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**CONSTRUCTION OF AN INTEGRAL MANIFOLD
FOR LINEAR DIFFERENTIAL-DIFFERENCE
EQUATIONS**

Abstract. In this paper we establish sufficient conditions for the existence of an asymptotic integral manifold of solutions of a linear system of differential-difference equations with a small parameter. This integral manifold is described by a linear system of differential equations without deviating argument.

Keywords: system with deviating argument, integral manifold of solutions, fundamental matrix, exponential dichotomy.

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1. INTRODUCTION AND PRELIMINARIES

The theory of linear differential equations with deviating argument is well established. There are numerous important papers on the subject. One of the classical works here is [2]. For a recent account of the theory, we refer the reader to [1] and the references given there.

We consider the linear system of differential equations with deviating argument

$$\begin{aligned} \frac{dX(t)}{dt} &= A(t)X(t) + \mu \sum_{k=1}^n (A_k(t)X(t + \tau_k(t)) + B_k(t)Y(t + \tau_k(t))) \\ \frac{dY(t)}{dt} &= B(t)Y(t) + \mu \sum_{k=1}^n (C_k(t)X(t + \tau_k(t)) + D_k(t)Y(t + \tau_k(t))), \end{aligned} \tag{1}$$

where $t \geq 0$, μ is a small parameter, $\dim X(t) = p$, $\dim Y(t) = q$,

$$|\tau_k(t)| \leq \tau \quad (k = 1, \dots, n; t \geq 0). \tag{2}$$

We assume that the matrices in System (1) are bounded:

$$\begin{aligned} \|A(t)\| \leq \alpha_0, \quad \sum_{k=1}^n \|A_k(t)\| \leq \alpha, \\ \sum_{k=1}^n \|B_k(t)\| \leq \alpha, \quad \sum_{k=1}^n \|C_k(t)\| \leq \alpha, \quad \sum_{k=1}^n \|D_k(t)\| \leq \alpha, \end{aligned} \quad (3)$$

If the parameter $\mu = 0$, System (1) decouples into two independent subsystems. Let the system

$$\frac{dX(t)}{dt} = A(t)X(t)$$

have a fundamental matrix of solutions $P(t, s)$ normalized at $t = s$ and satisfy the condition

$$\|P(t, s)\| \leq c_1 e^{\varepsilon|t-s|} \quad (c_1 \geq 1, \varepsilon > 0) \quad (4)$$

Let the system

$$\frac{dY(t)}{dt} = B(t)X(t)$$

have a fundamental matrix of solutions $Q(t, s)$ normalized at $t = s$ and satisfy the condition

$$\|Q(t, s)\| \leq c_2 e^{\lambda|t-s|} \quad (c_2 \geq 1, \lambda > \varepsilon) \quad (5)$$

Thus, for $\mu = 0$, System (1) possesses an exponential dichotomy with an exponent σ , where $-\lambda < \sigma < -\varepsilon$ (see [3]).

We will construct an integral manifold of solutions of System (1) in the form of ([3,4])

$$\frac{dX(t)}{dt} = H(t, \mu)X(t), \quad Y(t) = K(t, \mu)X(t). \quad (6)$$

Let a fundamental matrix of solutions of System (6) be denoted by $N(t, s, \mu)$, it follows that

$$X(t) = N(t, s, \mu)X(s). \quad (7)$$

From this we obtain

$$\begin{aligned} X(t + \tau_k(t)) &= N(t + \tau_k(t), t, \mu)X(t), \\ Y(t + \tau_k(t)) &= K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu)X(t). \end{aligned} \quad (8)$$

If the solutions of System (6) satisfy (1), then

$$\begin{aligned} H(t, \mu) &= A(t) + \mu \sum_{k=1}^n (A_k(t)N(t + \tau_k(t), t, \mu) + \\ &+ B_k(t)K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu)) \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{\partial K(t, \mu)}{\partial t} + K(t, \mu)(A(t) + \mu \sum_{k=1}^n (A_k(t)N(t + \tau_k(t), t, \mu) + \\ + B_k(t)K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu))) = \\ = B(t)K(t, \mu) + \mu \sum_{k=1}^n (C_k(t)N(t + \tau_k(t), t, \mu) + \\ + D_k(t)K(t + \tau_k(t), \mu)N(t + \tau_k(t), t, \mu)). \end{aligned} \tag{10}$$

We now proceed to considering the auxiliary matrix differential equation

$$\frac{\partial K(t, \mu)}{\partial t} = B(t)K(t, \mu) - K(t, \mu)A(t) + F(t), \tag{11}$$

where $\|F(t)\| \leq b$ for $t \geq 0$. It is easy to check that the matrix

$$K(t, \mu) = \int_0^t Q(t, s)F(s)P(s, t)ds \tag{12}$$

is a solution of (11). In addition, under conditions (4) and (5), we obtain

$$\|K(t, \mu)\| = c_1 c_2 \int_0^t e^{-\lambda(t-s)} \sup \|F(s)\| e^{\varepsilon(t-s)} \leq \frac{c_1 c_2}{\lambda - \varepsilon} \sup \|F(s)\|, \quad t \geq 0. \tag{13}$$

Application of (11) and (12) enables us to write System (10) in the form

$$\begin{aligned} K(t, \mu) = \mu \int_0^t Q(t, s) \sum_{k=1}^n (C_k(s) + D_k(s)K(s + \tau_k(s), \mu) - \\ - K(s, \mu)A_k(s) - K(s, \mu)B_k(s)K(s + \tau_k(s), \mu))N(s + \tau_k(s), t, \mu)P(s, t)ds. \end{aligned} \tag{14}$$

Our purpose here is to give a proof that an integral manifold of solutions of System (1) exists in form (6).

2. SUCCESSIVE APPROXIMATIONS

System (9),(14) of matrix equations defines the matrices $H(t, \mu)$, $K(t, \mu)$ and it can be solved by the method of successive approximations. We start this process by letting $H_0(t, \mu) = 0$, $K_0(t, \mu) = 0$ and

$$\begin{aligned} H_{j+1}(t, \mu) = A(t) + \mu \sum_{k=1}^n (A_k(t)N_j(t + \tau_k(t), t, \mu) + \\ + B_k(t)K_j(t + \tau_k(t), \mu)N_j(t + \tau_k(t), t, \mu)), \\ K_{j+1}(t, \mu) = \mu \int_0^t Q(t, s) \sum_{k=1}^n (C_k(s) + D_k(s)K_j(s + \tau_k(s), \mu) - \\ - K_j(s, \mu)A_k(s) - K_j(s, \mu)B_k(s)K_j(s + \tau_k(s), \mu)) \times \\ \times N_j(s + \tau_k(s), t, \mu)P(s, t)ds, \end{aligned} \tag{15}$$

for $j=0,1,2,\dots$

Let $N_j(t, s, \mu)$ be a fundamental matrix of solutions of the system

$$\frac{dX(t)}{dt} = H_j(t, \mu)X(t) \quad (j = 0, 1, 2, \dots). \quad (16)$$

Supposing that the inequalities

$$\|H_j(t, \mu)\| \leq h_j, \quad \|K_j(t, \mu)\| \leq k_j,$$

take place, we find the estimates

$$\|N_j(t, s, \mu)\| \leq e^{h_j|t-s|}, \quad \|N_j(t + \tau_k(t), t, \mu)\| \leq e^{\tau h_j}.$$

And owing to System (15) it follows that

$$\begin{aligned} h_{j+1} &\leq \alpha_0 + |\mu| \alpha(1 + k_j)e^{\tau h_j}, \\ k_{j+1} &\leq |\mu| \alpha \beta(1 + k_j)^2 e^{\tau h_j}, \end{aligned} \quad (17)$$

where

$$\beta \equiv \frac{c_1 c_2}{\lambda - \varepsilon}.$$

In order for the sequences $\{h_j\}$ and $\{k_j\}$ to be bounded from below, i.e.

$$h_j \geq h > 0, \quad k_j \geq k > 0, \quad j = 0, 1, 2, \dots$$

it is necessary and sufficient that the system of nonlinear equations

$$\alpha_0 + |\mu| \alpha(1 + k)e^{\tau h} = h, \quad |\mu| \alpha \beta(1 + k)^2 e^{\tau h} = k \quad (18)$$

has a positive solution.

We need to find the largest value $\mu = \mu_0$ for which System (18) has multiple solutions. In this connection h and k achieve the maximum. We obtain

$$\begin{aligned} \mu_0 &= \frac{2(\tau + \sqrt{\tau^2 + 4\beta^2})}{\alpha(\tau + 2\beta + \sqrt{\tau^2 + 4\beta^2})} \exp \left\{ -\tau \left(\alpha + \frac{2}{\tau + 2\beta + \sqrt{\tau^2 + 4\beta^2}} \right) \right\} \\ h &= \alpha_0 + \frac{k}{\beta(1 + k)}, \quad k = \frac{2\beta}{\tau + \sqrt{\tau^2 + 4\beta^2}}. \end{aligned} \quad (19)$$

It follows that for $|\mu| \leq \mu_0$ the sequences of matrices $H_j(t, \mu)$ and $K_j(t, \mu)$ ($j=0,1,2,\dots$) are well defined and bounded in norm for $t \geq 0$.

3. CONVERGENCE

Let us turn to prove that the sequences of matrices $H_j(t, \mu)$ and $K_j(t, \mu)$ ($j = 0, 1, 2, \dots$) converge uniformly in t , for $t \geq 0$.

Let us introduce the notation

$$\|H_j(t, \mu) - H_{j-1}(t, \mu)\| \leq u_j, \quad \|K_j(t, \mu) - K_{j-1}(t, \mu)\| \leq v_j \quad (j = 1, 2, 3, \dots).$$

Then from System (15), for $\mu = \mu_0$ there follows:

$$\begin{aligned} u_{j+1} &\leq \ell(1+k)\tau u_j + \ell v_j, \\ v_{j+1} &\leq \ell\beta\tau(1+k)^2 u_j + 2\ell\beta(1+k)v_j, \end{aligned} \quad (20)$$

where

$$\ell \equiv \frac{1}{2\beta(1+k) + \tau}.$$

The matrix of coefficients of the expression on the right hand side of (20)

$$R(\mu) = \begin{pmatrix} \ell\tau(1+k) & \ell \\ \ell\beta\tau(1+k)^2 & 2\ell\beta \end{pmatrix}$$

has the largest eigenvalue (in terms of the absolute value)

$$\rho_{max} = \frac{(2\beta + \tau + \sqrt{\tau^2 + 4\beta^2})^2}{2(2\beta + \sqrt{\tau^2 + 4\beta^2})(\tau + \sqrt{\tau^2 + 4\beta^2})} \equiv 1.$$

For $|\mu| < \mu_0$, the absolute values of eigenvalues of the matrix $R(\mu)$ are less than 1 and, therefore, the terms of the series

$$H(t, \mu) = \sum_{j=0}^{\infty} (H_{j+1}(t, \mu) - H_j(t, \mu)), \quad (21)$$

$$K(t, \mu) = \sum_{j=0}^{\infty} (K_{j+1}(t, \mu) - K_j(t, \mu)). \quad (22)$$

are bounded from above by the terms of the decreasing geometric progression. It follows that series (21) and (22) converge uniformly for $|\mu| \leq \mu_1 < \mu_0$.

We can now formulate our main results.

Theorem 3.1. *Let System (1) of differential-difference equations be dichotomic for $\mu = 0$ and $t \geq 0$, caused by inequalities (4) and (5). If conditions (2) and (3) hold, then System (1) has an integral manifold of form (6), where the matrices H and K depend analytically on μ , provided $|\mu| < \mu_0$, where μ_0 is given by (19).*

The construction of integral manifolds in form (6) for systems of differential-difference equations can be used in the investigation of qualitative properties of this systems.

Example 3.2. *For the delay system*

$$\begin{aligned} \frac{dx(t)}{dt} &= \mu \cos tx(t) + \mu ay(t - \tau), \\ \frac{dy(t)}{dt} &= -y(t) + \mu bx(t - \tau), \end{aligned} \quad (23)$$

we are interested in the construction of an integral manifold of solutions in form (6)

$$\frac{dx(t)}{dt} = h(t, \mu)x(t), \quad y(t) = k(t, \mu)x(t).$$

The functions k and h are defined by the system

$$\begin{aligned} h(t, \mu) &= \mu \cos t + \mu a k(t - \tau, \mu) \exp \int_t^{t-\tau} h(r, \mu) dr, \\ k(t, \mu) &= \mu \int_0^t (e^{-(t-s)}(b - k(s, \mu)) \cos s - \\ &\quad - a k(s, \mu) k(s - \tau, \mu)) \exp \int_s^{s-\tau} h(r, \mu) dr) ds. \end{aligned} \quad (24)$$

Solving System (24) by the method of successive approximations, we find

$$h(t, \mu) = \mu \cos t + \mu^2 ab + O(\mu^3).$$

The zero solution of the differential equation

$$\frac{dx(t)}{dt} = (\mu \cos t + \mu^2 ab + O(\mu^3))x(t)$$

and also the zero solution of System (23) are asymptotically stable for a sufficiently small values of $\mu > 0$ and $ab < 0$, and unstable if $ab > 0$.

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