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A NOTE ON THE p-DOMINATION NUMBER OF TREES

Abstract. Let p be a positive integer and $G = (V(G), E(G))$ a graph. A p-dominating set of G is a subset S of $V(G)$ such that every vertex not in S is dominated by at least p vertices in S. The p-domination number $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Let T be a tree with order $n \geq 2$ and $p \geq 2$ a positive integer. A vertex of $V(T)$ is a p-leaf if it has degree at most $p-1$, while a p-support vertex is a vertex of degree at least p adjacent to a p-leaf. In this note, we show that $\gamma_p(T) \geq (n + |L_p(T)| - |S_p(T)|)/2$, where $L_p(T)$ and $S_p(T)$ are the sets of p-leaves and p-support vertices of T, respectively. Moreover, we characterize all trees attaining this lower bound.

Keywords: p-domination number, trees.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *Open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of adjacent vertices of v, and the *Closed neighborhood* $N_G[v] = N_G(v) \cup \{v\}$. Let $deg_G(v) = |N_G(v)|$ denote the degree of v. The *maximum degree* $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. For $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. For a pair of vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ between u and v is the length of the shortest uv-paths in G. The *diameter* of G is $d(G) = \max\{d_G(u, v) : u, v \in V(G)\}$.

Let T be a nontrivial tree and $p \geq 2$ a positive integer. A *p-leaf* of T is a vertex with degree at most $p-1$, while a *p-support vertex* of T is a vertex of degree at least p adjacent to a p-leaf. We denote the sets of p-leaves and p-support vertices of T by $L_p(T)$ and $S_p(T)$, respectively. Notice that if $p = 2$ then the 2-leaves (resp. 2-support vertices) are the usual leaves (resp. support vertices) of T, while $L_2(T)$ (resp. $S_2(T)$) is the set of leaves (resp. support vertices) of T. A tree T is a double star if it contains exactly two vertices that are not leaves. A double star with two support vertices a and b is denoted by $S_{a,b}$.

For notation and graph theory terminology we follow [2, 5, 6]. For a vertex v in a rooted tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$.

In [4], Fink and Jacobson introduced the concept of p-domination. Let p be a positive integer. A subset S of $V(G)$ is a *p-dominating set* of G if for every vertex $v \in V(G) - S$, $|S \cap N_G(v)| \geq p$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality among the p-dominating sets of G. Any p-dominating set of G with cardinality $\gamma_p(G)$ will be called a γ_p-set of G. Note that the 1-domination number $\gamma_1(G)$ is the classical domination number $\gamma(G)$. For any $S, T \subseteq V(G)$, S p-dominates T in G if for every vertex $v \in T - S$, $|S \cap N_G(v)| \geq p$.

Some bounds of the p-domination number of a tree T are given in literature. Let T be a nontrivial tree of order $n \geq 3$ and with l leaves and s support vertices. Lemańska [7] showed that $\gamma(T) \geq (n + 2 - l)/2$. Chellali [3] proved that $\gamma_2(T) \geq (n+l-s)/2$ and this lower bound is sharp. In [1], Blidia et al. proved that, for $p \geq 2$, $\gamma_p(T) \leq (n + |L_p(T)|)/2$.

In this note, we give a lower bound of $\gamma_p(T)$ in terms of $n, |L_p(T)|, |S_p(T)|$, that is: Let T be a tree of order n. Then

$$\gamma_p(T) \geq (n + |L_p(T)| - |S_p(T)|)/2$$

for $p \geq 2$, which generalizes the lower bounds of Chellali [3]. Moreover, we characterize all trees attaining this lower bound.

Note that Fink and Jacobson [4] also provided a lower bound of $\gamma_p(T)$ in terms of the order n of a tree T and p, that is: $\gamma_p(T) \geq \frac{(p-1)n+1}{p}$. And recently Volkmann [8] characterized the family of trees with $\gamma_p(T) = \lceil \frac{(p-1)n+1}{p} \rceil$. Now we show that our bound is better than Fink and Jacobson's in some cases. Let $X_p(T)$ denote the set of vertices with degree at least p in T. Let $l_p = |L_p(T)|$ and $x_p = |X_p(T)|$. Then $\frac{\sum_{v\in L_p(T)} deg_T(v)}{l_p} \geq 1$ and $\frac{\sum_{v\in X_p(T)} deg_T(v)}{x_p} \geq p$. Let T be a tree satisfying the following conditions:

$$\frac{\sum_{v\in L_p(T)} deg_T(v)}{l_p} \geq 1 + \alpha \text{ and } \frac{\sum_{v\in X_p(T)} deg_T(v)}{x_p} \geq p + \beta, \tag{1.1}$$

where α and β are any two nonnegative constants satisfying $(p-1)\alpha + \beta = 1$. Hence $2n - 2 = \sum_{v\in L_p(T)} deg_T(v) + \sum_{v\in X_p(T)} deg_T(v) \geq (1+\alpha)l_p + (p+\beta)x_p = (1+\alpha)n + p(1-\alpha)x_p$, the second equality holds since $l_p + x_p = n$ and $(p-1)\alpha + \beta = 1$. So $x_p \leq \frac{n}{p} - \frac{2}{p(1-\alpha)}$. Therefore,

$$\frac{n + |L_p(T)| - |S_p(T)|}{2} \geq \frac{n + l_p - x_p}{2} = n - x_p \geq n - \frac{n}{p} + \frac{2}{p(1-\alpha)} > \frac{(p-1)n+1}{p}.$$

This implies that our bound is better than the bound given by Fink and Jacobson for the trees satisfying condition (1.1). [Such trees satisfying condition (1.1) exist. For example, let T be a tree of order $n \geq 2$ and $cor_p(T)$ a tree obtained from T by adding p pendant edges to each vertex of T. Then $L_p(cor_p(T)) = V(cor_p(T)) - V(T)$ and

$X_p(cor_p(T)) = V(T)$. Let $T' = cor_p(T)$. Then we have $\frac{\sum_{v\in L_p(T')} deg_{T'}(v)}{l_p} = 1 = 1+0$ and $\frac{\sum_{v\in X_p(T')} deg_{T'}(v)}{x_p} = p + \frac{1}{n}\sum_{v\in V(T)} deg_T(v) = p + \frac{2n-2}{n} \geq p+1$.]

2. MAIN RESULTS

The following result is straightforward and can be found in [1].

Lemma 2.1. ([1]) *Every p-dominating set of a graph G contains any vertex of degree at most $p-1$.*

A vertex is a central vertex of a star $K_{1,t}$ $(t \geq 1)$ if either $t \geq 2$ and it is the support vertex or $t = 1$ and it is one of the two leaves. For convenience, we call an isolated vertex a star, denoted by $K_{1,0}$, and the only vertex is called the central vertex.

We define the family $\mathcal{T}_p$ as:

$\mathcal{T}_p = \{T : T$ is obtained from a sequence $T_1, T_2, \cdots, T_k$ $(k \geq 1)$ of trees, where $T_1 = K_{1,t}$ $(t \geq p)$, $T = T_k$, and, if $k \geq 2$, $T_{i+1}(1 \leq i \leq k-1)$ is obtained from T_i by using Operation $\mathcal{O}_j$ $(j = 1, 2$ or $3)$ listed below.$\}$

- **Operation** $\mathcal{O}_1$: Attach a copy of $K_{1,t}$ $(0 \leq t \leq p-2)$ by joining the central vertex to a p-support vertex of T_i or to a vertex of degree at most $p-2$ in T_i.
- **Operation** $\mathcal{O}_2$: Attach a copy of $K_{1,t}$ $(t \geq p)$ by joining the central vertex to a p-support vertex of T_i.
- **Operation** $\mathcal{O}_3$: Attach a copy of $K_{1,t}$ $(t \geq p-1)$ by joining the central vertex to a vertex of degree at most $p-2$ in T_i.

From the way in which a tree $T \in \mathcal{T}_p$ is constructed we make the following lemma.

Lemma 2.2. *Let T be a tree in the family $\mathcal{T}_p$ with $p \geq 2$. Then, $V(T) = L_p(T) \cup S_p(T)$ and each vertex of $S_p(T)$ is adjacent to at least p p-leaves in T.*

From Lemma 2.1 and Lemma 2.2, it is straightforward to obtain the follow result.

Lemma 2.3. *For any positive integer $p \geq 2$, if $T \in \mathcal{T}_p$, then $L_p(T)$ is the unique γ_p-set of T, and*

$$\gamma_p(T) = |L_p(T)| = (|V(T)| + |L_p(T)| - |S_p(T)|)/2.$$

Theorem 2.4. *Let T be a tree with order $n \geq 2$ and $p \geq 2$ a positive integer. Then*

$$\gamma_p(T) \geq (n + |L_p(T)| - |S_p(T)|)/2$$

with equality if and only if either $\Delta(T) \leq p-1$ or $T \in \mathcal{T}_p$.

Proof. Let $l_p = |L_p(T)|$ and $s_p = |S_p(T)|$. We proceed by induction on the order n. If $n = 2$, then $T = P_2$, and so $\Delta(T) \leq p-1$. The result holds. This establishes the base case. Assume that the result is true for every tree T' with order $2 \leq |V(T')| = n' < n$ and let T be a tree of order n.

If $\Delta(T) \le p-1$, then $l_p = n$, $s_p = 0$ and by Lemma 2.1, $\gamma_p(T) = n = (n+l_p-s_p)/2$. The result follows. Assume now that $\Delta(T) \ge p$, then $d(T) \ge 2$. If $d(T) = 2$, then $T = K_{1,t}$ $(t \ge p)$ belongs to $\mathcal{T}_p$. By Lemma 2.3, $\gamma_p(T) = (n + l_p - s_p)/2$. If $d(T) = 3$, then T is a double star $S_{a,b}$ with $deg_T(a) \ge p$ or $deg_T(b) \ge p$. Without loss of generality, assume $deg_T(a) \ge p$. If $2 \le deg_T(b) \le p-1$, then T can be obtained recursively from $K_{1,t}$ $(t = deg_T(a) \ge p)$ by attaching $deg_T(b) - 1$ vertices to one leaf of $K_{1,t}$. Hence T is obtained recursively from $K_{1,t}$ by using $\deg_T(b) - 1$ Operations $\mathcal{O}_1$. Hence $T \in \mathcal{T}_p$ and, by Lemma 2.3, $\gamma_p(T) = (n + l_p - s_p)/2$. If $deg_T(b) = p$, then, to p-dominate b, one of a, b must be contained in a p-dominating set of T. Hence $\gamma_p(T) \ge l_p + 1 = (n+l_p-s_p)/2+1 > (n+l_p-s_p)/2$. If $deg_T(b) > p$ and $deg_T(a) = p$, then, to p-dominate a, one of a, b must be contained in a p-dominating set of T. Hence $\gamma_p(T) \ge l_p + 1 = (n+l_p-s_p)/2+1 > (n+l_p-s_p)/2$. If $deg_T(b) > p$ and $deg_T(a) > p$, then T can be seen as a tree constructed from a star $K_{1,t}$ $(t = deg_T(a) - 1 \ge p)$ by using Operation $\mathcal{O}_2$ by attaching a star $K_{1,m}$ $(m = deg_T(b) - 1 \ge p)$ to vertex a. Hence $T \in \mathcal{T}_p$. By Lemma 2.3, $\gamma_p(T) = (n + l_p - s_p)/2$. Therefore, in the following we assume that T is a tree with $d(T) \ge 4$ and $\Delta(T) \ge p$.

We now root T at a vertex r of maximum eccentricity. Let $P = uvwxy \cdots r$ be a longest path such that $d(u, r) = d(T) \ge 4$ and $deg_T(v)$ is as large as possible. Then, each vertex in $D(w) - C(w)$ has degree one.

Case 1. $deg_T(v) \ge p$.

Let $T' = T - D[v]$ and S be a γ_p-set of T such that S contains the vertices of $D[v]$ as few as possible. Since every vertex of $D(v)$ has degree one, $D(v) \subseteq S$. Hence $v \notin S$ (otherwise, we can replace v by w and get a γ_p-set of T which contains fewer vertices of $D[v]$ than S, a contradiction). Thus $S \cap V(T')$ is a p-dominating set of T', and so

$$\gamma_p(T) = |S| = |D(v)| + |S \cap V(T')| \ge |D(v)| + \gamma_p(T').$$

Subcase 1.1. $deg_T(w) \ne p$.

Then $n' = n - |D[v]|$, $l'_p = l_p - |D(v)|$ and $s'_p = s_p - 1$. By the inductive hypothesis on T',

$$\gamma_p(T) \ge |D(v)| + \gamma_p(T') \ge |D(v)| + (n' + l'_p - s'_p)/2 = (n + l_p - s_p)/2.$$

Further, if $\gamma_p(T) = (n+l_p-s_p)/2$, then $\gamma_p(T') = (n'+l'_p-s'_p)/2$. By the inductive hypothesis on T', $\Delta(T') \le p-1$ or $T' \in \mathcal{T}_p$.

If $\Delta(T') \le p-1$, then $deg_T(w) = deg_{T'}(w)+1 \le \Delta(T')+1 \le p$. Since $deg_T(w) \ne p$, $deg_T(w) \le p-1$, and so v is the unique vertex with degree at least p in T. Hence T can be obtained recursively from a star $K_{1,t}$ $(t = deg_T(v) \ge p)$ by attaching $n-t-1$ isolated vertices. Hence T is obtained recursively from the star $K_{1,t}$ by using $n-t-1$ operations $\mathcal{O}_1$, and so $T \in \mathcal{T}_p$.

If $T' \in \mathcal{T}_p$, then, by Lemma 2.3, $S \cap V(T') = L_p(T')$. If $deg_T(w) \le p-1$, then $deg_{T'}(w) \le p-2$. Thus T is obtained from T' by using Operation $\mathcal{O}_3$ by attaching the star $K_{1,t}$ $(= T[D[v]],\ t = deg_T(v) - 1 \ge p-1)$ to w. Hence $T \in \mathcal{T}_p$. If $deg_T(w) \ge p+1$ and $deg_T(v) \ge p+1$, then $deg_{T'}(w) \ge p$. Thus T is obtained from T' by using Operation $\mathcal{O}_2$ by attaching the star $K_{1,t}$ $(= T[D[v]],\ t = deg_T(v) - 1 \ge p)$ to w. Hence $T \in \mathcal{T}_p$. If $deg_T(w) \ge p+1$ and $deg_T(v) = p$, we claim that the

equality doesn't hold in this case. Since $deg_T(w) \geq p+1$, then $deg_{T'}(w) \geq p$. Hence, $w \notin L_p(T') = S \cap V(T')$. Since $v \notin S$ and to p-dominates v, $w \in S$, a contradiction.
Subcase 1.2. $deg_T(w) = p$.

Then $n' = n - |D[v]|$ and $l'_p = l_p - |D(v)| + 1$. Now we count the number of the p-support vertices of T'. Note that a p-support vertex of T in $C(w) \setminus \{v\}$ is a p-support vertex of T', too. Define δ_1 to be equal to 1, if $w \in S_p(T)$; and 0, otherwise. Define δ_2 to be equal to 1, if $deg_T(x) \geq p$ and $x \notin S_p(T)$; and 0, otherwise. Then $s'_p = s_p - 1 - \delta_1 + \delta_2$. By the inductive hypothesis on T',

$$\begin{aligned}\gamma_p(T) &\geq |D(v)| + \gamma_p(T') \geq \\ &\geq |D(v)| + (n' + l'_p - s'_p)/2 = \\ &= (n + l_p - s_p)/2 + (\delta_1 + 1 - \delta_2)/2 \geq \\ &\geq (n + l_p - s_p)/2.\end{aligned}$$

We claim that the equality doesn't hold in this case. Suppose to the contrary that $\gamma_p(T) = (n + l_p - s_p)/2$. Then $\delta_1 + 1 = \delta_2$, and $\gamma_p(T') = (n' + l'_p - s'_p)/2$. By the definitions of δ_1 and δ_2, $\delta_1 = 0$ and $\delta_2 = 1$. So $deg_T(x) \geq p$ and $x \notin S_p(T)$. By inductive hypothesis on T', $T' \in \mathcal{T}_p$. By Lemma 2.3, $L_p(T')$ is the unique p-dominating set of T'. Since $deg_{T'}(w) = p - 1$, w is a p-leaf of T'. Hence x is a p-support vertex of T'. By Lemma 2.2, x be adjacent to at least p p-leaves of T', and so x must be adjacent to at least $p - 1$ (≥ 1) p-leaves in T. Thus x is a p-support vertex of T since $deg_T(x) \geq p$, which contradicts $x \notin S_p(T)$.
Case 2. $deg_T(v) \leq p - 1$

By our choice of the path $P = uvwxy \cdots r$, each vertex in $D(w) - C(w)$ has degree one, and for each vertex $a \in C(w)$, $deg_T(a) \leq deg_T(v) \leq p - 1$. Hence $D(w) \subseteq L_p(T)$.
Subcase 2.1. $deg_T(w) \leq p - 1$ or $deg_T(w) \geq p + 2$.

Let $T' = T - D[v]$. Let S be a γ_p-set of T. Then $n' = n - |D[v]|$, $l'_p = l_p - |D[v]|$ and $s'_p = s_p$. If $deg_T(w) \leq p - 1$, then, by Lemma 2.1, $w \in S$. Hence $S \cap V(T')$ is a p-dominating set of T'. If $deg_T(w) \geq p + 2$, then $C(w) \setminus \{v\} \subseteq D(w) \subseteq L_p(T) \subseteq S$. Hence $C(w) \setminus \{v\} \subseteq S \cap V(T')$. Since $|C(w)| - 1 = |deg_T(w)| - 2 \geq p$, w is p-dominated by $S \cap V(T')$ and hence $S \cap V(T')$ is a p-dominating set of T', too. By the induction on T',

$$\begin{aligned}\gamma_p(T) = |S| &= |D[v]| + |S \cap V(T')| \geq \\ &\geq |D[v]| + \gamma_p(T') \geq \\ &\geq |D[v]| + (n' + l'_p - s'_p)/2 = \\ &= (n + l_p - s_p)/2.\end{aligned}$$

Further if $\gamma_p(T) = (n + l_p - s_p)/2$, then $\gamma_p(T') = (n' + l'_p - s'_p)/2$. By the inductive hypothesis on T', $\Delta(T') \leq p - 1$ or $T' \in \mathcal{T}_p$. We claim that the equality does not hold for $\Delta(T') \leq p - 1$. If not, then $deg_T(w) = deg_{T'}(w) + 1 \leq \Delta(T') + 1 \leq p$. Since $deg_T(w) \leq p - 1$ or $deg_T(w) \geq p + 2$, we have $deg_T(w) \leq p - 1$. Thus $\Delta(T) \leq p - 1$, which contradicts the assumption that $\Delta(T) \geq p$.

Note that $deg_{T'}(w) \le p-2$ or $deg_{T'}(w) \ge p+1$ ($w \in S_p(T')$ since $D(w) \subseteq L_p(T)$). Since $T' \in \mathcal{T}_p$, T can be constructed from T' by Operation $\mathcal{O}_1$ by attaching the star $K_{1,t}$ $(= T[D[v]],\ t = deg_T(v) - 1 \le p-2)$ to w. Hence, $T \in \mathcal{T}_p$.

Subcase 2.2. $deg_T(w) = p$ and $deg_T(x) \ne p$ or $deg_T(w) = p+1$ and $deg_T(x) \ne p$.

Let $T' = T - D[w]$, then $n' = n - |D[w]|$ and $l'_p = l_p - |D(w)|$ (since $D(w) \subseteq L_p(T)$ and $deg_T(x) \ne p$). Since $deg_T(x) \ne p$, $S_p(T') = S_p(T) \setminus \{w\}$ and $s'_p = s_p - 1$. Let S be a γ_p-set of T that contains the vertices of $D[w]$ as few as possible. Then $w \notin S$ (otherwise, we can replace w by x). Hence $S \cap V(T')$ is a p-dominating set of T'. By the induction on T',

$$\begin{aligned}\gamma_p(T) = |S| = |D(w)| + |S \cap V(T')| \ge \\ \ge |D(w)| + \gamma_p(T') \ge \\ \ge |D(w)| + (n' + l'_p - s'_p)/2 = \\ = (n + l_p - s_p)/2.\end{aligned}$$

Further if $\gamma_p(T) = (n + l_p - s_p)/2$, then $\gamma_p(T') = (n' + l'_p - s'_p)/2 = |S \cap V(T')|$. Hence $S \cap V(T')$ is a γ_p-set of T'. By the induction on T', $\Delta(T') \le p-1$ or $T' \in \mathcal{T}_p$. If $\Delta(T') \le p-1$, then $deg_T(w) = deg_{T'}(w) + 1 \le \Delta(T') + 1 \le p$. Since $deg_T(w) = p$ or $p+1$, $deg_T(w) = p$. Thus w is a unique vertex of degree at least p in T. Hence T can be obtained recursively from $K_{1,t}$ $(t = deg_T(w) = p)$ by attaching $n-t-1$ isolated vertices. Hence T can be obtained recursively from $K_{1,t}$ by using $n-t-1$ operations $\mathcal{O}_1$, and so $T \in \mathcal{T}_p$. Assume now that $T' \in \mathcal{T}_p$. By Lemma 2.3, $S \cap V(T') = L_p(T')$.

If $deg_T(w) = p$ and $deg_T(x) \ge p+1$, we claim that the equality doesn't hold in this case. Since $deg_T(x) \ge p+1$, $deg_{T'}(x) = deg_T(x) - 1 \ge p$ and so $x \notin L_p(T') = S \cap V(T')$. Hence $x \notin S$. Note that $deg_T(w) = p$ and $w \notin S$, to p-dominate w, $x \in S$, a contradiction.

If $deg_T(w) = p+1$ and $deg_T(x) \ge p+1$, then let $T'' = T[V(T) - (D(w) - C(w))]$. Note that $deg_{T'}(x) = deg_T(x) - 1 \ge p$, by Lemma 2.2, $x \in S_p(T')$. Thus T'' is obtained from T' by using Operation $\mathcal{O}_2$ by attaching the star $K_{1,t}$ $(= T[C(w) \cup \{w\}],\ t = deg_T(w) - 1 = p)$ to x, and so $T'' \in \mathcal{T}_p$. By $2 \le deg_T(v) \le p-1$, $p \ge 3$. Since $T[D(w) - C(w)]$ consists of $|D(w) - C(w)|$ isolated vertices, T is obtained recursively from T'' by attaching $|D(w) - C(w)|$ isolated vertices to some vertices of $C(w)$. Hence T is obtained recursively from T'' by using $|D(w) - C(w)|$ operations $\mathcal{O}_1$, and so $T \in \mathcal{T}_p$.

If $deg_T(x) \le p-1$, then $p \ge 3$ and let $T'' = T[V(T) - (D(w) - C(w))]$. Since $deg_{T'}(x) = deg_T(x) - 1 \le p-2$. T'' is obtained from T' by using Operation $\mathcal{O}_3$ by attaching the star $K_{1,t}$ $(= T[C(w) \cup \{w\}],\ t = deg_T(w) - 1 \ge p-1)$ to x. So, $T'' \in \mathcal{T}_p$. Since $T[D(w) - C(w)]$ consists of $|D(w) - C(w)|$ isolated vertices, T is obtained recursively from T'' by attaching $|D(w) - C(w)|$ isolated vertices to some vertices of $C(w)$. Hence T is obtained recursively from T'' by using $|D(w) - C(w)|$ operations $\mathcal{O}_1$, and so $T \in \mathcal{T}_p$.

Subcase 2.3. $deg_T(w) = p+1$ and $deg_T(x) = p$.

Let $T' = T - D[v]$. Note that $D[v] \subseteq L_p(T)$, we have $n' = n - |D[v]|$, $l'_p = l_p - |D[v]|$ and $s'_p = s_p$. Let S be a γ_p-set of T, then by Lemma 2.1, $D[v] \subseteq S$. If $w \in S$, then $S \cap V(T')$ is a p-dominating set of T'. Since $deg_T(x) = p$ and S p-dominates w,

$w \in S$ or $x \in S$. If $w \notin S$, then, to p-dominate x, $x \in S$ since $deg_T(x) = p$. Hence $|(S \cap V(T')) \cap N_{T'}(w)| = |(C(w) - \{v\}) \cup \{x\}| = deg_T(w) - 1 = p$. So, w is p-dominated by $S \cap V(T')$. Thus $S \cap V(T')$ is a p-dominating set of T'. By the induction on T',

$$\begin{aligned}\gamma_p(T) &= |S| = |D[v]| + |S \cap V(T')| \geq \\ &\geq |D[v]| + \gamma_p(T') \geq \\ &\geq |D[v]| + (n' + l'_p - s'_p)/2 = \\ &= (n + l_p - s_p)/2.\end{aligned}$$

We claim that the equality doesn't hold in this case. Suppose to the contrary that $\gamma_p(T) = (n+l_p-s_p)/2$. Then $\gamma_p(T') = (n'+l'_p-s'_p)/2 = |S \cap V(T')|$. Hence $S \cap V(T')$ is a γ_p-set of T'. Since $deg_{T'}(w) = deg_T(w) - 1 = p$, $\Delta(T') \geq p$. By the inductive hypothesis on T', $T' \in \mathcal{T}_p$. By Lemma 2.3, $S \cap V(T') = L_p(T')$. Since $deg_{T'}(w) = p$ and $deg_{T'}(x) = deg_T(x) = p$, by Lemma 2.2, $w \notin L_p(T')$ and $x \notin L_p(T')$. Hence $w, x \notin S$. But, to p-dominate w and x, at least one of w, x is contained by S, a contradiction.

Subcase 2.4. $deg_T(w) = p$ and $deg_T(x) = p$.

Since $d(T) \geq 4$, the father y of x in the rooted tree T exists. Let $C_p(x)$ be the set of children of x with degree at least p. Since $deg_T(w) = p$, $w \in C_p(x)$. Let $C_p(x) = \{w_1, \cdots, w_t\}$ $(t \geq 1)$. Then, for $1 \leq i \leq t$, $D(w_i) \neq \emptyset$. By the choice of the path P, each vertex of $\cup_{i=1}^t D(w_i)$ has degree at most $p-1$. So $\cup_{i=1}^t D(w_i) \subseteq L_p(T)$ and $C_p(x) \subseteq S_p(T)$. Define δ to be equal to 1, if $deg_T(y) \geq p$ and $y \notin S_p(T)$; and 0, otherwise.

Let $T' = T - \cup_{i=1}^t D[w_i]$. Then we have $n' = n - |\cup_{i=1}^t D[w_i]|$, $l'_p = l_p - |\cup_{i=1}^t D(w_i)| + 1$ and $s'_p = s_p - t + \delta$. Let S be a γ_p-set of T that contains the vertices of $\cup_{i=1}^t D[w_i]$ as few as possible. Then $w_i \notin S$ for $i = 1, \cdots, t$ (otherwise, we can replace w_i by x). Hence, to p-dominate w, x must be in S for $deg_T(w) = p$. Thus $S \cap V(T')$ is a p-dominating set of T'. By the induction on T',

$$\begin{aligned}\gamma_p(T) &= |S| = |\cup_{i=1}^t D(w_i)| + |S \cap V(T')| \geq \\ &\geq \sum_{i=1}^t |D(w_i)| + \gamma_p(T') \geq \\ &\geq \sum_{i=1}^t |D(w_i)| + (n' + l'_p - s'_p)/2 = \\ &= (n + l_p - s_p)/2 + (1-\delta)/2 \geq \\ &\geq (n + l_p - s_p)/2.\end{aligned}$$

We claim that the equality doesn't hold in this case. If $\gamma_p(T) = (n + l_p - s_p)/2$, then $\delta = 1$ and $\gamma_p(T') = (n'+l'_p-s'_p)/2 = |S \cap V(T')|$. Hence $deg_{T'}(y) = deg_T(y) \geq p$, $y \notin S_p(T)$ and $S \cap V(T')$ is a γ_p-set of T'. Thus $\Delta(T') \geq p$ and $y \in S_p(T')$. By the inductive hypothesis on T', $T' \in \mathcal{T}_p$. By Lemma 2.3, $S \cap V(T') = L_p(T')$. Then $y \notin S \cap V(T')$. So $y \notin S$. Hence, to p-dominate y, there are at least p p-leaves in T' (and hence $p - 1 \geq 1$ p-leaves in T) that are adjacent to y. That is $y \in S_p(T)$, which contradicts to $y \notin S_p(T)$. □

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