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**ON PERIODIC AND STABLE SOLUTIONS
OF THE LASOTA EQUATION
IN DIFFERENT PHASE SPACES**

Abstract. We study properties of the Lasota partial differential equation in two different spaces: V_α (Hölder continuous functions) and L^p . The aim of this paper is to generalize the results of [1].

Keywords: partial differential equations, periodic solutions, stable solutions.

Mathematics Subject Classification: 35B10, 35B35, 37C75, 47D06.

1. INTRODUCTION

We consider the partial differential equation

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = \lambda(x)u, \quad t \geq 0, \quad 0 \leq x \leq 1 \quad (1.1)$$

with the initial condition

$$u(0, x) = v(x), \quad 0 \leq x \leq 1, \quad (1.2)$$

where v belongs to some normed vector space V of functions defined on $[0, 1]$ and $\lambda : [0, 1] \rightarrow \mathbb{R}$ is a given continuous function. Let a semidynamical system

$$T_t : V \rightarrow V$$

be given by the formula

$$(T_t v)(x) = u(t, x),$$

where u is the solution of (1.1), (1.2). It is clear that this unique solution is given by the formula

$$(T_t v)(x) = u(t, x) = e^{g(x)} e^{-g(xe^{-t})} v(xe^{-t}), \quad x \in [0, 1], \quad (1.3)$$

where

$$g(x) = - \int_x^1 \frac{\lambda(s)}{s} ds$$

with the condition

$$\int_0^1 \frac{\lambda(s)}{s} ds = \infty. \quad (1.4)$$

We wish to investigate some properties of system (1.3): periodic solutions, strong and exponential stability.

Definition 1.1. A function $v_0 \in V$ is a periodic point of the semigroup $(T_t)_{t \geq 0}$, with a period $t_0 \geq 0$ iff $T_{t_0} v_0 = v_0$. A number $t_0 > 0$, is called a principal period of a periodic point v_0 iff the set of all periods of v_0 is equal to $\mathbb{N}t_0$.

Definition 1.2. The semigroup $(T_t)_{t \geq 0}$ is strongly stable in V iff for every $v \in V$,

$$\lim_{t \rightarrow \infty} T_t v = 0 \quad \text{in } V.$$

Definition 1.3. The semigroup $(T_t)_{t \geq 0}$ is exponentially stable iff there exist $D < \infty$ and $\omega > 0$ such that

$$\|T_t\| \leq D e^{-\omega t}, \quad \text{for } t \geq 0.$$

The problem of the chaotic behaviour of a partial differential equation was considered by Lasota [5], Rudnicki [8], Łoskot [7] and Szarek [6]. In the papers [1–4] there were described properties of the partial differential equation, analogical to (1.1), but with a constant function λ :

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = \gamma u, \quad t \geq 0, \quad 0 \leq x \leq 1 \quad (1.5)$$

and with the initial condition

$$u(0, x) = v(x), \quad 0 \leq x \leq 1. \quad (1.6)$$

This work has been intended as an attempt at generalizing the results of [1]. In [1] there was described the chaotic and stability behaviour of the suitable semidynamical system

$$(\tilde{T}_t v)(x) = \tilde{u}(t, x) = e^{\gamma t} v(xe^{-t}), \quad x \in [0, 1] \quad (1.7)$$

in different phase spaces V . All properties depended on the value γ . We are interested in finding a connection between this two equations. It is easy to check that if u and \tilde{u} are the solutions of equation (1.1) and (1.5), respectively, then

$$\tilde{u}(t, x) = \kappa(x) u(t, x), \quad (1.8)$$

where

$$\kappa(x) = e^{\int_0^x \frac{\lambda(0) - \lambda(s)}{s} ds} \quad \text{and} \quad \gamma = \lambda(0). \quad (1.9)$$

Hence the diagram

$$\begin{array}{ccc}
 V & \xrightarrow{T_t} & V \\
 m_\kappa \downarrow & & \downarrow m_\kappa \\
 V & \xrightarrow{\tilde{T}_t} & V
 \end{array}$$

This substitution will be a useful tool. It will be used in the proofs of theorems on chaos and stability of system (1.3) in the spaces V_α and L^p .

2. PROPERTIES OF THE DYNAMICAL SYSTEM $(T_t)_{t \geq 0}$ IN THE SPACE V_α

Let v be a continuous function on $[0, 1]$ such that $v(0) = 0$. For every interval $A \subset [0, 1]$ and for every $\alpha \in (0, 1]$, define

$$H_{A,\alpha}(v) = \sup_{x,y \in A, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\alpha}.$$

A function v for which $H_{A,\alpha}(v) < \infty$ is called a Hölder continuous on the interval A with exponent α . Write

$$H_\alpha = H_{[0,1],\alpha}.$$

Definition 2.1. Denote by V_α the space of all Hölder continuous functions v on $[0, 1]$ with exponent α , vanishing at zero and satisfying the following condition

$$\lim_{x \rightarrow 0} H_{[0,x],\alpha}(v) = 0.$$

Certain properties of system (1.7) in the space V_α have been established. For $\gamma > \alpha$, there exist periodic solutions of problem (1.5) and the set of all periodic points is dense in V_α . Strong and exponential stability take place, provided that $\gamma \leq \alpha$ and $\gamma < \alpha$, respectively (see [1] for more details).

Theorem 2.2. Let

$$|\lambda(0) - \lambda(x)| \leq Cx, \quad C > 0, \quad x \in [0, 1] \tag{2.1}$$

hold. Then the function $u \in V_\alpha$ if and only if $\tilde{u} \in V_\alpha$.

Proof. The assumption $u \in V_\alpha$ means that u is a Hölder continuous function with exponent α , vanishing at zero and $\lim_{a \rightarrow 0} H_{[0,a],\alpha}(u) = 0$. This gives

$$\begin{aligned}
 H_{[0,a],\alpha}(\tilde{u}) &= \sup_{x,y \in [0,a], x \neq y} \frac{|\tilde{u}(t,x) - \tilde{u}(t,y)|}{|x - y|^\alpha} = \\
 &= \sup_{x,y \in [0,a], x \neq y} \frac{|\kappa(x)u(t,x) - \kappa(y)u(t,y)|}{|x - y|^\alpha} \leq \\
 &\leq \sup_{x \in [0,a]} |\kappa(x)| \cdot \sup_{x,y \in [0,a], x \neq y} \frac{|u(t,x) - u(t,y)|}{|x - y|^\alpha} + \\
 &+ \sup_{y \in [0,a]} |u(t,y)| \cdot \sup_{x,y \in [0,a], x \neq y} \frac{|\kappa(x) - \kappa(y)|}{|x - y|^\alpha}.
 \end{aligned}$$

Using (1.9) and (2.1), we obtain

$$|\kappa(x)| \leq e^{\int_0^x \frac{|\lambda(0)-\lambda(s)|}{s} ds} \leq e^{\int_0^x C ds} = e^{Cx}.$$

It remains to show that the function κ is Hölder continuous. If $\kappa^{\frac{1}{\alpha}}$ appears the Lipschitz function, it will mean that κ is Hölder continuous with exponent α . For $x \in [0, a]$

$$\begin{aligned} |(\kappa^{\frac{1}{\alpha}}(x))'| &= \left| e^{\frac{1}{\alpha} \int_0^x \frac{\lambda(0)-\lambda(s)}{s} ds} \cdot \frac{1}{\alpha} \cdot \frac{\lambda(0) - \lambda(x)}{x} \right| \leq \left| e^{\frac{1}{\alpha} \int_0^x C ds} \cdot \frac{C}{\alpha} \right| = \\ &= \left| e^{\frac{Cx}{\alpha}} \cdot \frac{C}{\alpha} \right| \leq e^{\frac{Ca}{\alpha}} \cdot \frac{C}{\alpha}. \end{aligned}$$

Summarizing,

$$H_{[0,a],\alpha}(\tilde{u}) \leq e^{Ca} \left(H_{[0,a],\alpha}(u) + \sup_{y \in [0,a]} |u(t, y)| \cdot \left(\frac{C}{\alpha} \right)^\alpha \right).$$

Therefore, $\lim_{a \rightarrow 0} H_{[0,a],\alpha}(\tilde{u}) = 0$ and finally $\tilde{u} \in V_\alpha$. The rest of the proof runs as before. We can draw the same conclusion for the function u , assuming that $\tilde{u} \in V_\alpha$. \square

Assumption (2.1) will be needed throughout this section and the above theorem will be crucial for next results.

Theorem 2.3. *If $\lambda(0) > \alpha \in (0, 1]$, then for any t_0 there exists such $v_0 \in V_\alpha$ that*

$$T_{t_0} v_0 = v_0 \tag{2.2}$$

and

$$T_t v_0 = v_0 \text{ if and only if } t = nt_0 \text{ for some positive integer } n. \tag{2.3}$$

Proof. Let w be an arbitrary Hölder continuous function with the exponent α defined on the interval $[e^{-t_0}, 1]$ and satisfying the following conditions:

$$e^{-g(e^{-t_0})} w(e^{-t_0}) = w(1), \tag{2.4}$$

$$\forall t \in (0, t_0) \quad e^{-g(e^{-t})} w(e^{-t}) \neq w(1). \tag{2.5}$$

Consider the following function v on the interval $(0, 1]$:

$$v(x) = e^{g(x)} e^{-g(xe^{nt_0})} w(xe^{nt_0}) \quad \text{for } x \in [e^{-(n+1)t_0}, e^{-nt_0}].$$

The function v is defined on the whole interval $(0, 1] = \bigcup_{n=0}^{\infty} (e^{-(n+1)t_0}, e^{-nt_0}]$ and come into being by squeezing the graph of the function w into each interval $(e^{-(n+1)t_0}, e^{-nt_0}]$.

By the assumption of the continuity of w on $[e^{-t_0}, 1]$, its boundedness follows, i.e.,

there is $M > 0$ such that $|w(x)| \leq M$ for each $x \in [e^{-t_0}, 1]$. By the above, for $x \in [e^{-(n+1)t_0}, e^{-nt_0}]$, the following estimate holds:

$$|v(x)| = e^{g(x)} e^{-g(xe^{nt_0})} |w(xe^{nt_0})| \leq M e^{g(x)} \cdot \sup_{x \in [e^{-t_0}, 1]} e^{-g(x)} \leq M_1 e^{g(x)},$$

where $M_1 = M \cdot \sup_{x \in [e^{-t_0}, 1]} e^{-g(x)}$. From assumption (1.4), $\lim_{x \rightarrow 0} e^{g(x)} = 0$, so we deduce that $v(0) = 0$. We obtain the continuous function v defined on the whole interval $[0, 1]$. Property (2.2) follows from (2.4), while property (2.3) from (2.5).

The assumption $\lambda(0) > \alpha$ yields $\tilde{v} \in V_\alpha$, see [1] for details. Under Theorem 2.2, we see at once that $v \in V_\alpha$. \square

Theorem 2.4. *For $\lambda(0) > \alpha$, the set of periodic points of (1.1) is dense in V_α .*

Proof. Let v be an arbitrary function belonging to V_α . Define

$$w(x) = e^{g(x)} \left(e^{-g(xe^{nt_0})} v(xe^{nt_0}) - \sum_{k=n+1}^{\infty} m e^{-g(xe^{kt_0})} \right), \tag{2.6}$$

where $m = e^{-g(e^{-t_0})} v(e^{-t_0}) - v(1)$ and $x \in [e^{-(n+1)t_0}, e^{-nt_0}]$. To show the correctness of this definition, it is sufficient to make the following observation

$$\begin{aligned} e^{g(e^{-(n+1)t_0})} \left(e^{-g(e^{-(n+1)t_0} e^{nt_0})} v(e^{-(n+1)t_0} e^{nt_0}) - \sum_{k=n+1}^{\infty} m e^{-g(e^{-(n+1)t_0} e^{kt_0})} \right) &= \\ &= e^{g(e^{-(n+1)t_0})} \left(v(1) + m - \sum_{k=n+1}^{\infty} m e^{-g(e^{-(n+1)t_0} e^{kt_0})} \right) = \\ &= e^{g(e^{-(n+1)t_0})} \left(e^{-g(e^{-(n+1)t_0} e^{(n+1)t_0})} v(e^{-(n+1)t_0} e^{(n+1)t_0}) - \right. \\ &\quad \left. - \sum_{k=n+2}^{\infty} m e^{-g(e^{-(n+1)t_0} e^{kt_0})} \right). \end{aligned}$$

The function w is continuous and vanishes at 0, which is a consequence of (2.6). Let $\epsilon > 0$. Since $v \in V_\alpha$ and $w \in V_\alpha$, there exists such t_0 that $H_{[0, e^{-t_0}], \alpha}(v) < \frac{\epsilon}{4}$ and $H_{[0, e^{-t_0}], \alpha}(w) < \frac{\epsilon}{4}$. From (1.8) we know that $w(x) = \frac{\tilde{w}(x)}{\kappa(x)}$, where \tilde{w} is the periodic solution of (1.5). For $\lambda(0) > \alpha$ the set of periodic points of (1.5) is dense in V_α (see [1]), so $H_\alpha(v - \tilde{w}) < \frac{\epsilon}{4}$ and $H_\alpha(w - \tilde{w}) < \frac{\epsilon}{4}$.

Thus

$$\begin{aligned} H_\alpha(v - w) &\leq H_{[e^{-t_0}, 1], \alpha}(v - w) + H_{[0, e^{-t_0}], \alpha}(v - w) \leq \\ &\leq H_\alpha(v - \tilde{w}) + H_\alpha(\tilde{w} - w) + H_{[0, e^{-t_0}], \alpha}(v) + H_{[0, e^{-t_0}], \alpha}(w) < \\ &< \epsilon. \end{aligned}$$

This completes the proof. \square

Theorem 2.5. *If $\lambda(0) \leq \alpha$ and $v \in V_\alpha$, then*

$$\lim_{t \rightarrow \infty} H_\alpha(T_t v) = 0.$$

Moreover, if $\lambda(0) < \alpha$, then the semigroup $(T_t)_{t \geq 0}$ is exponentially stable.

Proof. Take any $v \in V_\alpha$. Using Theorem 2.2 and proceeding analogously as in its proof, we compute

$$\begin{aligned} H_\alpha(T_t v) &= \sup_{x,y \in [0,1], x \neq y} \frac{|u(t,x) - u(t,y)|}{|x - y|^\alpha} = \sup_{x,y \in [0,1], x \neq y} \frac{\left| \frac{\tilde{u}(t,x)}{\kappa(x)} - \frac{\tilde{u}(t,y)}{\kappa(y)} \right|}{|x - y|^\alpha} \leq \\ &\leq \sup_{x \in [0,1]} \left| \frac{1}{\kappa(x)} \right| \cdot \sup_{x,y \in [0,1], x \neq y} \frac{|\tilde{u}(t,x) - \tilde{u}(t,y)|}{|x - y|^\alpha} + \\ &\quad + \sup_{y \in [0,1]} |\tilde{u}(t,y)| \cdot \sup_{x,y \in [0,1], x \neq y} \frac{\left| \frac{1}{\kappa(x)} - \frac{1}{\kappa(y)} \right|}{|x - y|^\alpha} \leq \\ &\leq e^C \cdot \left(H_\alpha(T_t \tilde{v}) + \sup_{y \in [0,1]} |\tilde{u}(t,y)| \cdot \left(\frac{C}{\alpha} \right)^\alpha \right). \end{aligned}$$

We know that $T_t \tilde{v} \rightarrow 0$ in V_α for every $\tilde{v} \in V_\alpha$. The claim $\lim_{t \rightarrow \infty} H_\alpha(T_t \tilde{v}) = 0$ is based on the results of paper [1]. From the same source, we have derived the estimate $\|T_t \tilde{v}\| \leq e^{(\gamma - \alpha)t} \|\tilde{v}\|$.

$$|\tilde{u}(t,y)| = |(T_t \tilde{v})(y)| = |(T_t \tilde{v})(y) - (T_t \tilde{v})(0)| \leq H_\alpha(T_t \tilde{v}) y^\alpha \leq H_\alpha(T_t \tilde{v}),$$

hence, since $\lambda(0) < \alpha$, there follows the exponential stability of the semigroup $(T_t)_{t \geq 0}$ with $D = e^C \left(1 + \left(\frac{C}{\alpha} \right)^\alpha \right)$ and $\omega = \alpha - \lambda(0)$. □

3. CHAOS AND STABILITY OF THE SYSTEM $(T_t)_{t \geq 0}$ IN THE SPACE L^p

Theorem 3.1. *Assume that*

$$\exists C, q > 0 \quad \forall x \in [0, 1] \quad |\lambda(0) - \lambda(x)| \leq Cx^q. \tag{3.1}$$

The function u belongs to the space L^p if and only if $\tilde{u} \in L^p$.

Proof. Using (3.1), substitution (1.8) discussed in Section 1 and assuming that $u \in L^p$, we obtain

$$\begin{aligned} \|\tilde{u}(t,x)\|^p &= \int_0^1 |\kappa(x)u(t,x)|^p dx \leq \int_0^1 e^{p \int_0^x \frac{|\lambda(0) - \lambda(s)|}{s} ds} |u(t,x)|^p dx \leq \\ &\leq \int_0^1 e^{\frac{Cp}{q} x^q} |u(t,x)|^p dx \leq e^{\frac{Cp}{q}} \int_0^1 |u(t,x)|^p dx = \\ &= e^{\frac{Cp}{q}} \|u(t,x)\|^p < \infty \end{aligned}$$

In the same manner, we can establish the inverse implication. □

The above theorem will be significant in obtaining the next one. It enables the results of [1] to be used and generalized. From now on, we assume (3.1).

Theorem 3.2. *For $\lambda(0) > -\frac{1}{p}$ there exists a periodic solution of (1.1).*

Proof. For any t_0 , define the following function v :

$$v(x) = e^{g(x)} e^{-g(xe^{nt_0})} w(xe^{nt_0}), \quad x \in [e^{-(n+1)t_0}, e^{-nt_0}], \tag{3.2}$$

$$v(0) = 0,$$

where w is an arbitrary function from the space L^p . We have shown that such function v is a periodic solution, so it is sufficient to prove that $v \in L^p$. The function v is the solution of equation (1.1), so we can express it using the function \tilde{v} , the solution of (1.5), $v(x) = \frac{\tilde{v}(x)}{\kappa(x)}$. Brzeźniak and one of the authors [1] showed that \tilde{v} is a periodic solution and belongs to L^p when $\gamma > -\frac{1}{p}$. Our assumption and Theorem 3.1 guarantee the same conclusion for the function v . \square

Theorem 3.3. *If $\lambda(0) > -\frac{1}{p}$, then the set of periodic points is dense in L^p .*

Proof. Let $w \in L^p$ and $\epsilon > 0$. Fix t_0 such that

$$\left[\int_0^{e^{-t_0}} |w(x)|^p dx \right]^{\frac{1}{p}} < \frac{\epsilon}{2}$$

and

$$e^{\frac{C}{q}} \|\tilde{v}\| < \frac{\epsilon}{2}, \quad C > 0$$

where \tilde{v} is a periodic solution of (1.5).

The function v is defined by formula (3.2). The function v belongs to the set of periodic points due to Theorem 3.2. It is sufficient to estimate $\|v - w\|$. Since $v(x) = w(x)$ for $x \in [e^{-t_0}, 1]$, it is obvious that $\|v - w\| = \|(v - w)1_{[0, e^{-t_0}]}\|$, where $1_{[0, e^{-t_0}]}$ denotes the indicator of the set $[0, e^{-t_0}]$. Applying the estimate from Theorem 3.1 and substitution (1.8), we can assert that

$$\|v - w\| \leq \|v1_{[0, e^{-t_0}]}\| + \|w1_{[0, e^{-t_0}]}\| < \epsilon.$$

\square

Theorem 3.4. *If $\lambda(0) \leq -\frac{1}{p}$, and $v \in L^p$ then*

$$\lim_{t \rightarrow \infty} \|T_t v\| = 0.$$

Moreover, for $\lambda(0) < -\frac{1}{p}$ the semigroup $(T_t)_{t \geq 0}$ is exponentially stable on L^p .

Proof. Let $v \in L^p$ be an arbitrary function.

$$\begin{aligned} \|T_t v\|^p &= \int_0^1 |u(t, x)|^p dx = \int_0^1 \left| \frac{\tilde{u}(t, x)}{\kappa(x)} \right|^p dx = \\ &= \int_0^1 \left| \frac{1}{\kappa(x)} (T_t \tilde{v})(x) \right|^p dx \leq e^{\frac{Cp}{q}} \|T_t \tilde{v}\|^p, \end{aligned}$$

where $C > 0$. We know that $\|T_t \tilde{v}\| \rightarrow 0$, as $t \rightarrow \infty$ (see [1]), which proves the first part of the Theorem. From [1] we know that $\|T_t \tilde{v}\|^p \leq e^{(\gamma p + 1)t} \|\tilde{v}\|^p$. It gives the exponential stability of the semigroup $(T_t)_{t \geq 0}$ with $D = e^{\frac{C}{q}}$ and $\omega = -\frac{1}{p}(\lambda(0)p + 1)$. \square

Acknowledgements

The second author acknowledges the support from Białystok Technical University (Grant No. S/WI/1/08).

REFERENCES

- [1] Z. Brzeźniak, A.L. Dawidowicz, *On the periodic solution to the von Foerster-Lasota equation*, to appear in Semigroup Forum.
- [2] A.L. Dawidowicz, *On the existence of an invariant measure for a quasi-linear partial differential equation*, Zeszyty Naukowe UJ, Prace Matematyczne 23 (1982), 117–123.
- [3] A.L. Dawidowicz, *On the existence of an invariant measure for the dynamical system generated by partial differential equation*, Ann. Polon. Math. XLI (1983), 129–137.
- [4] A.L. Dawidowicz, N. Haribash, *On the periodic solutions of von Foerster type equation*, Universitatis Iagellonicae Acta Mathematica (1999) 37, 321–324.
- [5] A. Lasota, G. Pianigiani, *Invariant measures on topological spaces*, Boll. Un. Mat. Ital. (5) 15-B (1977), 592–603.
- [6] A. Lasota, T. Szarek, *Dimension of measures invariant with respect to Ważewska partial differential equation*, J. Differential Equations **196** (2004) 2, 448–465.
- [7] K. Łoskot, *Turbulent solutions of first order partial differential equation*, J. Differential Equations **58** (1985) 1, 1–14.
- [8] R. Rudnicki, *Invariant measures for the flow of a first order partial differential equation*, Ergodic Theory and Dynamical Systems, **5** (1985) 3, 437–443.

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Received: March 3, 2008.

Accepted: October 10, 2008.