{l+\alpha}(\bar{G}) : \forall x \in \partial G : \mathbf{g}(x) = g(x)\}$$

and

$$|g|_{H^{2,p}(\partial G)} := \inf\{|\mathbf{g}|_{H^{2,p}(G)} : \mathbf{g} \in H^{2,p}(G) \cap C(\bar{G}) : \forall x \in \partial G : \mathbf{g}(x) = g(x)\}.$$

We denote $z = (z^i)_{i \in S} \in C_S(\bar{G})$ if $z : \bar{G} \rightarrow \mathcal{B}(S)$ and $z^i : \bar{G} \rightarrow \mathbb{R}$ ($i \in S$) is a continuous function and $\sup_{i \in S} |z^i|_{C(\bar{G})} < \infty$. The space $C_S(\bar{G})$ is a Banach space with the norm

$$\|z\|_{C_S(\bar{G})} := \sup_{i \in S} |z^i|_{C(\bar{G})}$$

and the partly order $z \leq \tilde{z}$ defined as $z^i(x) \leq \tilde{z}^i(x)$ for every $x \in \bar{G}$, $i \in S$. The space $C_S^{l+\alpha}(\bar{G})$ is a space of all functions $(z^i)_{i \in S}$ such that $z^i \in C^{l+\alpha}(\bar{G})$ for every $i \in S$ and $\sup_{i \in S} |z^i|_{C^{l+\alpha}(\bar{G})} < \infty$. In this space the norm is defined as

$$\|z\|_{C_S^{l+\alpha}(\bar{G})} = \sup_{i \in S} |z^i|_{C^{l+\alpha}(\bar{G})}.$$

We will write that $z = (z^i)_{i \in S} \in L_S^p(G)$ if $z^i \in L^p(G)$ for every $i \in S$ and $\sup_{i \in S} |z^i|_{L^p(G)} < \infty$. A notation $z = (z^i)_{i \in S} \in H_S^{l,p}(G)$ means that $z^i \in H^{l,p}(G)$ for every $i \in S$ and $\sup_{i \in S} |z^i|_{H^{l,p}(G)} < \infty$. In these spaces the norms are defined as

$$\|z\|_{L_S^p(G)} = \sup_{i \in S} |z^i|_{L^p(G)}$$

and

$$\|z\|_{H_S^{l,p}(G)} = \sup_{i \in S} |z^i|_{H^{l,p}(G)}.$$

We consider the Dirichlet problem for an infinite weakly coupled system of semilinear differential-functional equations of the following form

$$-\mathcal{L}^i[u^i](x) = f^i(x, u(x), u), \quad \text{for } x \in G, i \in S \quad (1)$$

and

$$u^i(x) = h^i(x), \quad \text{for } x \in \partial G, i \in S, \quad (2)$$

where

$$\mathcal{L}^i[u^i](x) := \sum_{j,k=1}^m a_{jk}^i(x) u_{x_j x_k}^i(x) + \sum_{j=1}^m b_j^i(x) u_{x_j}^i(x),$$

which are strongly uniformly elliptic in \bar{G} ,

$$f^i: \bar{G} \times \mathcal{B}(S) \times C_S(\bar{G}) \ni (x, y, z) \mapsto f^i(x, y, z) \in \mathbb{R}$$

for every $i \in S$. The notation $f(x, u(x), u)$ means that the dependence of f on the second variable is a function-type dependence and $f(x, u(x), \cdot)$ is a functional-type dependence.

A function u is said to be *regular in \bar{G}* if $u \in C_S(\bar{G}) \cap C_S^2(G)$. A function u is said to be a *classical (regular) solution* of the problem (1), (2) in \bar{G} if u is regular in \bar{G} and fulfills the system of equations (1) in G with the condition (2). A function u is said to be a *weak solution* of the problem (1), (2) in \bar{G} if $u \in L_S^2(G)$ such that $\mathcal{L}^i[u^i] \in L^2(G)$ and

$$-\int_G \mathcal{L}^i[u^i](x)\xi(x) dx = \int_G f^i(x, u(x), u)\xi(x) dx$$

for every $i \in S$ and for any test function $\xi \in C_0^\infty(\bar{G})$.

We would like to find assumptions which guarantee existence of the classical solutions of the problem (1), (2)

$$u: \bar{G} \rightarrow \mathcal{B}(S).$$

Regular functions $u_0 = u_0(x)$ and $v_0 = v_0(x)$ in \bar{G} satisfying the infinite systems of inequalities:

$$\begin{cases} -\mathcal{L}^i[u_0^i](x) \leq f^i(x, u_0(x), u_0) & \text{for } x \in G, i \in S, \\ u_0^i(x) \leq h^i(x) & \text{for } x \in \partial G, i \in S, \end{cases} \tag{3}$$

$$\begin{cases} -\mathcal{L}^i[v_0^i](x) \geq f^i(x, v_0(x), v_0) & \text{for } x \in G, i \in S, \\ v_0^i(x) \geq h^i(x) & \text{for } x \in \partial G, i \in S \end{cases} \tag{4}$$

are called a *lower* and an *upper function* for the problem (1), (2), respectively.

If $u_0 \leq v_0$, we define

$$\mathcal{K} := \{(x, y, z): x \in \bar{G}, y \in [m_0, M_0], z \in \langle u_0, v_0 \rangle\},$$

where $m_0 := (m_0^i)_{i \in S}$, $m_0^i := \min_{x \in \bar{G}} u_0^i(x)$, $M_0 := (M_0^i)_{i \in S}$, $M_0^i := \max_{x \in \bar{G}} v_0^i(x)$ and $\langle u_0, v_0 \rangle := \{\zeta \in C_S(\bar{G}): u_0(x) \leq \zeta(x) \leq v_0(x) \text{ for } x \in \bar{G}\}$.

Assumptions. We make the following assumptions.

- (a) \mathcal{L} is a strongly uniformly elliptic operator in \bar{G} , i.e., there exists a constant $\mu > 0$ such that

$$\sum_{j,k=1}^m a_{jk}^i(x)\xi_j\xi_k \geq \mu \sum_{j=1}^m \xi_j^2, \quad i \in S,$$

for all $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $x \in G$.

- (b) The functions a_{jk}^i, b_j^i for $i \in S, j, k = 1, \dots, m$ are functions of class $C^{0+\alpha}(\bar{G})$ and $a_{jk}^i(x) = a_{kj}^i(x)$ for every $x \in \bar{G}$.
- (c) $h^i \in C^{2+\alpha}(\partial G)$ for every $i \in S$ and $\sup_{i \in S} |h^i|_{C^{2+\alpha}(\partial G)} < \infty$.
- (d) There exists at least one ordered pair u_0, v_0 of a lower and an upper function for the problem (1), (2) in \bar{G} such that

$$u_0(x) \leq v_0(x) \quad \text{for } x \in \bar{G}.$$

- (e) $f(\cdot, y, z) \in C_S^{0+\alpha}(\bar{G})$ for $y \in \mathcal{B}(S), z \in C_S(\bar{G})$.
- (f) For every $i \in S, x \in \bar{G}, y, \tilde{y} \in \mathcal{B}(S), z \in C_S(\bar{G})$

$$|f^i(x, y, z) - f^i(x, \tilde{y}, z)| \leq L_f \|y - \tilde{y}\|_{\mathcal{B}(S)},$$

where $L_f > 0$ is a constant independent of i and

$$|f^i(x, y^1, \dots, y^{i-1}, y^i, y^{i+1}, \dots, z) - f^i(x, y^1, \dots, y^{i-1}, \tilde{y}^i, y^{i+1}, \dots, z)| \leq k^i |y^i - \tilde{y}^i|,$$

where $k^i > 0$ is a constant and there exists $\mathbf{k} < \infty$ such that $k^i \leq \mathbf{k}$ for every $i \in S$.

- (g) $f(x, \cdot, \cdot)$ is a continuous function for every $x \in \bar{G}$.
- (h) $f^i(x, \cdot, z)$ is a quasi-increasing function for every $i \in S, x \in \bar{G}, z \in C_S(\bar{G})$ i.e., for every $i \in S$ for arbitrary $y, \tilde{y} \in \mathcal{B}(S)$ if $y^j \leq \tilde{y}^j$ for all $j \in S$ such that $j \neq i$ and $y^i = \tilde{y}^i$, then $f^i(x, y, z) \leq f^i(x, \tilde{y}, z)$ for $x \in \bar{G}, z \in C_S(\bar{G})$.
- (i) $f^i(x, y, \cdot)$ is an increasing function for every $i \in S, x \in \bar{G}, y \in \mathcal{B}(S)$.

2. AUXILIARY RESULTS

From the assumption (f) we have $k := (k^i)_{i \in S} \in \mathcal{B}(S)$. Let $\beta = (\beta^i)_{i \in S} \in C_S^{0+\alpha}(\bar{G})$. We define the operator

$$\mathcal{P}: C_S^{0+\alpha}(\bar{G}) \ni \beta \mapsto \gamma \in C_S^{2+\alpha}(\bar{G}),$$

where $\gamma = (\gamma^i)_{i \in S}$ is the solution (supposedly unique) of the following problem

$$\begin{cases} -(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i](x) = f^i(x, \beta(x), \beta) + k^i \beta^i(x) & \text{for } x \in G, i \in S, \\ \gamma^i(x) = h^i(x) & \text{for } x \in \partial G, i \in S. \end{cases} \quad (5)$$

We remark that the problem (5) is a system of separate problems with only one equation, so $\mathcal{P}[\beta]$ is a collection of solutions of these problems.

Lemma 1. *The operator $\mathcal{P}: C_S^{0+\alpha}(\bar{G}) \rightarrow C_S^{2+\alpha}(\bar{G})$ is a continuous and bounded operator. If the operator \mathcal{P} maps $C_S^{0+\alpha}(\bar{G})$ into $C_S^{0+\alpha}(\bar{G})$, then it is a compact operator.*

Proof. Let $\beta \in C_S^{0+\alpha}(\bar{G})$, so

$$|\beta^i(x) - \beta^i(\tilde{x})| \leq H_\beta \|x^j - \tilde{x}^j\|_{\mathbb{R}^m}^\alpha,$$

where $H_\beta > 0$ is some constant independent of i and $\|x\|_{\mathbb{R}^m} = (\sum_{j=1}^m x_j^2)^{1/2}$.

We define the operator

$$\mathbf{F} = (\mathbf{F}^i)_{i \in S}: C_S^{0+\alpha}(\bar{G}) \ni \beta \mapsto \delta \in C_S^{0+\alpha}(\bar{G})$$

such that for every $i \in S$

$$\mathbf{F}^i[\beta](x) = \delta^i(x) := f^i(x, \beta(x), \beta) + k^i \beta^i(x).$$

For arbitrary $i \in S$ we have:

$$\begin{aligned} |\delta^i(x) - \delta^i(\tilde{x})| &= |f^i(x, \beta(x), \beta) + k^i \beta^i(x) - f^i(\tilde{x}, \beta(\tilde{x}), \beta) - k^i \beta^i(\tilde{x})| \leq \\ &\leq |f^i(x, \beta(x), \beta) - f^i(\tilde{x}, \beta(x), \beta)| + |f^i(\tilde{x}, \beta(x), \beta) - f^i(\tilde{x}, \beta(\tilde{x}), \beta)| + \\ &\quad + k^i |\beta^i(x) - \beta^i(\tilde{x})| \leq (H_f + L_f H_\beta + \mathbf{k} H_\beta) \|x - \tilde{x}\|_{\mathbb{R}^m}^\alpha, \end{aligned}$$

where $H_f + L_f H_\beta + \mathbf{k} H_\beta$ is some constant independent of i .

By the properties of f , we see that the operator \mathbf{F} is a continuous and bounded operator.

Now, we have our problem for arbitrary $i \in S$ in the following form

$$\begin{cases} -(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i](x) = \delta^i(x) & \text{for } x \in G, \\ \gamma^i(x) = h^i(x) & \text{for } x \in \partial G, \end{cases} \quad (6)$$

which satisfies the assumptions of the Schauder theorem ([7], p. 115), so the problem (6) for every $i \in S$ has a unique solution $\gamma^i \in C^{2+\alpha}(\bar{G})$ and the following estimate

$$|\gamma^i|_{C^{2+\alpha}(\bar{G})} \leq C \left(|\delta^i|_{C^{0+\alpha}(\bar{G})} + |h^i|_{C^{2+\alpha}(\partial G)} \right) \quad (7)$$

holds, where $C > 0$ is independent of δ , h and i .

Let us introduce the operator

$$\mathbf{G} = (\mathbf{G}^i)_{i \in S}: C_S^{0+\alpha}(\bar{G}) \ni \delta \mapsto \gamma \in C_S^{2+\alpha}(\bar{G}).$$

The function

$$\gamma^i = \mathbf{G}^i[\delta^i] = \mathbf{G}_1^i[h^i] + \mathbf{G}_2^i[\delta^i],$$

where

$$\mathbf{G}_1^i: C^{2+\alpha}(\bar{G}) \ni h^i \mapsto \mathbf{G}_1^i[h^i] \in C^{2+\alpha}(\bar{G})$$

and $\mathbf{G}_1^i[h^i]$ is the unique solution of the problem (6) with $\delta^i(x) = 0$ in \bar{G} , and

$$\mathbf{G}_2^i: C^{0+\alpha}(\bar{G}) \ni \delta^i \mapsto \mathbf{G}_2^i[\delta^i] \in C^{2+\alpha}(\bar{G})$$

and $\mathbf{G}_2^i[\delta^i]$ is the unique solution of the problem (6) with $h^i(x) = 0$ on ∂G . The operator \mathbf{G}^i is a continuous operator because the operator \mathbf{G}_1^i is independent of δ^i , and \mathbf{G}_2^i is a continuous operator (from (7)) with respect to δ^i . By (7), we have

$$|\gamma^i|_{C^{2+\alpha}(\bar{G})} = |\mathbf{G}^i \circ \mathbf{F}^i[\beta^i]|_{C^{2+\alpha}(\bar{G})} \leq C \left(|\delta^i|_{C^{2+\alpha}(\bar{G})} + |h^i|_{C^{2+\alpha}(\partial G)} \right),$$

where $C > 0$ is independent of δ , h and i . Thus the operator $\mathbf{G} \circ \mathbf{F}$ is a continuous and bounded operator.

Since $\partial G \in C^{2+\alpha}$, the imbedding operator

$$\mathbf{I}: C_S^{2+\alpha}(\bar{G}) \rightarrow C_S^{0+\alpha}(\bar{G})$$

is a compact operator ([1], p. 11). So $\mathcal{P} = \mathbf{I} \circ \mathbf{G} \circ \mathbf{F}$ is a compact operator. □

Next, let us consider the operator \mathcal{P} as a operator mapping $L_S^p(G)$.

Lemma 2. *The operator \mathcal{P} is a compact operator mapping $L_S^p(G)$ into $L_S^p(G)$.*

Proof. We define δ and the operator \mathbf{F} such in the proof of Lemma 1 but on an element of $L_S^p(G)$.

The operator $\mathbf{F}: L_S^p(G) \rightarrow L_S^p(G)$ and \mathbf{F} is a continuous and bounded operator by arguing as [6] (Th. 2.1, Th. 2.2 and Th. 2.3, pp. 31–37) and [9] (Th. 19.1, p. 204).

Now, we have our problem for arbitrary $i \in S$ in the following form

$$\begin{cases} -(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i](x) = \delta^i(x) & \text{for } x \in G, \\ \gamma^i(x) = h^i(x) & \text{for } x \in \partial G, \end{cases} \tag{8}$$

which satisfies the assumptions of Agmon–Douglis–Nirenberg theorem for arbitrary $i \in S$, so the problem (8) has a unique weak solution $\gamma^i \in H^{2,p}(G)$ and the following estimate

$$|\gamma^i|_{H^{2,p}(G)} \leq C \left(|\delta^i|_{L^p(G)} + |h^i|_{H^{2,p}(\partial G)} \right) \tag{9}$$

holds, where $C > 0$ is independent of δ , h and i .

Let us introduce the operator

$$\mathbf{G} = (\mathbf{G}^i)_{i \in S}: L_S^p(G) \ni \delta \mapsto \gamma \in H_S^{2,p}(G).$$

The function

$$\gamma^i = \mathbf{G}^i[\delta^i] = \mathbf{G}_1^i[h^i] + \mathbf{G}_2^i[\delta^i],$$

where

$$\mathbf{G}_1^i: H^{2,p}(G) \ni h^i \mapsto \mathbf{G}_1^i[h^i] \in H^{2,p}(G)$$

and $\mathbf{G}_1^i[h^i]$ is the unique weak solution of the problem (8) with $\delta^i(x) = 0$ in \bar{G} , and

$$\mathbf{G}_2^i: L^p(G) \ni \delta^i \mapsto \mathbf{G}_2^i[\delta^i] \in H^{2,p}(G)$$

and $\mathbf{G}_2^i[\delta^i]$ is the unique solution of the problem (8) with $h^i(x) = 0$ on ∂G . The operator \mathbf{G}^i is a continuous operator because the operator \mathbf{G}_1^i is independent of δ^i , and \mathbf{G}_2^i is a continuous operator (from (9)) with respect to δ^i . Also we know that

$$|\gamma^i|_{H^{2,p}(G)} = |\mathbf{G}^i \circ \mathbf{F}^i[\beta^i]|_{H^{2,p}(G)} \leq C \left(|\delta^i|_{L^p(G)} + |h^i|_{H^{2,p}(\partial G)} \right),$$

where $C > 0$ is independent of δ, h and i . Thus the operator $\mathbf{G} \circ \mathbf{F}$ is a continuous and bounded operator. Since $\partial G \in C^{2+\alpha}$, the imbedding operator

$$\mathbf{I}: H_S^{2,p}(G) \rightarrow L_S^p(G)$$

is a compact operator ([1], p. 97), and $\mathcal{P} = \mathbf{I} \circ \mathbf{G} \circ \mathbf{F}$ is also a compact operator. \square

Now, we prove next some properties of the operator \mathcal{P} .

Lemma 3. *The operator \mathcal{P} is an increasing operator.*

Proof. Let $\beta(x) \leq \tilde{\beta}(x)$ in \bar{G} , so for all $i \in S$, $\beta^i(x) \leq \tilde{\beta}^i(x)$ in \bar{G} . Let $\gamma := \mathcal{P}[\beta]$ and $\tilde{\gamma} := \mathcal{P}[\tilde{\beta}]$. For arbitrary $i \in S$

$$\begin{cases} -(\mathcal{L}^i - k^i \mathcal{I})[\tilde{\gamma}^i - \gamma^i](x) = \\ = f^i(x, \tilde{\beta}(x), \tilde{\beta}) - f^i(x, \beta(x), \beta) + k^i(\tilde{\beta}^i(x) - \beta^i(x)) & \text{for } x \in G, \\ (\tilde{\gamma}^i - \gamma^i)(x) = 0, & \text{for } x \in \partial G. \end{cases}$$

By assumption (h), (i) and (f),

$$\begin{aligned} -(\mathcal{L}^i - k^i \mathcal{I})[\tilde{\gamma}^i - \gamma^i](x) &\geq \\ &\geq \left[f^i(x, \beta^1(x), \dots, \beta^{i-1}(x), \tilde{\beta}^i(x), \beta^{i+1}(x), \dots, \beta) - f^i(x, \beta(x), \beta) \right] + \\ &\quad + k^i(\tilde{\beta}^i(x) - \beta^i(x)) \geq 0. \end{aligned}$$

So for every $i \in S$

$$\begin{cases} -(\mathcal{L}^i - k^i \mathcal{I})[\tilde{\gamma}^i - \gamma^i](x) \geq 0 & \text{for } x \in G, \\ \tilde{\gamma}^i(x) - \gamma^i(x) = 0 & \text{for } x \in \partial G. \end{cases}$$

By the maximum principle ([8], p. 64)

$$\tilde{\gamma}^i(x) - \gamma^i(x) \geq 0 \text{ in } \bar{G}.$$

We have for all $i \in S$ $\gamma^i(x) \leq \tilde{\gamma}^i(x)$, so

$$\gamma(x) \leq \tilde{\gamma}(x) \text{ in } \bar{G}. \quad \square$$

Lemma 4. *If β is an upper (resp. a lower) function for the problem (1), (2) in \bar{G} , then $\mathcal{P}[\beta] \leq \beta$ (resp. $\mathcal{P}[\beta] \geq \beta$) in \bar{G} and $\mathcal{P}[\beta]$ is an upper (resp. a lower) function for problem (1), (2) in \bar{G} .*

Proof. Let $\gamma = \mathcal{P}[\beta]$. By (5) we have for every $i \in S$

$$\begin{aligned} -(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i - \beta^i](x) &= -(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i](x) + (\mathcal{L}^i - k^i \mathcal{I})[\beta^i](x) = \\ &= f^i(x, \beta(x), \beta) + k^i \beta^i(x) + \mathcal{L}^i[\beta^i](x) - k^i \beta^i(x) = f^i(x, \beta(x), \beta) + \mathcal{L}^i[\beta^i](x) \end{aligned}$$

and from (4)

$$f^i(x, \beta(x), \beta) + \mathcal{L}^i[\beta^i](x) \leq 0$$

and

$$(\gamma^i - \beta^i)(x) = h^i(x) - \beta^i(x) \leq 0.$$

So for every $i \in S$

$$\begin{cases} -(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i - \beta^i](x) \leq 0 & \text{for } x \in G, \\ (\gamma^i - \beta^i)(x) \leq 0 & \text{for } x \in \partial G. \end{cases}$$

Now, by using the maximum principle ([8], th. 6, p. 64) separately for every $i \in S$

$$\gamma^i(x) - \beta^i(x) \leq 0 \text{ in } \bar{G}.$$

So

$$\gamma(x) \leq \beta(x) \text{ in } \bar{G}.$$

From Lemma 1 it follows that $\gamma \in C_S^{2+\alpha}(\bar{G})$ and from (5) and the assumption (f), we get for every $i \in S$

$$\begin{aligned} -\mathcal{L}^i[\gamma^i](x) - f^i(x, \gamma(x), \gamma) &= -(\mathcal{L}^i - k^i \mathcal{I})[\gamma^i](x) - f^i(x, \gamma(x), \gamma) - k^i \gamma^i(x) = \\ &= f^i(x, \beta(x), \beta) + k^i \beta^i(x) - f^i(x, \gamma(x), \gamma) - k^i \gamma^i(x) \geq \\ &\geq (f^i(x, \gamma^1(x), \dots, \gamma^{i-1}(x), \beta^i(x), \gamma^{i+1}(x), \dots, \gamma) - f^i(x, \gamma(x), \gamma)) + \\ &\quad + k^i(\beta^i(x) - \gamma^i(x))) \geq 0 \text{ in } \bar{G}, \end{aligned}$$

so it is a upper the function for problem (1), (2) in \bar{G} . \square

3. MAIN RESULT

Theorem. *If the assumptions (a)–(i) hold, then the problem (1), (2) has at least one classical solution u such that $u \in \langle u_0, v_0 \rangle$.*

Proof. By induction, we define two sequences of functions $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ by setting:

$$\begin{aligned} u_1 &= \mathcal{P}[u_0], & u_n &= \mathcal{P}[u_{n-1}], \\ v_1 &= \mathcal{P}[v_0], & v_n &= \mathcal{P}[v_{n-1}]. \end{aligned}$$

Because u_0 and v_0 are regular functions, these sequences are well defined by Lemma 1. The sequence $\{u_n\}_{n=0}^\infty$ is increasing and $\{v_n\}_{n=0}^\infty$ is decreasing by Lemma 4:

$$\begin{aligned} u_1(x) &= \mathcal{P}[u_0](x) \geq u_0(x) && \text{in } \bar{G}, \\ v_1(x) &= \mathcal{P}[v_0](x) \leq v_0(x) && \text{in } \bar{G}, \end{aligned}$$

and by induction:

$$\begin{aligned} u_n(x) &= \mathcal{P}[u_{n-1}](x) \geq u_{n-1}(x) && \text{in } \bar{G}, \quad n = 1, 2, \dots \\ v_n(x) &= \mathcal{P}[v_{n-1}](x) \leq v_{n-1}(x) && \text{in } \bar{G}, \quad n = 1, 2, \dots \end{aligned}$$

Since the operator \mathcal{P} is increasing and by the assumption (d) we have

$$u_1(x) = \mathcal{P}[u_0](x) \leq \mathcal{P}[v_0](x) = v_1(x) \text{ in } \bar{G}$$

and consequently by induction

$$u_n(x) \leq v_n(x) \text{ in } \bar{G}.$$

Therefore

$$u_0(x) \leq u_1(x) \leq \dots \leq u_n(x) \leq \dots \leq v_n(x) \leq \dots \leq v_1(x) \leq v_0(x).$$

The sequences $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ are monotone and bounded, so they have pointwise limits and we can define:

$$\underline{u}(x) := \lim_{n \rightarrow \infty} u_n(x), \quad \bar{v}(x) := \lim_{n \rightarrow \infty} v_n(x)$$

for every $x \in \bar{G}$.

The functions $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ are functions of class $L_S^p(G)$. Let be $p \in (m, \infty)$ (we need this assumption to can use a imbedding theorem). Because $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ are bounded functions in $L_S^p(G)$ and \mathcal{P} is an increasing compact operator in $L_S^p(G)$ (from Lemma 2), $\{\mathcal{P}u_n\}$ and $\{\mathcal{P}v_n\}$ are converging sequences in $L_S^p(G)$ and

$$\begin{aligned} \underline{u}(x) &= \lim_{n \rightarrow \infty} \mathcal{P}[u_n](x) = \lim_{n \rightarrow \infty} \mathcal{P}[\mathcal{P}[u_{n-1}]](x) = \mathcal{P}[\underline{u}](x), \\ \bar{v}(x) &= \lim_{n \rightarrow \infty} \mathcal{P}[v_n](x) = \lim_{n \rightarrow \infty} \mathcal{P}[\mathcal{P}[v_{n-1}]](x) = \mathcal{P}[\bar{v}](x). \end{aligned}$$

Since $\underline{u}, \bar{v} \in L_S^p(G)$ and:

$$\underline{u} = \mathcal{P}[\underline{u}], \quad \bar{v} = \mathcal{P}[\bar{v}] \tag{10}$$

by the Agmon–Douglis–Nirenberg theorem we have

$$\underline{u}, \bar{v} \in H_S^{2,p}(G).$$

Because the Sobolev space $H^{2,p}(\bar{G})$ for $p > m$ is continuously imbedding in $C^{0+\alpha}(\bar{G})$ and

$$|u^i|_{C^{0+\alpha}(\bar{G})} \leq C |u^i|_{H^{2,p}(G)},$$

where C is independent of i ([1], p. 97–98), we get

$$\underline{u}, \bar{v} \in C_S^{0+\alpha}(\bar{G}). \quad (11)$$

Applying the Schauder theorem to (10) separately for every $s \in S$ for (11) we get

$$\underline{u}, \bar{v} \in C_S^{2+\alpha}(\bar{G}).$$

From the proof we know that

$$u_0(x) \leq \underline{u}(x) \leq \bar{v}(x) \leq v_0(x). \quad \square$$

Corollary. *The solutions \underline{u}, \bar{v} are minimal and maximal solution of the problem (1), (2) in $\langle u_0, v_0 \rangle$.*

Proof. If w is a solution of the problem (1), (2) then $w(x) = \mathcal{P}[w](x)$ and $u_0(x) \leq w(x) \leq v_0(x)$. Because \mathcal{P} is an increasing operator, we have

$$u_1(x) = \mathcal{P}[u_0](x) \leq \mathcal{P}[w](x) = w(x) = \mathcal{P}[w](x) \leq \mathcal{P}[v_0](x) = v_1(x)$$

and by induction we get

$$u_n(x) \leq w(x) \leq v_n(x).$$

Thus

$$\underline{u}(x) = \lim_{n \rightarrow \infty} u_n(x) \leq w(x) \leq \lim_{n \rightarrow \infty} v_n(x) = \bar{v}(x). \quad \square$$

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