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UNIQUENESS OF SOLUTIONS OF A GENERALIZED CAUCHY PROBLEM FOR A SYSTEM OF FIRST ORDER PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. The paper is concerned with weak solutions of a generalized Cauchy problem for a nonlinear system of first order differential functional equations. A theorem on the uniqueness of a solution is proved. Nonlinear estimates of the Perron type are assumed. A method of integral functional inequalities is used.

Keywords: functional differential equations, comparison methods, estimates of the Perron type.

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1. INTRODUCTION

Differential inequalities find numerous applications in the theory of first order partial differential equations. Such problems as: estimates of solutions of initial or initial boundary value problems, estimates of the domain of classical or generalized solutions, estimates of the difference between two solutions, criterion of uniqueness, are classical examples, however, not the only ones. The theory of partial differential inequalities has been described extensively in the monographs [1, 2, 5]. Hyperbolic functional differential inequalities generated by initial problems or by mixed problems have been studied in the monograph [3]. In particular, uniqueness results for initial problems on the Haar pyramid with nonlinear estimates of the Perron type were obtained as consequences of suitable comparison theorems for differential functional inequalities. Uniqueness criteria for a classical Cauchy problem and solutions considered on unbounded domains can be found in [3, 4] Chapter 4.

The aim of this paper is to give sufficient conditions for the uniqueness of solutions to a generalized Cauchy problem for a nonlinear system of first order partial differential functional equations. We formulate our functional differential problem. For any metric spaces X and Y, by $C(X,Y)$ we denote the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequality hold between their corresponding components. Let us denote by $\mathbb{N}$ the set of natural numbers. Set

$$E_i = [a_i, c] \times \mathbb{R}^n \text{ for } i = 1, \dots, k \quad \text{and} \quad B = [-b_0, 0] \times [-b, b],$$

where $c > a_i \geq 0$, $b_0 \in \mathbb{R}_+$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n_+$ and $\mathbb{R}_+ = [0, +\infty)$. Let $E = [0, c] \times \mathbb{R}^n$. Suppose that $\psi_0 : [0, c] \to \mathbb{R}$ and $\psi' = (\psi_1, \dots, \psi_n) : E \to \mathbb{R}^n$ are given functions. We assume that there is $c_0 \in \mathbb{R}_+$ such that $-c_0 \leq \psi_0(t)$ and $\psi_0(t) \leq t$ for $t \in [0, c]$. Write $\psi(t, x) = (\psi_0(t), \psi'(t, x))$ and $d_0 = b_0 + c_0$. For a function $z : [-d_0, c] \times \mathbb{R}^n \to \mathbb{R}^k$ and a point $(t, x) \in [-c_0, c] \times \mathbb{R}^n$, we define a function $z_{(t,x)} : B \to \mathbb{R}^k$ in the following way

$$z_{(t,x)}(\tau, y) = z(t + \tau, x + y), \quad (\tau, y) \in B.$$

Write

$$\Omega_i = E_i \times C(B, \mathbb{R}^k) \times \mathbb{R}^n \quad \text{and } E_{0.i} = [-d_0, a_i] \times \mathbb{R}^n, \quad i = 1, \dots, k,$$

where $0 \leq a_i < c$ for $1 \leq i \leq k$. Suppose that $f_i : \Omega_i \to \mathbb{R}^k$ and $\phi_i : E_{0.i} \to \mathbb{R}$, $1 \leq i \leq k$, are given functions. Let us denote by $z = (z_1, \dots, z_k)$ an unknown function in the variables (t, x), $x = (x_1, \dots, x_n)$. We consider the system of functional differential equations

$$\partial_t z_i(t, x) = f_i\left(t, x, z_{\psi(t,x)}, \partial_x z_i(t, x)\right), \quad 1 \leq i \leq k, \tag{1}$$

with the initial condition

$$z_i(t, x) = \phi_i(t, x) \quad \text{on} \quad E_{0.i} \quad \text{for} \quad 1 \leq i \leq k, \tag{2}$$

where $\partial_x z_i = (\partial_{x_1} z_i, \dots, \partial_{x_n} z_i)$. Note that $z_{\psi(t,x)}$ is a restriction of z to the set

$$[\psi_0(t) - b_0, \psi_0(t)] \times [\psi'(t, x) - b, \psi'(t, x) + b]$$

and this restriction is shifted to the set B. A function $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_k) : [-d_0, c] \times \mathbb{R}^n \to \mathbb{R}^k$ is a solution of problem (1)–(2) if:

(i) $\tilde{z} \in C([-d_0, c] \times \mathbb{R}^n, \mathbb{R}^k)$ and $\partial_x \tilde{z}_i$ exist on $[a_i, c] \times \mathbb{R}^n$ for $1 \leq i \leq k$,
(ii) for each $1 \leq i \leq k$ and $x \in \mathbb{R}^n$, the function $\tilde{z}_i(\cdot, x) : [a_i, c] \to \mathbb{R}$ is absolutely continuous,
(iii) for each $x \in \mathbb{R}^n$ and for $1 \leq i \leq k$, the i-th equation (1) is satisfied for almost all $t \in [a_i, c]$ and condition (2) holds.

System (1) with initial condition (2) is called a generalized Cauchy problem. If $a_i = 0$ for $i = 1, \dots, k$, then (1)–(2) reduces to the classical Cauchy problem. Then the results for classical initial problems presented in [3,4] are not applicable to (1)–(2).

We give examples of systems which can be derived from (1) by specializing f and ψ.

Example 1.1. *Suppose that $F_i : E_i \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k$, $i = 1, \ldots, k$, are given functions. Write*

$$f_i(t, x, w, q) = F_i(t, x, w(0, \theta), q) \quad on \quad \Omega_i \quad for \quad i = 1, \ldots, k,$$

where $\theta = (0, \ldots, 0) \in \mathbb{R}^n$. Then (1) reduces to the system with deviated variables

$$\partial_t z_i(t, x) = F_i(t, x, z(\psi(t, x)), \partial_x z_i(t, x)), \quad i = 1, \ldots, k.$$

Example 1.2. *For the above F_i, $1 \le i \le k$, we put*

$$f_i(t, x, w, q) = F_i(t, x, \int_B w(\tau, y) dy d\tau, q) \quad on \quad \Omega_i \quad for \quad i = 1, \ldots, k$$

and assume that $\psi(t, x) = (t, x)$ on E. Then (1) reduces to the integral differential system

$$\partial_t z_i(t, x) = F_i(t, x, \int_B z(t + \tau, x + y) dy d\tau, \partial_x z_i(t, x)), \quad (t, x) \in E, \quad i = 1, \ldots, k.$$

It is clear that more complicated differential systems with deviated variables and differential integral problem can be obtained from (1) by suitable choice of f and ψ. Sufficient conditions for the existence of classical solutions of a generalized Cauchy problem can be found in [6].

The paper is organized as follows. In Section 2, we prove that there exists a maximal solution of the generalized Cauchy problem for ordinary differential systems. Carathéodory solutions are considered. We also prove a theorem on integral functional inequalities generated by the above Cauchy problem. Applications of integral functional inequalities are presented in Section 3. Suppose that there exists a comparison problem of the Perron type for (1)–(2). Then the solution of (1)–(2) is unique in the class of bounded and uniformly continuous functions. This is the main result of the paper.

2. INTEGRAL FUNCTIONAL INEQUALITIES

By $L[[t_1, t_2], \mathbb{R}_+]$ for $[t_1, t_2] \subset \mathbb{R}$ we denote the class of all functions $\gamma : [t_1, t_2] \to \mathbb{R}_+$ that are integrable over $[t_1, t_2]$. Write $I = [-b_0, 0]$. For a function $\omega : [-d_0, c] \to \mathbb{R}^k$ and a point $t \in [-c_0, c]$ we define a function $\omega_t : I \to \mathbb{R}^k$ by $\omega_t(\tau) = \omega(t + \tau)$ for $\tau \in I$. Given the functions $\sigma_i : [a_i, c] \times C(I, \mathbb{R}^k_+) \to \mathbb{R}_+$ and $\eta_i : [-d_0, a_i] \to \mathbb{R}$ for $i = 1, \ldots, k$, we consider the Cauchy problem

$$\omega_i'(t) = \sigma_i(t, \omega_{\psi_0(t)}), \quad i = 1, \ldots, k, \tag{3}$$

$$\omega_i(t) = \eta_i(t), \quad t \in [-d_0, a_i], \quad i = 1, \ldots, k. \tag{4}$$

We consider Carathéodory solutions of problem (3)–(4). Note that if $a_i = 0$ for $i = 1, \ldots, k$ and $\Psi_0(t) = t$, then (3)–(4) reduces to the classical Cauchy problem.

The function $\sigma = (\sigma_1, \ldots, \sigma_k)$ is said to satisfy the Carathéodory conditions if for $i = 1, \ldots, k$ there is:

(i) $\sigma_i(t,\cdot) : C(I,\mathbb{R}^k_+) \to \mathbb{R}_+$ is continuous for almost all $t \in [a_i, c]$,
(ii) $\sigma_i(\cdot,\eta) : [a_i,c] \to \mathbb{R}_+$ is measurable for every $\eta \in C(I,\mathbb{R}^k_+)$ and there exists $m_{\sigma_i} \in L([a_i,c],\mathbb{R}_+)$ such that $\sigma_i(t,\eta) \le m_{\sigma_i}(t)$ for $\eta \in C(I,\mathbb{R}^k_+)$ and for almost all $t \in [a_i,c]$.

We say that $\sigma_i(t,\cdot) : C(I,\mathbb{R}^k_+) \to \mathbb{R}_+$ is a nondecreasing function if for $\eta, \tilde\eta \in C(I,\mathbb{R}^k_+)$ such that $\eta(s) \le \tilde\eta(s)$ for $s \in I$, there holds

$$\sigma_i(t,\eta) \le \sigma_i(t,\tilde\eta).$$

We prove that there exists a maximal solution of (3)–(4).

Lemma 2.1. *Suppose that:*

1) $\eta_i \in C([-d_0,a_i],\mathbb{R}_+)$ *for* $i = 1,\dots,k$,
2) $\sigma = (\sigma_1,\dots,\sigma_k)$ *satisfies Carathéodory conditions,*
3) *for almost all* $t \in [a_i,c]$ *the functions* $\sigma_i(t,\cdot) : C(I,\mathbb{R}^k_+) \to \mathbb{R}_+$, $i = 1,\dots,k$, *are nondecreasing,*
4) $\psi_0 : [0,a] \to \mathbb{R}$ *is continuous and* $-c_0 \le \psi_0(t) \le t$ *for* $t \in [a_i,c]$.

Then problem (3)–(4) *admits a maximal solution on* $[-d_0,c]$.

Proof. Let $\gamma = (\gamma_1,\dots,\gamma_k) : [-d_0,c] \to \mathbb{R}^k_+$ be a subsolution of problem (3)–(4), i.e.,

1) γ_i is continuous on $[-d_0,a_i]$ and absolutely continuous on $[a_i,c]$ for $i = 1,\dots,k$,
2) $\gamma_i'(t) \le \sigma_i(t,\gamma_{\psi_0(t)})$ for almost all $t \in [a_i,c]$ and $\gamma_i(t) \le \eta_i(t)$ for $t \in [-d_0,a_i]$ and $i = 1,\dots,k$.

Function γ can be given by its coordinates as $\gamma_i(t) = \eta_i(t)$ for $t \in [-d_0,a_i]$, $\gamma_i(t) = \eta_i(a_i)$ for $t \in [a_i,c]$, where $i = 1,\dots,k$.

Let us define the sequence

$$\{\varphi^{(m)}\}_{m\in\mathbb{N}}, \quad \varphi^{(m)} = (\varphi_1^{(m)},\dots,\varphi_k^{(m)}) : [-d_0,c] \to \mathbb{R}^k_+$$

as follows

$$\varphi_i^{(0)}(t) = \eta_i(t) + 2 \quad \text{for} \quad t \in [-d_0,a_i], \quad i = 1,\dots,k,$$
$$\varphi_i^{(0)}(t) = \eta_i(a_i) + 2 + \int_{a_i}^t \Big[m_{\sigma_i}(\tau) + 2\Big]d\tau \quad \text{for} \quad t \in [a_i,c],\ i = 1,\dots,k,$$

and

$$\varphi_i^{(m)}(t) = \eta_i(t) + \frac{1}{m} \quad \text{for} \quad t \in [-d_0,a_i], \tag{5}$$

$$\varphi_i^{(m)}(t) = \eta_i(a_i) + \frac{1}{m} + \int_{a_i}^t \Big[\sigma_i(\tau,(\varphi^{(m-1)})_{\psi_0(\tau)}) + \frac{1}{m}\Big]d\tau \quad \text{for} \quad t \in [a_i,c], \tag{6}$$

where $i = 1,\dots,k$ and $m \in \mathbb{N}$. First we will show that

$$\varphi^{(m)}(t) < \varphi^{(m-1)}(t) \text{ for } m \ge 1 \text{ and } t \in [-d_0,c]. \tag{7}$$

Let $m = 0$. Then for $t \in [-d_0, a_i]$ there is $\eta_i(t) + 2 > \eta_i(t) + 1$, $i = 1, \dots, k$.

If $t \in [a_i, c]$, then from Carathéodory condition (ii) there follows

$$\varphi_i^{(1)}(t) = \eta_i(a_i) + 1 + \int_{a_i}^{t} \Big[\sigma_i(\tau, (\varphi^{(0)})_{\psi_0(\tau)}) + 1\Big] d\tau <$$
$$< \eta_i(a_i) + 2 + \int_{a_i}^{t} \Big[m_{\sigma_i}(\tau) + 2\Big] d\tau = \varphi_i^{(0)}(t), \quad t \in [a_i, c], \quad i = 1, \dots, k.$$

This implies that $\varphi^{(0)}(t) > \varphi^{(1)}(t)$ for $t \in [-d_0, c]$.

Suppose that the inequality

$$\varphi^{(m)}(t) < \varphi^{(m-1)}(t), \quad t \in [-d_0, c],$$

holds for a fixed $m \geq 0$. Then for $m+1$ and $i = 1, \dots, k$:

$$\varphi_i^{(m)}(t) > \varphi_i^{(m+1)}(t) \quad \text{for} \quad t \in [-d_0, a_i].$$

From the assumption on the monotonicity of the function σ, there follows that

$$\varphi_i^{(m+1)}(t) = \eta_i(a_i) + \frac{1}{m+1} + \int_{a_i}^{t} \Big[\sigma_i(\tau, (\varphi^{(m)})_{\psi_0(\tau)}) + \frac{1}{m+1}\Big] d\tau <$$
$$< \eta_i(a_i) + \frac{1}{m} + \int_{a_i}^{t} \Big[\sigma_i(\tau, (\varphi^{(m-1)})_{\psi_0(\tau)}) + \frac{1}{m}\Big] d\tau =$$
$$= \varphi_i^{(m)}(t), \qquad t \in [a_i, c], \qquad i = 1, \dots, k.$$

Therefore, $\varphi^{(m+1)}(t) < \varphi^{(m)}(t)$ for $t \in [-d_0, c]$. Then the proof of (7) is completed by induction. Now we will show the inequality

$$\gamma(t) < \varphi^{(m)}(t) \quad \text{for} \quad t \in [-d_0, c] \quad \text{and for} \quad m \in \mathbb{N}. \tag{8}$$

Let $m \in \mathbb{N}$ be fixed. If assertion (8) is false, then there are $\tilde{t} \in (-d_0, c]$ and $i = 1, \dots, k$ such that

$$\gamma(t) < \varphi^{(m)}(t) \quad \text{for} \quad t \in [-d_0, \tilde{t})$$

and

$$\gamma_i(\tilde{t}) = \varphi_i^{(m)}(\tilde{t}). \tag{9}$$

It follows that $\tilde{t} > a_i$. Then

$$\gamma_i(\tilde{t}) \leq \gamma_i(a_i) + \int_{a_i}^{\tilde{t}} \sigma_i\big(\tau, \gamma_{\psi_0(\tau)}\big) d\tau \leq \eta_i(a_i) + \int_{a_i}^{\tilde{t}} \sigma_i\big(\tau, \varphi_{\psi_0(\tau)}^{(m)}\big) d\tau < \varphi_i^{(m)}(\tilde{t}).$$

which contradicts (9). This completes the proof of (8).

It is easy to see that the functional sequence $\{\varphi^{(m)}\}_{m \in \mathbb{N}}$ is equicontinuous and uniformly bounded on $[-d_0, c]$. It follows from Arzela-Ascoli theorem that the sequence $\{\varphi^{(m)}\}_{m \in \mathbb{N}}$ is uniformly convergent to a continuous function $\varphi = (\varphi_1, \dots, \varphi_k) \in$

$C([-d_0,c],\mathbb{R}^k)$. Moreover, for $i=1,\dots,k$ and $t\in[-d_0,a_i]$, there is $\varphi_i(t)=\eta_i(t)$ and φ_i is absolutely continuous on $[a_i,c]$. From (5)–(6) we obtain, in the limit, that φ is the solution of problem (3)–(4). Since $\gamma(t)<\varphi^{(m)}(t)$ for $t\in[-d_0,c]$ and $m\in\mathbb{N}$, we obtain $\gamma(t)\le\varphi(t)$ for $t\in[-d_0,c]$. Hence the function φ is the solution of (3)–(4). Now we show that φ is the maximal solution of (3)–(4). Let ω be solution of (3)–(4). Then $\omega_i(t)=\eta_i(t)$ for $t\in[-d_0,a_i]$ and $i=1,\dots,k$. Also

$$\omega_i'(t)=\sigma_i(t,\omega_{\psi_0(t)})\quad\text{for}\quad t\in[a_i,c]\quad\text{and}\quad i=1,\dots,k.$$

Similarly as for function γ, we show that $\omega(t)<\varphi^{(m)}(t)$ for $t\in[-d_0,c]$ and $m\in\mathbb{N}$. Then for $t\in[-d_0,c]$ we obtain

$$\omega(t)\le\lim_{m\to\infty}\varphi^{(m)}(t)=\varphi(t).$$

This implies that φ is the maximal solution of (3)–(4). □

We prove a theorem on integral functional inequalities generated by (3)–(4).

Theorem 2.1. *Suppose that:*

1) $\sigma=(\sigma_1,\dots,\sigma_k)$ *satisfies Carathéodory conditions,*
2) *for almost all* $t\in[a_i,c]$ *the functions* $\sigma_i(t,\cdot):C(I,\mathbb{R}^k_+)\to\mathbb{R}_+$, $i=1,\dots,k$ *are nondecreasing,*
3) $\eta_i\in C([-d_0,a_i],\mathbb{R}_+)$ *for* $i=1,\dots,k$,
4) $\varphi:[-d_0,c]\to\mathbb{R}^k$ *is the maximal solution of* (3)–(4),
5) *the function* $\gamma\in C([-d_0,c],\mathbb{R}^k_+)$ *satisfies an initial estimate* $\gamma_i(t)\le\eta_i(t)$ *for* $t\in[-d_0,a_i]$, *and integral inequalities*

$$\gamma_i(t)\le\eta_i(a_i)+\int_{a_i}^t\sigma_i(\tau,\gamma_{\psi_0(\tau)})d\tau\quad t\in[a_i,c],\ i=1,\dots,k$$

   hold,
6) $\psi_0:[0,a]\to\mathbb{R}$ *is continuous and* $-c_0\le\psi_0(t)\le t$ *for* $t\in[a_i,c]$.

Then $\gamma(t)\le\varphi(t)$ *for* $t\in[-d_0,c]$.

Proof. From assumption 4) it follows that $\gamma_i(t)\le\varphi_i(t)$ for $t\in[-d_0,a_i]$ and for $i=1,\dots,k$. Let us consider the sequence $\{\varphi^{(m)}\}$ defined in Lemma 2.1. Then $\lim_{m\to\infty}\varphi^{(m)}(t)=\varphi(t)$ uniformly on $[-d_0,c]$.

Let us consider the function $u:[-d_0,c]\to\mathbb{R}^k_+$ defined by its coordinates

$$u_i(t)=\eta_i(t)\quad\text{for}\quad t\in[-d_0,a_i],$$

$$u_i(t)=\eta_i(a_i)+\int_{a_i}^t\sigma_i(\tau,\gamma_{\psi_0(\tau)})d\tau\quad\text{for}\quad t\in[a_i,c],$$

where $i=1,\dots,k$. Then $\gamma_i(t)\le u_i(t)$ for $t\in[-d_0,c]$ and for $i=1,\dots,k$. From the assumption on the monotonicity of the function σ there follows that

$$u_i(t)\le\eta_i(a_i)+\int_{a_i}^t\sigma_i(\tau,u_{\psi_0(\tau)})d\tau,\quad t\in[a_i,c].$$

Moreover,

$$\varphi_i^{(m)}(t) > \int_{a_i}^{t} \sigma_i(\tau, \varphi_{\psi_0(\tau)}^{(m)}) d\tau \quad \text{for } t \in [a_i, c] \text{ and for } i = 1, \ldots, k.$$

Similarly as in Lemma (2.1) we show that $u(t) < \varphi^{(m)}(t)$ for $t \in [-d_0, c]$ and $m \in \mathbb{N}$. Consequently, $u(t) \leq \varphi(t)$ for $t \in [-d_0, c]$. Then $\gamma(t) \leq \varphi(t)$ for $t \in [-d_0, c]$. □

3. GENERALIZED CAUCHY PROBLEM

We will need the following operator $V : C(B, \mathbb{R}) \to C(I, \mathbb{R}_+)$ defined as follows:

$$V[\beta](t) = \max\{|\beta(t, x)| : x \in [-b, b]\}, \; t \in [-b_0, 0],$$

where $\beta \in C(B, \mathbb{R})$. For a function $\omega = (\omega_1, \ldots, \omega_k) \in C(B, \mathbb{R}^k)$, we write

$$V[\omega](t) = (V[\omega_1](t), \ldots, V[\omega_k](t)).$$

We start with the formulation of assumptions on σ and f.

Assumption H[σ]. *The functions $\sigma_i : [-d_0, c] \times C(I, \mathbb{R}_+^k) \to \mathbb{R}_+$ for $i = 1, \ldots, k$ satisfy the conditions:*

1) *the function $\sigma = (\sigma_1, \ldots, \sigma_k)$ satisfies Carathéodory conditions,*
2) *for almost all $t \in [a_i, c]$, the function $\sigma_i(t, \cdot) : C(I, \mathbb{R}_+^k) \to \mathbb{R}_+$ is nondecreasing.*

Assumption H[f]. *The functions $f_i : \Omega_i \to \mathbb{R}^k$, for each i, $1 \leq i \leq k$, satisfy the conditions:*

1) *the function $f_i(t, \cdot)$ is continuous for almost all $t \in [a_i, c]$ and function $f_i(\cdot, x, \omega, q)$ is measurable for every $(x, \omega, q) \in \mathbb{R}^n \times C(B, \mathbb{R}^k) \times \mathbb{R}^n$,*
2) *the partial derivatives*

$$\partial_q f_i = [\partial_{q_j} f_i]_{j=1,\ldots,n}$$

exist on $\mathbb{R}^n \times C(B, \mathbb{R}^k) \times \mathbb{R}^n$ for almost all $t \in [a_i, c]$; moreover, $\partial_q f_i(t, \cdot) : \mathbb{R}^n \times$ $\times C(B, \mathbb{R}^k) \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous for almost all $t \in [a_i, c]$ and

$$\partial_q f_i(\cdot, x, \omega, q) = (\partial_{q_1} f_i(\cdot, x, \omega, q), \ldots, \partial_{q_n} f_i(\cdot, x, \omega, q))$$

are measurable for every $(x, \omega, q) \in \mathbb{R}^n \times C(B, \mathbb{R}^k) \times \mathbb{R}^n$,

3) *there exist $m_{q_i} \in L([a_i, c], \mathbb{R}_+)$ such that*

$$\|\partial_q f_i(t, x, \omega, q)\| \leq m_{q_i}(t), \quad i = 1, \ldots, k,$$

on $\mathbb{R}^n \times C(B, \mathbb{R}^k) \times \mathbb{R}^n$ for almost all $t \in [a_i, c]$.

We give a theorem on the estimate of the difference between two solutions of system (1).

Theorem 3.1. *Suppose that Assumptions* $\mathbf{H}[\sigma]$, $\mathbf{H}[f]$ *are satisfied and:*

1) *the function* $u : [-d_0, c] \times \mathbb{R}^n \to \mathbb{R}^k$ *is a solution of problem* (1)–(2), *and the function* $v : [-d_0, c] \times \mathbb{R}^n \to \mathbb{R}^k$ *is solution of system* (1) *with the initial condition*
$$z_i(t,x) = \chi_i(t,x), \quad (t,x) \in E_{0.i} \quad 1 \le i \le k, \tag{10}$$
where $\chi_i : E_{0.i} \to \mathbb{R}$ *for* $i = 1, \dots, k$ *are given functions,*
2) u *and* v *are bounded and uniformly continuous on* $[-d_0, c] \times \mathbb{R}^n$,
3) *for every* $i = 1, \dots, k$, *the initial inequality*
$$|\phi_i(t,x) - \chi_i(t,x)| \le \eta_i(t), \quad (t,x) \in [-d_0, a_i]$$
is satisfied, where $\eta_i \in C([-d_0, a_i], \mathbb{R}_+)$,
4) *for every* $i = 1, \dots, k$, *the estimate*
$$|f_i(t,x,\omega,q) - f_i(t,x,\tilde{\omega},q)| \le \sigma_i(t, V[\omega - \tilde{\omega}])$$
is satisfied on $[a_i, c] \times \mathbb{R}^n \times C(B, \mathbb{R}^k) \times \mathbb{R}^n$,
5) $\psi : E \to \mathbb{R}^{n+1}$ *is continuous and* $-c_0 \le \psi_0(t) \le t$ *for* $t \in [a_i, c]$,
6) *the function* $\omega = (\omega_1, \dots, \omega_k) : [-d_0, c] \to \mathbb{R}^k_+$ *is the maximal solution of problem* (3)–(4).

Under these assumptions,
$$|(u_i - v_i)(t,x)| \le \omega_i(t), \ i = 1, \dots, k, \tag{11}$$
on $[-d_0, c] \times \mathbb{R}^n$.

Proof. It follows from assumption 3) that inequality (11) holds for $(t,x) \in [-d_o, a_i] \times \mathbb{R}^n$. Let
$$\Phi_i(t) = \sup\{|(u_i - v_i)(t,x)| : x \in \mathbb{R}^n\}, \quad -d_0 \le t \le c, \ i = 1, \dots, k.$$
Now assertion (11) is equivalent to the estimates $\Phi_i(t) \le \omega_i(t)$ for $t \in [-d_0, c]$. It follows from (1) that
$$\partial_t u_i(t,x) - \partial_t v_i(t,x) = f_i\big(t,x,u_{\psi(t,x)}, \partial_x u_i(t,x)\big) - f_i\big(t,x,v_{\psi(t,x)}, \partial_x v_i(t,x)\big), \ i = 1, \dots, k.$$
Applying the Hadamard theorem to the difference
$$f_i\left(t,x,v_{\psi(t,x)}, \partial_x u_i(t,x)\right) - f_i\left(t,x,v_{\psi(t,x)}, \partial_x v_i(t,x)\right)$$
we obtain
$$\begin{aligned}
&\partial_t (u_i - v_i)(t,x) - \\
&-\sum_{j=1}^{n} \Big[\int_0^1 \partial_{q_j} f_i\left(t,x,v_{\psi(t,x)}, \partial_x v_i(t,x) + s[\partial_x u_i(t,x) - \partial_x v_i(t,x)]\right) ds\Big] \partial_{x_j}(u_i - v_i)(t,x) = \\
&= f_i\left(t,x,u_{\psi(t,x)}, \partial_x u_i(t,x)\right) - f_i\left(t,x,v_{\psi(t,x)}, \partial_x u_i(t,x)\right), \quad i = 1, \dots, k.
\end{aligned}$$

Let us denote by $g_i(\cdot, t, x)$ the solution of the Cauchy problem

$$y'(s) = -\int_0^1 \partial_q f_i \left(s, y(s), v_{\psi(s,y(s))}, \partial_x v_i(s, y(s)) + \zeta[\partial_x u_i(s, y(s)) - \partial_x v_i(s, y(s))]\right) d\zeta,$$
$$y(t) = x.$$

Then

$$\frac{d}{ds}\Big((u_i - v_i)(s, g_i(s,t,x))\Big) = f_i \left(s, g_i(s,t,x), u_{\psi(s,g_i(s,t,x))}, \partial_x u_i(s, g_i(s,t,x))\right) - \\ - f_i \left(t, x, v_{\psi(s,g_i(s,t,x))}, \partial_x u_i(s, g_i(s,t,x))\right),$$

and consequently

$$(u_i - \tilde{u}_i)(t,x) - (u_i - \tilde{u}_i)(a_i, g_i(a_i,t,x)) = \\ = \int_{a_i}^t \Big[f_i \left(s, g_i(s,t,x), u_{\psi(s,g_i(s,t,x))}, \partial_x u_i(s, g_i(s,t,x))\right) - \\ - f_i \left(t, x, v_{\psi(s,g_i(s,t,x))}, \partial_x u_i(s, g_i(s,t,x))\right)\Big] ds,$$

where $i = 1, \ldots, k$. It follows from assumption 4) that

$$\big|(u_i - \tilde{u}_i)(t,x)\big| \le \big|(u_i - \tilde{u}_i)(a_i, g_i(a_i,t,x))\big| + \\ + \int_{a_i}^t \sigma_i(s, V[u_{\psi(s,g_i(s,t,x))} - v_{\psi(s,g_i(s,t,x))}]) ds, \quad i = 1, \ldots, k.$$

In virtue of Assumption $\mathbf{H}[\sigma]$ and definition of function Φ, the following integral functional inequalities hold:

$$\Phi_i(t) \le \Phi_i(a_i) + \int_{a_i}^t \sigma_i(s, \Phi_{\psi_0(s)}) ds, \quad i = 1, \ldots, k.$$

From Theorem 2.1, there follows $\Phi_i(t) \le \omega_i(t)$ for $t \in [a_i, c]$ and $i = 1, \ldots, k$. Consequently,

$$\Phi(t) \le \omega(t) \quad t \in [-d_0, c].$$

This completes the proof. □

Theorem 3.2. *Suppose that Assumptions* $\mathbf{H}[\sigma]$, $\mathbf{H}[f]$ *are satisfied and:*

1) *for every* $i = 1, \ldots, k$ *the inequality*

$$|f_i(t,x,\omega,q) - f_i(t,x,\tilde{\omega},q)| \le \sigma_i(t, V[\omega - \tilde{\omega}]) \tag{12}$$

holds on $[a_i, c]$,

2) $\psi : E \to \mathbb{R}^{n+1}$ *is continuous and* $-c_0 \le \psi_0(t) \le t$ *for* $t \in [a_i, c]$,
3) *the maximal solution of problem* (3), *with the initial condition* (4) *and* $\eta_i(t) = 0$ *for* $t \in [-d_0, a_i]$, *is the null function.*

Then problem (1)–(2) *admits exactly one bounded and uniformly continuous solution on* $[-d_0, c]$.

Proof. This theorem is a consequence of Theorem 3.1. □

It is important that we consider functional differential systems (3)–(4) as comparison problems for (1)–(2). We show that there is the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t^\beta)), \quad \omega(0) = 0, \tag{13}$$

where $\beta > 0$, such that $\tilde{\omega}(t) = 0$ for $t \in [0, 1]$ is the maximal solution of (13) and the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)), \quad \omega(0) = 0$$

has a positive solution on $(0, 1]$.

Example 3.1. *Suppose that* $a_i = 0$ *for* $1 \leq i \leq k$ *and* $0 < c \leq 1$. *Consider the Cauchy problem*

$$\begin{cases} \omega_i'(t) = A_i \sqrt[\alpha]{\omega_1(t^\beta) + \ldots + \omega_k(t^\beta)}, \\ \omega_i(0) = 0, \qquad i = 1, \ldots, k, \end{cases} \tag{14}$$

where $1 < \alpha < \beta$ *and* $A_i > 0$ *for* $i = 1, \ldots, k$. *Let* $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_k) : [0, \tilde{a}] \to \mathbb{R}^n_+$ *be the maximal solution of* (14). *We prove that* $\tilde{\omega}(t) = 0$ *for* $t \in [0, c]$. *Write*

$$\tilde{A} = \sqrt[\alpha]{k} \max\{A_i : 1 \leq i \leq k\},$$

$$A = \max\{1, \tilde{A}\}.$$

By $\tilde{y} : [0, \tilde{c}] \to \mathbb{R}_+$ *for* $0 < \tilde{c} \leq c$ *let us denote the maximal solution of the scalar Cauchy problem*

$$y'(t) = A \sqrt[\alpha]{y(t^\beta)}, \quad y(0) = 0.$$

It is easy to see that

$$\tilde{\omega}_i(t) \leq \tilde{y}(t) \quad \textit{for } t \in [0, \min\{\tilde{c}, \tilde{a}\}] \quad i = 1, \ldots, k.$$

The function $\tilde{y}$ *satisfies the condition*

$$\tilde{y}(t) \leq A^k t^k, \quad t \in [0, \tilde{c}], k \in \mathbb{N}.$$

Then $\tilde{y}(t) = 0$ *for* $t \in [0, \frac{1}{A})$ *and, consequently,* $\tilde{\omega}(t) = 0$ *for* $t \in [0, \frac{1}{A})$. *It is clear that* $\tilde{\omega}(t) = 0$ *for* $t \in [0, c]$.

Example 3.2. *Suppose that* $\alpha > 1$ *and* $A_i \in \mathbb{R}_+$ *for* $1 \leq i \leq k$ *and*

$$\sum_{i=1}^{k} A_i > 0.$$

Then the maximal solution $\tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_k) : [0, c] \to \mathbb{R}^k_+$ *of the Cauchy problem*

$$\begin{cases} \omega_i'(t) = A_i \sqrt[\alpha]{\omega_1(t) + \ldots + \omega_k(t)}, \\ \omega_i(0) = 0, \qquad i = 1, \ldots, k. \end{cases} \tag{15}$$

satisfies the condition

$$\sum_{i=1}^{k} \tilde{\omega}_i(t) > 0 \quad \textit{for } t \in (0, c]. \tag{16}$$

Write

$$y(t) = \sum_{i=1}^{k} \tilde{\omega}_i(t), \quad t \in [0, c], \quad A = \sum_{i=1}^{k} A_i.$$

It is easily seen that

$$y(t) \geq \left(\frac{\alpha - 1}{\alpha} At \right)^{\frac{\alpha}{\alpha-1}}, \quad t \in [0, c]$$

and assertion (16) *follows.*

REFERENCES

[1] V. Lakshmikantham, S. Köksal, *Monotone Flows and Rapid Convergence for Nonlinear Partial Differential Equations,* Taylor & Francis, London, 2003.

[2] V. Lakshmikantham, S. Leela, *Differential and Integral Inequalities: Theory and Applications,* Academic Press, New York, 1996.

[3] Z. Kamont, *Hyperbolic Functional Differential Inequalities and Applications,* Kluwer Academic Publishers, Dordrecht, 1999.

[4] Z. Kamont, A. Salvadori, *Uniqueness of solutions to hyperbolic functional-differential problems,* Nonlinear Anal. **30** (1997), 4585–4594.

[5] J. Szarski, *Differential Inequalities,* Państwowe Wydawnictwo Naukowe, Warsaw, 1965.

[6] J. Szarski, *Generalized Cauchy problem for differential-functional equations with first order partial derivatives,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), 575–580.

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