Opuscula Mathematica • Vol. 26 • No. 2 • 2006

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CONTINUOUS DEPENDENCE OF SOLUTIONS OF ELLIPTIC BVPs ON PARAMETERS

Abstract. The continuous dependence of solutions for a certain class of elliptic PDE on functional parameters is studied in this paper. The main result is as follow: the sequence $\{x_k\}_{k\in N}$ of solutions of the Dirichlet problem discussed here (corresponding to parameters $\{u_k\}_{k\in N}$) converges weakly to x_0 (corresponding to u_0) in $W_0^{1,q}(\Omega, R)$, provided that $\{u_k\}_{k\in N}$ tends to u_0 a.e. in Ω. Our investigation covers both sub and superlinear cases. We apply this result to some optimal control problems.

Keywords: continuous dependence on parameters, elliptic Dirichlet problems, optimal control problem.

Mathematics Subject Classification: 49K40, 49K20, (35J20, 35J60).

1. INTRODUCTION

This paper is devoted to the continuous dependence (on parameters) of solutions for the Dirichlet problem associated with the PDE of elliptic type

$$\begin{cases} -\text{div } (k(y)|\nabla x(y)|^{q-2}\nabla x(y)) = G_x(y, x(y), u(y)) & \text{for a.e. } y \in \Omega, \\ x \in W_0^{1,p}(\Omega, R), \end{cases} \tag{1.1}$$

where $q \geq 2$, $k \in C^1(\overline{\Omega}, R_+)$, G_x denotes the derivative of G with respect to x and functional parameters $u \in U \subset L^p(\Omega, R^m)$, $m \geq 1$, $p \in (1, \infty)$. We also investigate an optimal control problem governed by (1.1) with a certain integral cost functional. We propose an approach based on the following assumptions:

(Ω) *Ω is a bounded domain in R^n with a locally $C^{1,1}$ boundary;*

(K) *$k \in C^1(\overline{\Omega}, R)$, $\overline{k}_0 \geq k(y) \geq k_0 > 0$ for all $y \in \Omega$;*

(G1) *for each* $u \in U$, *there exist* $\overline{z}_u$, $z_{0u} \in \mathbf{W}_0^{2,\infty}(\Omega, R)$, $\mathbf{W}_0^{2,\infty} := W_0^{1,\infty} \cap W^{2,\infty}$, *such that* $0 < z_{0u}(y) < \overline{z}_u(y)$ *for a.e.* $y \in \Omega$ *and*

$$-G_x(y, \overline{z}_u(y), u(y)) \geq \operatorname{div}(k\,(y)\,|\nabla z_{0u}(y)|^{q-2} \nabla z_{0u}(y));$$

(G2) *there exists* $M > 0$ *such that for all* $u \in U$, $\operatorname*{ess\,sup}_{y\in\Omega} \overline{z}_u(y) \leq M$;

(G3) $G : \Omega \times I \times R^m \to R$, *where* I *is some closed neighborhood of the interval* $[0, M]$,

a. $G(\cdot, x, u)$ *is measurable on* Ω *for all* $(x, u) \in I \times R^m$,

b. $G_x(y, \cdot, \cdot)$ *is continuous in* $I \times R^m$ *for a.a.* $y \in \Omega$,

c. *for a.a.* $y \in \Omega$ *and all* $u \in R^m$, $G_x(y, \cdot, u)$ *is nonnegative and increasing in* I;

(G4) *for all* $u \in U$, $0 < \left| \int_\Omega G(y, 0, u(y))dy \right| < \infty$;

(G5) *there exits* $\varphi \in L^{q'}(\Omega, R_+)$ *such that for a.a.* $y \in \Omega$ *and all* $u \in U$,

$$G_x(y, M, u(y)) \leq \varphi(y).$$

In the 1990s, some papers concerning the continuous dependence of solutions for various systems of ODE and PDE on parameters were published. In [11] the problem

$$\begin{cases} \dfrac{d}{dt} f_{x'}(t, x, x', u) = f_x(t, x, x', u) & \text{for a.e. } t \in (0, \pi) \\ x(0) = x(\pi) = 0 \end{cases} \tag{1.2}$$

for $f : [0, \pi] \times R^n \times R^n \times R^m \to R$, is investigated with variational methods under the assumptions which guarantee, the coercivity of the functional of action associated with (1.2). The author gives sufficient conditions for the existence of solutions of (1.2) and their continuous dependence on the parameter u, when u converges to u_0 in the strong or weak -* topology in L^∞. Similar problems are also widely discussed by D. Idczak in [1] and [2]. Applying the dual principle of minimal action and generalizing the perturbation method described, e.g., in [7], [8], D. Idczak studies the continuous dependence on functional parameters for (1.2), also in the non-coercive case. Next, these results are applied to infer the existence of solutions for the optimal control problem with constraints (1.2) and the cost functional $F(x, u) = \int_0^\pi f^0(t, x(t), x'(t), u(t))dt$, which is Fréchet differentiable with respect to x in certain Sobolev space. Similar problems are also discussed in [9], where f has the special form $f(t, x, x', u) = L(t, x') - V(t, x, u)$, it is concave-convex in (x, x') and it is not necessarily C^1. These results cover both sub- and superlinear cases.

The continuous dependence of solution for PDE on parameters has been studied, e.g., in [3–6, 11–13]. More precisely, the main result of [13] is the following: if $G : \Omega \times R^N \times A \to R$ satisfies the inequalities $a < pG(y, x, u) \leq \langle G_x(y, x, u), x\rangle$, for given constants $a > 0$, $p > 2$ and $|x|$ sufficiently large, and some technical assumptions, then the strong (or weak) convergence of the sequence of parameters $\{u_k\}_{k\in N}$ to u_0 in $L^p(\Omega, R^m)$ implies strong convergence of the sequence $\{x_k\}_{k\in N}$ of solutions for the Dirichlet problem

$$\triangle x_i(y) = G_{x_i}(y, x(y), u(y)) \text{ a.e. in } \Omega, \quad x_i(y) = 0 \text{ for } y \in \partial\Omega,$$

(corresponding to parameters $\{u_k\}_{k\in N}$ to x_0 (corresponding to u_0) in $H_0^1(\Omega, R^N)$. Papers [3–6] are devoted to the hyperbolic and elliptic control systems investigated with variational and topological methods, e.g., [5] describes the optimal control problem governed by the elliptic system

$$\begin{cases} \triangle x(y) = G_z^1(y, x(y)) + \left\langle G_z^2(y, x(y)), u(y)\right\rangle \\ x(y) = v(y) \text{ on } \partial\Omega \end{cases}$$

with the integral cost functional

$$J(x, u, v) = \int_\Omega \Phi^1(y, x(y), \nabla x(y, z), u(y, z))dy + \int_{\partial\Omega} \Phi^2(y, v(y))d\mu(y),$$

where $\Omega \subset R^n$ is a bounded domain with the Lipschitze boundary, $G^1 : R^n \times R^N \to R$, $G^2 : R^n \times R^N \to R^N$, for each $i = 1, 2$, G^i, G_z^i, Φ^i, are Carathéodory functions satisfying some growth conditions and $\Phi^1(y, x, \cdot, \cdot)$, $\Phi^2(y, \cdot)$ are convex. The authors proved the existence of at least one optimal control process for the above system.

The results presented in the papers mentioned above are based on the global assumptions made on the nonlinearity of PDE and a function which appears in the cost functional. This paper is devoted to similar problems for PDE with only local regularities concerning the right-hand side of the differential equation. Our approach can be applied for sublinear and superlinear cases, which is due to the fact that we do not impose any growth conditions on the nonlinearity of the equation. Moreover, we need conditions concerning the behaviour of $G(y, \cdot, u)$ (smoothness and convexity) in certain interval only. It is associated with the fact that for each $u \in U$, we consider solutions of (1.1) from a pre-specified subset of $W_0^{1,q}(\Omega, R)$. The existence of such solutions was proved in [10] (Theorem 4.1):

Theorem 1.1. *Assume that:*

(Q) *q is even and $q \geq 2$;*

(Ω) *Ω is a bounded domain in R^n with a locally $C^{1,1}$ boundary;*

(F1) *there exist $\overline{z}$, $z_0 \in \mathbf{W}_0^{2,\infty}(\Omega, R)$ such that $0 < z_0(y) < \overline{z}(y)$ for a.e. $y \in \Omega$ and*

$$-F_x(y, \overline{z}(y)) \geq \operatorname{div}(k\,(y)\,|\nabla z_0(y)|^{q-2}\nabla z_0(y)) \tag{1.3}$$

a.e. in Ω;

(F2) $F(y,\cdot)$ *is convex and* $C^1(\tilde{I},R)$ *for a.e.* $y\in\Omega$, $G(\cdot,x)$ *is measurable on* Ω *for all* $x\in\tilde{I}$, *where* $\tilde{I}$ *is some closed neighborhood* $\tilde{I}$ *of the interval* $I=\left[0, ess\sup_{y\in\Omega}\overline{z}(y)\right]$ *for a.e.* $y\in\Omega$;

(F3) F_x *is nonnegative in* $\tilde{I}$ *for a.e.* $y\in\Omega$;

(F4) $\left|\int_\Omega F(y,0)dy\right|<\infty$. *Then there exists a solution* $x_0\in X_{\overline{z}}$ *for*

$$\begin{cases}-\mathrm{div}\left(k\left(y\right)\left|\nabla x(y)\right|^{q-2}\nabla x(y)\right)=F_x(y,x(y)) & \text{for a.e. } y\in\Omega\\ x\in W_0^{1,q}(\Omega,R)\end{cases}\tag{1.4}$$

with

$$X_{\overline{z}}=\Big\{\, x\in W_0^{1,q}(\Omega,R), 0\le x(y)\le\overline{z}(y) \quad \text{a.e. on} \quad \Omega \\ \text{and } \mathrm{div}(k(y)|\nabla x(y)|^{q-2}\nabla x(y))\in L^\infty(\Omega,R)\Big\}. \tag{1.5}$$

In [10] (Section 6) we studied the continuous dependence of solutions for (1.1) on functional parameters in the case of $G(y,x,u)=F(y,x)+g(y,u)x$ with F satisfying assumptions **(F1)–(F4)** and $g:\Omega\times R^m\to R$ being a Carathéodory function such that $\Omega\ni y\to g(y,u(y))$ belongs to $L^\infty(\Omega,R_+)$ for each $u\in U$. Now we want to expand these results and consider function G in a general form. Finally, we will apply them to the optimal control problem

$$F(x,u)=\int_0^\pi \widetilde{f}(t,x(t),x'(t),u(t))dt\to\min,$$

$\widetilde{f}:\Omega\times I\times R^m\to R$, with constraints (1.1). Since we are able to work with the solutions of (1.1) from a bounded set we can investigate the existence of optimal solutions under assumptions weaker than in the previous papers, namely, when all conditions associated with the behaviour of $\widetilde{f}(y,\cdot,u)$ concern the interval I only.

2. DEPENDENCE OF SOLUTIONS ON FUNCTIONAL PARAMETERS

Define X as follows

$$X=\Big\{x\in W_0^{1,q}(\Omega,R),\ 0\le x(y)\le M \text{ a.e. on } \Omega \\ \text{and } \mathrm{div}(k|\nabla x|^{q-2}\nabla x)\in L^{q'}(\Omega,R)\Big\}.$$

Theorem 2.1. *Assume conditions* (Ω), (K), (G1)–(G5) *and the pointwise convergence of* $\{u_m\}_{m\in N} \subset U$ *to* $u_0 \in U$ *a.e. in* Ω. *For each* $m \in N$, *let* $x_m \in X$ *denote a solution of (1.1) dependent on* u_m, *namely*

$$-\operatorname{div}\left(k(y)|\nabla x_m(y)|^{q-2}\nabla x_m(y)\right) = G_x(y, x_m(y), u_m(y)) \text{ for a.e. } y \in \Omega. \tag{2.1}$$

Then $\{x_m\}_{m\in N}$ *(or its subsequence) tends weakly to* $x_0 \in X$ *in* $W_0^{1,q}(\Omega, R)$, *where* x_0 *is a solution of the following equation*

$$-\operatorname{div}\left(k(y)|\nabla x(y)|^{q-2}\nabla x(y)\right) = G_x(y, x(y), u_0(y)) \text{ for a.e. } y \in \Omega. \tag{2.2}$$

Proof. We start our proof with the observation that for each $m \in N$, Theorem 4.1 from [10] yields the existence of $x_m \in X_{u_m}$ such that (2.1) holds, where

$$X_{u_m} = \Big\{x \in W_0^{1,q}(\Omega, R),\ 0 \le x(y) \le \overline{z}_{u_m}(y) \text{ a.e. on } \Omega \\ \text{and } \operatorname{div}\left(k|\nabla x|^{q-2}\nabla x\right) \in L^\infty(\Omega, R)\Big\}.$$

It is clear that for each $m \in N$, $X_{u_m} \subset X$ (condition (G2)). Thus, by definition of X, $0 \le x_m(y) \le M$ a.e. $y \in \Omega$ and for each $m \in N$,

$$\int_\Omega |\nabla x_m(y)|^q dy \le \frac{1}{k_0}\int_\Omega \left\langle k(y)|\nabla x_m(y)|^{q-2}\nabla x_m(y), \nabla x_m(y)\right\rangle dy = \\ = \frac{1}{k_0}\int_\Omega G_x(y, x_m(y), u_m(y))x_m(y)dy \le \frac{M}{k_0}\int_\Omega \varphi(y)dy.$$

Thus certain subsequence of $\{x_m\}_{m\in N}$, again denoted by $\{x_m\}_{m\in N}$, tends weakly to $x_0 \in W_0^{1,q}(\Omega, R)$ and, consequently, $x_m \underset{m\to\infty}{\to} x_0$ in $L^q(\Omega, R)$. Let us consider $\{p_m\}_{m\in N} \subset L^{q'}(\Omega, R^n)$ given by

$$p_m(y) = k(y)|\nabla x_m(y)|^{q-2}\nabla x_m(y), \text{ a.e. on } \Omega.$$

From the properties of $\{x_m\}_{m\in N}$ we infer that (up to a subsequence) $p_m \rightharpoonup p_0 \in L^{q'}(\Omega, R^n)$ (weakly), as $m \to \infty$, in $L^{q'}(\Omega, R^n)$. Moreover, assertion (2.1) and assumptions (G2) and (G5) imply the boundedness of the sequence $\{\operatorname{div}_{p_m}\}_{m\in N}$ in $L^{q'}(\Omega, R)$. We will show that $\operatorname{div} p_0$ exists in the weak sense, belongs to $L^{q'}(\Omega, R)$ and $\operatorname{div} p_m \rightharpoonup \operatorname{div} p_0$, as $m \to \infty$, in $L^{q'}(\Omega, R^n)$. To this end, we have to note that

$$\int_\Omega \langle p_0(y), \nabla h(y)\rangle\, dy = \lim_{m\to\infty}\int_\Omega \langle p_m(y), \nabla h(y)\rangle\, dy = \\ = -\lim_{m\to\infty}\int_\Omega \operatorname{div} p_m(y)h(y)dy = \lim_{m\to\infty}\int_\Omega G_x(y, x_m(y), u_m(y))h(y)dy$$

for any $h \in C_0^\infty(\Omega, R)$. (The last equality holds also for all $h \in L^q(\Omega, R)$). On the other hand, the smoothness of $G_x(y, \cdot, \cdot)$ and monotonicity of $G_x(y, \cdot, u)$ for each $u \in U$ and a.a. $y \in \Omega$, assumption (G5) and the facts that $u_m(y) \to u_0(y)$ and $x_m(y) \to x_0(y)$ a.e. in Ω, as $m \to \infty$, imply the following equality

$$\lim_{m\to\infty} \int_\Omega G_x(y, x_m(y), u_m(y))h(y)dy = \int_\Omega G_x(y, x_0(y), u_0(y))h(y)dy$$

for any $h \in L^q(\Omega, R)$. Combining both assertions, for any $h \in C_0^\infty(\Omega, R)$, we derive

$$\int_\Omega \langle p_0(y), \nabla h(y)\rangle\, dy = \int_\Omega G_x(y, x_0(y), u_0(y))h(y)dy \tag{2.3}$$

and

$$\lim_{m\to\infty} \int_\Omega \operatorname{div} p_m(y)h(y)dy = -\int_\Omega G_x(y, x_0(y), u_0(y))h(y)dy \tag{2.4}$$

for each $h \in L^q(\Omega, R)$. (2.3), (2.4) and the Euler–Lagrange lemma lead to $\operatorname{div} p_m \rightharpoonup \operatorname{div} p_0$ in $L^{q'}(\Omega, R^n)$, as $m \to \infty$, and

$$\operatorname{div} p_0(y) = -G_x(y, x_0(y), u_0(y)) \text{ for a.e. } y \in \Omega. \tag{2.5}$$

Moreover, there is

$$0 = \lim_{m\to\infty} \int_\Omega \left\{ \frac{1}{q'(k(y))^{\frac{q'}{q}}} |p_m(y)|^{q'} + \frac{1}{q} k(y)|\nabla x_m(y)|^q - \langle p_m(y), \nabla x_m(y)\rangle \right\} dy \geq$$

$$\geq \int_\Omega \left\{ \frac{1}{q'(k(y))^{\frac{q'}{q}}} |p_0(y)|^{q'} + \frac{1}{q} k(y)|\nabla x_0(y)|^q - \langle p_0(y), \nabla x_0(y)\rangle \right\} dy$$

and finally,

$$p_0(y) = k(y)|\nabla x_0(y)|^{q-2}\nabla x_0(y)) \text{ for a.e. } y \in \Omega. \tag{2.6}$$

Substituting (2.6) into (2.5) yields that the weak limit x_0 of $\{x_m\}_{m\in N}$ in $W_0^{1,q}(\Omega, R)$ belongs to X and is a solution of (2.2). □

3. APPLICATION TO THE OPTIMAL CONTROL PROBLEM

In this section we derive sufficient conditions for the optimal control problem

$$F(x, u) = \int_\Omega \widetilde{f}(y, x(y), u(y))dy \to \min \tag{3.1}$$

subject to

$$-\operatorname{div}\left(k(y)|\nabla x(y)|^{q-2}\nabla x(y)\right) = G_x(y, x(y), u(y)) \text{ for a.e. } y \in \Omega. \tag{3.2}$$

We look for an optimal pair (x_0, u_0) in the set $X\widetilde{U}$ defined as

$$X\widetilde{U} := \Big\{(x,u) \in W_0^{1,q}(\Omega, R) \times \widetilde{U},\ 0 \le x(y) \le M \text{ a.e. on } \Omega \\ \text{and } \operatorname{div}(k|\nabla x|^{q-2}\nabla x) \in L^{q'}(\Omega, R) \text{ and } x \text{ is a solution of 3.2} \\ \text{corresponding to parameter } u\Big\},$$

where $\widetilde{U} := \{\ u : \Omega \to A,\ u$ satisfies the Lipschitz condition with a fixed constant $L\}$, $L > 0$, A is a compact subset of R^m. We assume (Ω), (K), (G1)–(G5) with $U = \widetilde{U}$ and the following additional conditions

(f1) $\widetilde{f} : \Omega \times I \times R^m \to R$ *is measurable with respect to the first variable for all* $(x,u) \in I \times R^m$ *and* $\widetilde{f}(y,\cdot,\cdot)$ *is continuous in* $I \times R^m$ *for a.a.* $y \in \Omega$;

(f2) *there exists* $\alpha \in L^1(\Omega, R_+)$ *such that for all* $u \in \widetilde{U}$ *and* $x \in I$

$$|\widetilde{f}(y, x, u(y))| \le \alpha(y)$$

a.e. in Ω.

Theorem 3.1. *Under the above assumptions, there exists* $(x_0, u_0) \in X\widetilde{U}$ *such that*

$$F(x_0, u_0) = \min_{(x,u)\in X\widetilde{U}} F(x,u).$$

Proof. Let $\{(x_m, u_m)\}_{m\in N} \subset X\widetilde{U}$ be a minimizing sequence of $F : X\widetilde{U} \to R$. Since $\{u_m\}_{m\in N}$ is bounded and equicontinuous in Ω, the Arzela–Ascoli theorem guarantees the existence of a subsequence, still denoted by $\{u_m\}_{m\in N}$, which converges uniformly to $u_0 \in \widetilde{U}$. Thus, by Theorem 2.1, we state that a subsequence of $\{x_m\}_{m\in N}$, again denoted by $\{x_m\}_{m\in N}$, converges weakly to $x_0 \in X$ in $W_0^{1,q}(\Omega, R)$, where x_0 is a solution of 3.2 corresponding to u_0. Taking into account the pointwise convergence of $\{(x_m, u_m)\}_{m\in N}$ to $(x_0, u_0) \in X\widetilde{U}$ a.e. in Ω and assumption (f1), we obtain

$$\widetilde{f}(y, x_m(y), u_m(y)) \underset{m\to\infty}{\to} \widetilde{f}(y, x_0(y), u_0(y))$$

a.e. in Ω and, by (f2)

$$|\widetilde{f}(y, x_m(y), u_m(y))| \le \alpha(y)$$

a.e. in Ω. Therefore, the Lebesgue dominated convergence theorem yields

$$\int_\Omega \widetilde{f}(y, x_m(y), u_m(y))dy \underset{m\to\infty}{\to} \int_\Omega \widetilde{f}(y, x_0(y), u_0(y))dy.$$

Finally,

$$\min_{(x,u)\in X\widetilde{U}} F(x,u) = \liminf_{m\to\infty} F(x_m, u_m) = F(x_0, u_0).$$

□

REFERENCES

[1] Idczak D., *Optimal Control of a Coercive Dirichlet Problem*, SIAM J. Control Optim. **36** (1998) 4, 1250–1267.

[2] Idczak D., *Stability in Semilinear Problems*, J. Differential Equations **162** (2000) 1, 64–90.

[3] Idczak D., Majewski M., Walczak S., *Stability analysis of one and two-dimensional continuous systems with parameters*, Proceedings of the Fourteenth International Symposium of Mathematical Theory of Networks and Systems MTNS 2000, June 19–23, 2000 Perpignan, France.

[4] Idczak D., Majewski M., Walczak S., *Stability of solutions to an optimal control problem for a continuous Fornasini-Marchesini system*, The Second International Workshop on Multidimensional (nD) Systems (Czocha Castle, 2000), 201–208, Tech. Univ. Press, Zielona Góra, 2000.

[5] Idczak D., Majewski M., Walczak S., *N-dimensional continuous systems with the Darboux-Goursat and Dirichlet boundary data*, Proceedings of the 9th IEEE International Conference on Electronics, Circuits and Systems. September 15–18, 2002, Dubrovnik, Croatia.

[6] Ledzewicz U., Schättler H., Walczak S., *Stability of elliptic optimal control problems*, Comput. Math. Appl. **41** (2001) 10–11, 1245–1256.

[7] Mawhin J., *Problèmes de Dirichlet Variationnels Non Linéares*, Les Presses de l'Université dr Montréal, 1987.

[8] Mawhin J., Willem M., *Critical point theory and Hamiltonian systems*, New York 1989.

[9] Nowakowski A., Rogowski A., *Dependence on Parameters for the Dirichlet problem with Superlinear Nonlinearities*, Topological Methods in Nonlinear Analysis, 16 (2000) 1, 145–160.

[10] Orpel A., *On the existence of positive solutions and their continuous dependence on functional parameters for some class of elliptic problems,* J. Differential Equations, **204** (2004), pp. 247–264

[11] Walczak S., *On the continuous dependence on parameters of solutions of the Dirichlet problem. Part I Coercive case, Part II The case of saddle points*, Bull. Classe Sci. l'Acad. Royale Belgique, **7–12** (1995), 263–273.

[12] Walczak S., *Continuous dependence on parameters and boundary data for nonlinear P.D.E., coercive case*, Differential and Integral Equations **11** (1998), 35–46.

[13] Walczak S., *Superlinear variational and boundary value problems with parameters*, Nonlinear Anal. 43 (2001) 2. Ser. A: Theory Methods 183–198.

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Received: September 9, 2005.