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SENSITIVITY ANALYSIS IN PIECEWISE LINEAR FRACTIONAL PROGRAMMING PROBLEM WITH NON-DEGENERATE OPTIMAL SOLUTION

Abstract. In this paper, we study how changes in the coefficients of objective function and the right-hand-side vector of constraints of the piecewise linear fractional programming problems affect the non-degenerate optimal solution. We consider separate cases when changes occur in different parts of the problem and derive bounds for each perturbation, while the optimal solution is invariant. We explain that this analysis is a generalization of the sensitivity analysis for LP, LFP and PLP. Finally, the results are described by some numerical examples.

Keywords: piecewise linear fractional programming, fractional programming, piecewise linear programming, sensitivity analysis.

Mathematics Subject Classification: 90C31.

1. INTRODUCTION

In practice, numerical results are subject to errors and the exact solution of the problem under consideration is not known. The results obtained by some methods, although being approximations of the solutions of the problem, could be considered as the exact results of the corresponding perturbed problem and this is the motivation to investigate the sensitivity analysis. We would like to know the effect of data perturbation on the optimal solution. Hence, the study of sensitivity analysis is of great importance. Generally, independent and simultaneous perturbations are investigated. The materials presented in the rest of this section are selected from [10].

The piecewise linear fractional programming problem ($PLFP$) is defined as follows:

$$\min \quad Z(x) = \frac{P(\mathbf{x})}{D(\mathbf{x})} = \frac{\alpha_0 + \sum_{j=1}^{n} f_j(x_j)}{\beta_0 + \sum_{j=1}^{n} g_j(x_j)}$$

$$\text{s.t:} \quad \mathbf{Ax} = \mathbf{b} \qquad\qquad (PLFP)$$
$$0 \le \mathbf{x} \le \mathbf{u},$$

where $f_j(x_j)$ and $g_j(x_j)$, $j = 1, 2, \ldots, n$, are continuous piecewise linear convex and concave functions, respectively, such that $\beta_0 + \sum_{j=1}^{n} g_j(x_j) > 0$ for any feasible solution $\mathbf{x}$, $\mathbf{A}$ is an $m \times n$ matrix of full row rank, $\mathbf{b}$ is an m-vector and $\mathbf{u}$ is an n-vector.

Let $0 = \delta_0^j < \delta_1^j < \ldots < \delta_{\tau_j}^j < \delta_{\tau_j+1}^j = u_j$ be an ascending order of the breakpoints of both $f_j(x_j)$ and $g_j(x_j)$. Then within each subinterval $[\delta_i^j, \delta_{i+1}^j]$, $i = 0, 1, \ldots, \tau_j$, both $f_j(x_j)$ and $g_j(x_j)$ are linear functions. Therefore $f_j(x_j)$ and $g_j(x_j)$ can be written as

$$f_j(x_j) = c_i^j x_j + \alpha_i^j, \quad \delta_i^j \le x_j \le \delta_{i+1}^j; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{1.1}$$

and

$$g_j(x_j) = d_i^j x_j + \beta_i^j, \quad \delta_i^j \le x_j \le \delta_{i+1}^j; \quad i = 0, 1, 2, \ldots, \tau_j, \tag{1.2}$$

for some real numbers c_i^j, α_i^j, d_i^j and β_i^j, $i = 0, 1, \ldots, \tau_j$, $j = 1, 2, \ldots, n$.

The following lemmas determine the convexity and the concavity conditions for a continuous piecewise linear function [6].

Lemma 1.1. *A continuous piecewise linear function is convex if and only if its slope is nondecreasing with respect to x_j; that is, $c_0^j \le c_1^j \le \ldots \le c_{\tau_j}^j$, $j = 1, 2, \ldots, n$.*

Lemma 1.2. *A continuous piecewise linear function is concave if and only if its slope is non-increasing with respect to x_j; that is, $d_0^j \ge d_1^j \ge \ldots \ge d_{\tau_j}^j$, $j = 1, 2, \ldots, n$.*

Let $\mathbf{x}^0$ be an optimal solution to $PLFP$. For each $j = 1, 2, \ldots, n$, choose an index j_i such that $\delta_{j_i}^j \le x_j^0 \le \delta_{j_i+1}^j$. Then any optimal solution to the LFP problem:

$$(LFP) \qquad \min \qquad \frac{\alpha^* + \sum_{j=1}^{n} c_{j_i}^j x_j}{\beta^* + \sum_{j=1}^{n} d_{j_i}^j x_j}$$

$$\text{s.t:} \quad \mathbf{Ax} = \mathbf{b}$$
$$\delta_{j_i}^j \le x_j \le \delta_{j_i+1}^j, \quad j = 1, 2, \ldots, n,$$

is also an optimal solution to the $PLFP$ where $\alpha^* = \alpha_0 + \sum_{j=1}^{n} \alpha_{j_i}^j, \beta^* = \beta_0 + \sum_{j=1}^{n} \beta_{j_i}^j$.

Definition of a basic feasible solution (BFS) for $PLFP$ is introduced as follows:

Let $\mathbf{A} = [A_{.1}, \ldots, A_{.n}]$ be the coefficients matrix and $B = \{B_1, \ldots, B_m\} \subset \{1, \ldots, n\}$ be a subset of the indices of the columns of the matrix $\mathbf{A}$, such that $\mathbf{B} = [A_{.B_1}, \ldots, A_{.B_m}]$ is a non-singular matrix with inverse $\mathbf{B}^{-1} = [\beta_{ij}]$. Let $N = \{1, 2, \ldots, n\} \setminus B$. The variables x_{B_i}, $i = 1, \ldots, m$, are called basic variables and x_j, $j \in N$, are referred to as non-basic variables. These vectors are denoted by $\mathbf{x}_B$ and $\mathbf{x}_N$, respectively. Consequently, the solution $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$, such that

$$x_j = \delta^j_{\nu_j}, \quad j \in N, \qquad \nu_j \in \{0, 1, \ldots, \tau_j + 1\},$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in N} \mathbf{B}^{-1} A_{.j} x_j, \tag{1.3}$$

is called a basic solution. If, in addition $0 \leq \mathbf{x}_B \leq \mathbf{u}_B$, then $\mathbf{x}$ is a basic feasible solution (BFS). Moreover, if $x_{B_i} \in \{\delta^{B_i}_0, \delta^{B_i}_1, \ldots, \delta^{B_i}_{\tau_{B_i}+1}\}$ for some i, then $\mathbf{x}$ is a degenerate BFS. If $x_{B_i} \notin \{\delta^{B_i}_0, \delta^{B_i}_1, \ldots, \delta^{B_i}_{\tau_{B_i}+1}\}$ for any i, then it is a non-degenerate BFS.

It is showed in [10] that there exists an optimal solution of $PLFP$ which is a BFS. The optimality criterion given by Punnen and Pandey [10] for $PLFP$ using the simplex algorithm is stated as follows:

Let $\mathbf{B}$ denote the optimal basis matrix and let $\mathbf{x}^* = (\mathbf{x}^*_B, \mathbf{x}^*_N)$ be the corresponding non-degenerate basic feasible solution for $PLFP$. This solution will be optimal if

$$\eta^-_j(\mathbf{x}^*) = (c^j_{\nu_j-1} - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\mathbf{x}^*)(d^j_{\nu_j-1} - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \leq 0,$$

and

$$\eta^+_j(\mathbf{x}^*) = (c^j_{\nu_j} - \mathbf{c}_B \mathbf{B}^{-1} A_{.j}) - Z(\mathbf{x}^*)(d^j_{\nu_j} - \mathbf{d}_B \mathbf{B}^{-1} A_{.j}) \geq 0,$$

for $j = 1, 2, \ldots, n$, where $Z(\mathbf{x}^*)$ is the objective function value at the optimal solution $\mathbf{x}^*$, $\mathbf{c}_B$ and $\mathbf{d}_B$ are the sub-vectors of $\mathbf{c}$ and $\mathbf{d}$ such that their ith coordinates corresponding to $\mathbf{B}$ are $c^{B_i}_{\mu(B_i)}$ and $d^{B_i}_{\mu(B_i)}$, respectively. If $\nu_j = \tau_j + 1$ then η^+_j is defined as 0. Similarly, when $\nu_j = 0$ then η^-_j is defined as 0. Note that $\mu(B_i)$ denotes the index for which $\delta^{B_i}_{\mu(B_i)} \leq x^*_{B_i} \leq \delta^{B_i}_{\mu(B_i)+1}$.

The sensitivity analysis has been done for linear fractional programming [1, 2]. These results have been extended to the variations for both numerator and denominator of the objective function as well as with right-hand-side of the constraints. Then a primal-dual algorithm proposed to parametric right-hand-side analysis and this algorithm suggests a branch-bound method for integer linear programming [4]. An alternative procedure studied for multi-parametric sensitivity analysis in linear programming by the concept of a maximum volume in the tolerance region, which is bounded by a symmetrically rectangular parallelepiped and can be solved by a maximization problem [13]. For the example of linear fractional programming problem we refer the reader to the examples given in [3]. In Example 2 of [3] let the goods be two sets like (i)-beans, lentils and pea, (ii)-celery, lettuce and cabbages, the prices of which can vary in two different policies. Thus the problem is how we can manage this problem after it has been solved before the changes occur and this leads to piecewise linear fractional problem. In [8,9], the sensitivity analysis with the maximum volume in the tolerance region is provided for $PLFP$ when the variations include both numerator and denominator of the objective function, right-hand-side and the coefficients matrix.

In the present paper, sensitivity analysis investigated in [1,2] for the $PLFP$ has been extended. Therefore, we consider separate cases when changes occur in different parts of the problem and derive bounds for each perturbation, while the optimal solution is invariant. Since linear programming (LP)[5], piecewise linear programming problems (PLP)[7] and linear fractional programming problems (LFP)[3, 11, 12] are all special cases of the $PLFP$, therefore a unified framework of sensitivity analysis is presented which covers almost all approaches that have appeared in the literature.

The paper is organized as follows. In Section 2, we obtain bounds for the parameter when the right hand side vector is perturbed. In Section 3 we consider the perturbation in the coefficients of the numerator of the objective function. Section 4 contains changes in the coefficients of the denominator of the objective function.

2. CHANGES IN RHS VECTOR $\mathbf{b}$

Let us replace the entry b_γ by $b_\gamma^{'} = b_\gamma + \delta$ in the RHS vector $\mathbf{b} = (b_1, \ldots, b_\gamma, \ldots, b_m)^T$ and investigate how the optimal basis $\mathbf{B}$, optimal solution $\mathbf{x}^*$ and the optimal value of objective function $Z(\mathbf{x})$ are affected. So from (1.3) we will have

$$\begin{aligned}\bar{\mathbf{x}}_B &= \mathbf{B}^{-1}(b_1, \ldots, b_\gamma + \delta, \ldots, b_m)^T - \sum_{j\in N} \mathbf{B}^{-1} A_{.j}\delta^j_{\nu_j} = \\ &= \mathbf{B}^{-1}\mathbf{b} - \sum_{j\in N} \mathbf{B}^{-1} A_{.j}\delta^j_{\nu_j} + \delta\beta_{.\gamma} = \mathbf{x}^*_B + \delta\beta_{.\gamma},\end{aligned}$$

where $\beta_{.\gamma}$ is the ith column of the matrix $\mathbf{B}^{-1}$. Now the ith component of $\bar{\mathbf{x}}_B$ is given by

$$\bar{x}_{B_i} = x^*_{B_i} + \delta\beta_{i\gamma}, \qquad i = 1, \ldots, m.$$

This new basic solution $\bar{\mathbf{x}}_B$ will be feasible if

$$\delta^{B_i}_{\mu(B_i)} \leq x^*_{B_i} + \delta\beta_{i\gamma} \leq \delta^{B_i}_{\mu(B_i)+1}, \qquad i = 1, \ldots, m.$$

Therefore, we obtain the following range for δ:

$$\begin{aligned}&\max\left\{ \max_{\substack{\beta_{i\gamma}<0\\1\leq i\leq m}} \frac{\delta^{B_i}_{\mu(B_i)+1} - x^*_{B_i}}{\beta_{i\gamma}}, \max_{\substack{\beta_{i\gamma}>0\\1\leq i\leq m}} \frac{\delta^{B_i}_{\mu(B_i)} - x^*_{B_i}}{\beta_{i\gamma}} \right\} \leq \delta \leq \\ &\leq \min\left\{ \min_{\substack{\beta_{i\gamma}>0\\1\leq i\leq m}} \frac{\delta^{B_i}_{\mu(B_i)+1} - x^*_{B_i}}{\beta_{i\gamma}}, \min_{\substack{\beta_{i\gamma}<0\\1\leq i\leq m}} \frac{\delta^{B_i}_{\mu(B_i)} - x^*_{B_i}}{\beta_{i\gamma}} \right\}.\end{aligned} \tag{2.1}$$

The new solution $\bar{\mathbf{x}}$ is an optimal solution for the perturbed $PLFP$ problem if

$$\eta^+_j(\bar{\mathbf{x}}) = (c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}) - Z(\bar{\mathbf{x}})(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}) \geq 0, \quad j \in N, \tag{2.2}$$

and

$$\eta_j^-(\bar{\mathbf{x}}) = (c^j_{\nu_j-1} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}) - Z(\bar{\mathbf{x}})(d^j_{\nu_j-1} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}) \leq 0, \quad j \in N. \tag{2.3}$$

Consider formulas (2.2) and (2.3). Observe that the reduced costs $c^j_{\nu_j-1} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}$, $d^j_{\nu_j-1} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}$, $c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}$ and $d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}$ do not depend on $\mathbf{b}$ and $\bar{\mathbf{x}}$ directly. So, any change in $\mathbf{b}$ may affect only the value of the objective function $Z(\mathbf{x})$. Hence, we have

$$Z(\bar{\mathbf{x}}) = \frac{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}' + \sum_{j\in N}(c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j})\delta^j_{\nu_j} + \alpha_0}{\mathbf{d}_B\mathbf{B}^{-1}\mathbf{b}' + \sum_{j\in N}(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j})\delta^j_{\nu_j} + \beta_0} = \frac{P(\mathbf{x}^*) + \delta\mathbf{c}_B\beta_{.\gamma}}{D(\mathbf{x}^*) + \delta\mathbf{d}_B\beta_{.\gamma}}. \tag{2.4}$$

By the assumption, $D(\mathbf{x}) > 0$ for any feasible solution $\mathbf{x}$. Thus, to preserve this condition we need to have

$$D(\mathbf{x}^*) + \delta\mathbf{d}_B\beta_{.\gamma} > 0, \tag{2.5}$$

which implies

$$\delta \begin{cases} > \dfrac{-D(\mathbf{x}^*)}{\mathbf{d}_B\beta_{.\gamma}}, & \text{if} \quad \mathbf{d}_B\beta_{.\gamma} > 0, \\ < \dfrac{-D(\mathbf{x}^*)}{\mathbf{d}_B\beta_{.\gamma}}, & \text{if} \quad \mathbf{d}_B\beta_{.\gamma} < 0. \end{cases} \tag{2.6}$$

Moreover, by using (2.4) we can re-write (2.2) in the following form

$$\eta_j^+(\bar{\mathbf{x}}) = (c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}) - \frac{P(\mathbf{x}^*) + \delta\mathbf{c}_B\beta_{.\gamma}}{D(\mathbf{x}^*) + \delta\mathbf{d}_B\beta_{.\gamma}}(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}) \geq 0. \tag{2.7}$$

From (2.5), the relation (2.7) is satisfied if

$$(c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j})(D(\mathbf{x}^*) + \delta\mathbf{d}_B\beta_{.\gamma}) - (P(\mathbf{x}^*) + \delta\mathbf{c}_B\beta_{.\gamma})(d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}) \geq 0,$$

which implies

$$\delta(\Delta_j'\mathbf{d}_B\beta_{.\gamma} - \Delta_j''\mathbf{c}_B\beta_{.\gamma}) \geq -D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*), \quad j \in N,$$

where $\Delta_j' = c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}$ and $\Delta_j'' = d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}$.

From the latter relation we obtain

$$\begin{aligned} &\max_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\Delta_j'\mathbf{d}_B\beta_{.\gamma} - \Delta_j''\mathbf{c}_B\beta_{.\gamma}} : \ \Delta_j'\mathbf{d}_B\beta_{.\gamma} - \Delta_j''\mathbf{c}_B\beta_{.\gamma} > 0\right\} \leq \delta \leq \\ &\leq \min_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\Delta_j'\mathbf{d}_B\beta_{.\gamma} - \Delta_j''\mathbf{c}_B\beta_{.\gamma}} : \ \Delta_j'\mathbf{d}_B\beta_{.\gamma} - \Delta_j''\mathbf{c}_B\beta_{.\gamma} < 0\right\}. \end{aligned} \tag{2.8}$$

Similarly, if $\eta_j^-(\bar{\mathbf{x}}) \le 0$ we obtain

$$\begin{aligned} &\max_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\bar{\Delta}_j^{'}\mathbf{d}_B\beta_{.\gamma}-\bar{\Delta}_j^{''}\mathbf{c}_B\beta_{.\gamma}}:\ \bar{\Delta}_j^{'}\mathbf{d}_B\beta_{.\gamma}-\bar{\Delta}_j^{''}\mathbf{c}_B\beta_{.\gamma}<0\right\}\le\delta\le \\ &\le\min_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\bar{\Delta}_j^{'}\mathbf{d}_B\beta_{.\gamma}-\bar{\Delta}_j^{''}\mathbf{c}_B\beta_{.\gamma}}:\ \bar{\Delta}_j^{'}\mathbf{d}_B\beta_{.\gamma}-\bar{\Delta}_j^{''}\mathbf{c}_B\beta_{.\gamma}>0\right\}, \end{aligned} \tag{2.9}$$

where $\bar{\Delta}_j^{'} = c_{\nu_j-1}^j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}$ and $\bar{\Delta}_j^{''} = d_{\nu_j-1}^j - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}$. Thus, we have proved the following theorem:

Theorem 2.1. *If δ satisfies* (2.1), (2.6), (2.8) *and* (2.9) *then* $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_B, \bar{\mathbf{x}}_N)$ where $\bar{\mathbf{x}}_B = \mathbf{x}_B^* + \delta\beta_{.\gamma}$ *is an optimal solution of the perturbed $PLFP$ problem (with $b_\gamma \to b_\gamma^{'} = b_\gamma + \delta$).*

Remark 2.2. Lower and upper bounds given in Theorem 2.1 are generalizations of the corresponding bounds for LP, PLP and LFP. Indeed,

1. If $\beta_0 = 1$ and $g_j(x_j) = 0, j = 1,2,\ldots,n$, then the $PLFP$ reduces to PLP and this means that $D(\mathbf{x}^*) = 1$, $\Delta_j^{''} = d_{\nu_j}^j - \mathbf{d}_B\mathbf{B}^{-1}A_{.j} = 0$, $\eta_j^+(\bar{\mathbf{x}}) = c_{\nu_j}^j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} = \Delta_j^{'}$, $\eta_j^-(\bar{\mathbf{x}}) = c_{\nu_j-1}^j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} = \bar{\Delta}_j^{'}, j \in N$, and $Z(\bar{\mathbf{x}}) = P(\mathbf{x}^*) + \delta\mathbf{c}_B\beta_{.\gamma}$. Thus, bounds (2.1) in the current form are valid for PLP too, and restrictions (2.8) and (2.9) are not present in the bounds since $\Delta_j^{'}\mathbf{d}_B\beta_{.\gamma} - \Delta_j^{''}\mathbf{c}_B\beta_{.\gamma} = \bar{\Delta}_j^{'}\mathbf{d}_B\beta_{.\gamma} - \bar{\Delta}_j^{''}\mathbf{c}_B\beta_{.\gamma} = 0$. Therefore, if δ satisfies (2.1) then $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_B, \bar{\mathbf{x}}_N)$ where $\bar{\mathbf{x}}_B = \mathbf{x}_B^* + \delta\beta_{.\gamma}$ is an optimal solution of the perturbed PLP problem (when $b_\gamma \to b_\gamma^{'} = b_\gamma + \delta$).
2. If $\beta_0 = 1$, $g_j(x_j) = 0$ and $f_j(x_j), j = 1,2,\ldots,n$, are linear functions then the $PLFP$ reduces to LP with bounded variables. In this case, optimality conditions (2.8) and (2.9) and feasibility condition (2.1) are respectively as follows

$$\eta_j^+(\mathbf{x}^*) = c_{\nu_j}^j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} = c_j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} \ge 0, \quad \text{if } x_j = 0,$$

$$\eta_j^-(\mathbf{x}^*) = c_{\nu_j-1}^j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} = c_j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} \le 0, \quad \text{if } x_j = u_j,$$

$$\max\left\{\max_{\substack{\beta_{i\gamma}<0\\1\le i\le m}}\frac{u_{B_i}-x_{B_i}^*}{\beta_{i\gamma}},\ \max_{\substack{\beta_{i\gamma}>0\\1\le i\le m}}\frac{-x_{B_i}^*}{\beta_{i\gamma}}\right\}\le\delta\le\min\left\{\min_{\substack{\beta_{i\gamma}>0\\1\le i\le m}}\frac{u_{B_i}-x_{B_i}^*}{\beta_{i\gamma}},\ \min_{\substack{\beta_{i\gamma}<0\\1\le i\le m}}\frac{-x_{B_i}^*}{\beta_{i\gamma}}\right\}.$$

3. If both $g_j(x_j)$ and $f_j(x_j), j = 1,2,\ldots,n$, are linear functions then the $PLFP$ reduces to LFP and this means that $c_{\nu_j}^j = c_j$, $d_{\nu_j}^j = d_j$, $\Delta_j^{'} = c_j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}$ and $\Delta_j^{''} = d_j - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}$. Therefore (2.6) in the current form is valid for LFP and feasibility and optimality conditions are respectively as follows

$$\max_{\beta_{i\gamma}>0}\frac{-x_{B_i}^*}{\beta_{i\gamma}} \le \delta \le \min_{\beta_{i\gamma}<0}\frac{-x_{B_i}^*}{\beta_{i\gamma}}, \quad i = 1,2,\ldots,m,$$

$$\max_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j(\mathbf{x}^*)}{\Delta_j^{'}\mathbf{d}_B\beta_{.\gamma}-\Delta_j^{''}\mathbf{c}_B\beta_{.\gamma}}:\ \Delta_j^{'}\mathbf{d}_B\beta_{.\gamma}-\Delta_j^{''}\mathbf{c}_B\beta_{.\gamma}>0\right\}\le\delta\le$$
$$\le\min_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j(\mathbf{x}^*)}{\Delta_j^{'}\mathbf{d}_B\beta_{.\gamma}-\Delta_j^{''}\mathbf{c}_B\beta_{.\gamma}}:\ \Delta_j^{'}\mathbf{d}_B\beta_{.\gamma}-\Delta_j^{''}\mathbf{c}_B\beta_{.\gamma}<0\right\},$$

where $\eta_j(\mathbf{x}^*)=(c_j-\mathbf{c}_B\mathbf{B}^{-1}A_{.j})-Z(\mathbf{x}^*)(d_j-\mathbf{d}_B\mathbf{B}^{-1}A_{.j})$.

Example 2.3. Consider the problem $(PLFP)$:

$$\begin{array}{llllll}
\min & Z(\mathbf{x})=\dfrac{\sum_{j=1}^{4}f_j(x_j)}{\sum_{j=1}^{4}g_j(x_j)} & & & & \\
s.t & 3x_1+ & 4x_2+ & x_3+ & 2x_4= & 21\\
 & x_1+ & 3x_2+ & x_3+ & 3x_4= & 13\\
 & 2x_1+ & x_2+ & 2x_3+ & 3x_4= & 14\\
 & 0\le x_1\le 5, & 0\le x_2\le 3, & 0\le x_3\le 5, & 0\le x_4\le 5 &
\end{array}$$

where

$$f_1(x_1)=\begin{cases}3x_1, & 0\le x_1\le 1,\\ 4x_1-1, & 1\le x_1\le 5,\end{cases}\qquad g_1(x_1)=\begin{cases}4x_1+1, & 0\le x_1\le 1,\\ 3x_1+2, & 1\le x_1\le 5,\end{cases}$$

$$f_2(x_2)=\begin{cases}2x_2+1, & 0\le x_2\le 1,\\ 3x_2, & 1\le x_2\le 3,\end{cases}\qquad g_2(x_2)=\begin{cases}3x_2+1, & 0\le x_2\le 1,\\ 2x_2+2, & 1\le x_2\le 3,\end{cases}$$

$$f_3(x_3)=\begin{cases}x_3+3, & 0\le x_3\le 2,\\ 2x_3+1, & 2\le x_3\le 3,\\ 3x_3-2, & 3\le x_3\le 5,\end{cases}\qquad g_3(x_3)=\begin{cases}3x_3+1, & 0\le x_3\le 2,\\ 2x_3+3, & 2\le x_3\le 3,\\ x_3+6, & 3\le x_3\le 5,\end{cases}$$

$$f_4(x_4)=\begin{cases}x_4+1, & 0\le x_4\le 1,\\ 2x_4, & 1\le x_4\le 3,\\ 3x_4-3, & 3\le x_4\le 5,\end{cases}\qquad g_4(x_4)=\begin{cases}4x_4+1, & 0\le x_4\le 1,\\ 2x_4+3, & 1\le x_4\le 3,\\ x_4+6, & 3\le x_4\le 5.\end{cases}$$

Using the simplex algorithm of Punnen and Pandey [10], the initial and the final simplex tables are given as follows (see Tab. 1 and 2).

Table 1. Initial simplex

c_B	d_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{b}$
M	0	x_5	3	4	1	2	1	0	0	21
M	0	x_6	1	3	1	3	0	1	0	13
M	0	x_7	2	1	2	3	0	0	1	14
η_j^+			$2-54M$	$\frac{-7-176M}{4}$	$\frac{-11-160M}{4}$	$4-56M$	0	0	0	$z=\frac{5+48M}{4}$
η_j^-			0	0	0	0	0	0	0	
x_j			0	0	0	0	21	13	14	

Table 2. Final simplex

c_B	d_B	x_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\bar{b}$
3	2	x_2	0	1	$-3/10$	0	$3/20$	$1/4$	$-7/20$	$3/2$
1	4	x_4	0	0	$1/2$	1	$-1/4$	$1/4$	$1/4$	$3/2$
4	3	x_1	1	0	$2/5$	0	$3/10$	$-1/2$	$3/10$	4
η_j^+			0	0	1.031	0	$M-2.623$	$M+1$	$M-1.438$	$z=.885$
η_j^-			0	0	$-.854$	0	0	0	0	
x_j			$32/10$	$21/10$	2	$1/2$	0	0	0	

The optimal solution is $x^* = (32/10, 21/10, 2, 1/2, 0, 0, 0)^t$. Here $B = \{2, 4, 1)\}$ and the matrix of the optimal basis is $\begin{pmatrix} 4 & 2 & 3 \\ 3 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix}$ and its inverse $\beta = \mathbf{B}^{-1} = \begin{pmatrix} 3/20 & 1/4 & -7/20 \\ -1/4 & 1/4 & 1/4 \\ 3/10 & -1/2 & 3/10 \end{pmatrix}$.

If $b_1 \to b_1^{'} = b_1 + \delta$, then by using (2.1), (2.6), (2.8) and (2.9) we get

$$\max\Big\{\max\{-2\},\ \max\{\tfrac{-22}{3}, \tfrac{-22}{3}\}\Big\} \le \delta \le \min\Big\{\min\{6,6\},\ \min\{2\}\Big\} \Rightarrow -2 \le \delta \le 2,$$

$$\mathbf{d}_B\beta_{.1} = \tfrac{1}{5} > 0 \Rightarrow \delta > \frac{\frac{-278}{10}}{\frac{1}{5}} = -139,$$

$$\Delta_3^{'}\mathbf{d}_B\beta_{.1} - \Delta_3^{''}\mathbf{c}_B\beta_{.1} = 1 > 0 \Rightarrow \delta \ge \frac{\frac{-278}{10}(1.031)}{1} = -28.66,$$

$$\bar{\Delta}_3^{'}\mathbf{d}_B\beta_{.1} - \bar{\Delta}_3^{''}\mathbf{c}_B\beta_{.1} = \tfrac{-3}{5} < 0 \Rightarrow \delta \ge \frac{\frac{-278}{10}(-0.854)}{\frac{-3}{5}} = -39.57.$$

Therefore, the following range is obtained for δ,

$$-2 \le \delta \le 2.$$

3. CHANGES IN THE COEFFICIENTS OF NUMERATOR OF THE OBJECTIVE

In this section our goal is to determine the lower and upper bounds for δ, which guarantee that the replacement $c_i^j \to c'^j_i = c_i^j + \delta$ does not affect the optimal basis, and the original optimal solution $\mathbf{x}^*$ remains feasible and optimal.

By this replacement, we have to distinguish the following two cases:

Case 1. $c_i^j \in \Big\{c_{\nu_j}^j : \nu_j \in \{0, 1, \ldots, \tau_j\}\Big\}$,

Case 2. $c_i^j \in \Big\{c_{\mu(B_1)}^{B_1}, \ldots, c_{\mu(B_m)}^{B_m}\Big\}$.

Case 1. c_i^j is the coefficient of a non-basic variable. Thus, this change of the coefficient does not affect feasibility of the vector $\mathbf{x}^*$. However, it may affect the optimal value of $Z(\mathbf{x})$ and hence, can change the reduced costs $\eta_j^+(\mathbf{x}^*)$ and $\eta_j^-(\mathbf{x}^*)$. So, by replacing $c_{\nu_\gamma}^\gamma \to c_{\nu_\gamma}^\gamma + \delta$ we have

$$\bar{Z}(\mathbf{x}^*) = \frac{P(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)}. \tag{3.1}$$

Now, the optimal solution $\mathbf{x}^*$ of the original $PLFP$ problem remains optimal for the perturbed $PLFP$ problem if we have

$$\bar{\eta}_j^+(\mathbf{x}^*) = \begin{cases} (c_{\nu_j}^j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}) - \dfrac{P(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)}(d_{\nu_j}^j - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}) \geq 0, \ j \neq \gamma, \ j \in N, \\ (c_{\nu_\gamma}^\gamma + \delta - \mathbf{c}_B\mathbf{B}^{-1}A_{.\gamma}) - \dfrac{P(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)}(d_{\nu_\gamma}^\gamma - \mathbf{d}_B\mathbf{B}^{-1}A_{.\gamma}) \geq 0, \ j = \gamma, \end{cases}$$

or

$$\begin{cases} \Delta_j^{'} - \dfrac{P(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)}\Delta_j^{''} \geq 0, & j \neq \gamma \ , \ j \in N, \\ \Delta_\gamma^{'} + \delta - \dfrac{P(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}{D(\mathbf{x}^*)}\Delta_\gamma^{''} \geq 0, & j = \gamma. \end{cases} \tag{3.2}$$

From $D(\mathbf{x}^*) > 0$, the relation (3.2) is satisfied if

$$\begin{cases} \Delta_j^{'}D(\mathbf{x}^*) - (P(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma)\Delta_j^{''} \geq 0, & j \neq \gamma, \ j \in N, \\ (\Delta_\gamma^{'} + \delta)D(\mathbf{x}^*) - (P(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma)\Delta_\gamma^{''} \geq 0, & j = \gamma. \end{cases} \tag{3.3}$$

Therefore, we will get

$$\max_{\substack{j\in N \\ j\neq\gamma}} \left\{ \frac{D(\mathbf{x}^*) \ \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta_j^{''}} : \ \Delta_j^{''} < 0 \right\} \leq \delta \leq \min_{\substack{j\in N \\ j\neq\gamma}} \left\{ \frac{D(\mathbf{x}^*) \ \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \Delta_j^{''}} : \ \Delta_j^{''} > 0 \right\}, \tag{3.4}$$

and

$$\delta \begin{cases} \geq \dfrac{-D(\mathbf{x}^*) \ \eta_\gamma^+(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta_\gamma^{''}}, & \text{if} \quad D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta_\gamma^{''} > 0, \\ \leq \dfrac{-D(\mathbf{x}^*) \ \eta_\gamma^+(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta_\gamma^{''}}, & \text{if} \quad D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \Delta_\gamma^{''} < 0. \end{cases} \tag{3.5}$$

Similarly, if $\bar{\eta}_j^-(\mathbf{x}^*) \leq 0$, we will get

$$\max_{\substack{j\in N \\ j\neq\gamma}} \left\{ \frac{D(\mathbf{x}^*) \ \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}_j^{''}} : \ \bar{\Delta}_j^{''} > 0 \right\} \leq \delta \leq \min_{\substack{j\in N \\ j\neq\gamma}} \left\{ \frac{D(\mathbf{x}^*) \ \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma \bar{\Delta}_j^{''}} : \ \bar{\Delta}_j^{''} < 0 \right\}, \tag{3.6}$$

and

$$\delta \begin{cases} \leq \dfrac{-D(\mathbf{x}^*) \ \eta_\gamma^-(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}_\gamma^{''}}, & \text{if} \quad D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}_\gamma^{''} > 0, \\ \geq \dfrac{-D(\mathbf{x}^*) \ \eta_\gamma^-(\mathbf{x}^*)}{D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}_\gamma^{''}}, & \text{if} \quad D(\mathbf{x}^*) - \delta_{\nu_\gamma}^\gamma \bar{\Delta}_\gamma^{''} < 0. \end{cases} \tag{3.7}$$

Therefore, we have proved the following theorem.

Theorem 3.1. *If δ satisfies (3.4), (3.5), (3.6), (3.7) and the convexity condition for $f_j(x_j)$ holds, then optimal solution $\mathbf{x}^*$ of the original $PLFP$ problem is also an optimal solution of the perturbed $PLFP$ problem (where $c^{\gamma}_{\nu_\gamma} \to c^{\gamma}_{\nu_\gamma} + \delta$).*

Remark 3.2. Lower and upper bounds given in Theorem 2.1 are generalizations of the corresponding bounds for LP, PLP and LFP. Indeed,

1. If both $f_j(x_j)$ and $g_j(x_j)$, $j = 1, 2, \ldots n$, are linear functions then the $PLFP$ reduces to LFP. Therefore from (3.4) and (3.6) we conclude that $-\infty \leq \delta \leq \infty$ and from (3.5) and (3.7) it follows that $\delta \leq -\eta_\gamma(\mathbf{x}^*)$ where $\eta_\gamma(\mathbf{x}^*) = c_\gamma - \mathbf{c}_B\mathbf{B}^{-1}A_{.\gamma} - Z(\mathbf{x}^*)(d_\gamma - \mathbf{d}_B\mathbf{B}^{-1}A_{.\gamma})$.
2. If $\beta_0 = 1$ and $g_j(x_j) = 0, j = 1, 2, \ldots, n$, then the $PLFP$ reduces to PLP. In this case, from (3.4), (3.6) and from (3.5), (3.7) we have, respectively,

$$-\infty \leq \delta \leq \infty,$$

$$-\Delta^{'}_\gamma \leq \delta \leq -\bar{\Delta}^{'}_\gamma.$$

Case 2. c^j_i is the coefficient of a basic variable. Then the replacement $c^{B_k}_{\mu(B_k)} \to c^{'B_k}_{\mu(B_k)} = c^{B_k}_{\mu(B_k)} + \delta$ affects the optimal value of $P(\mathbf{x})$ as well as $Z(\mathbf{x})$

$$\begin{aligned}\tilde{P}(\mathbf{x}^*) &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} + \delta\beta_{k.}\mathbf{b} + \sum_{j\in N}(c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j})\delta^j_{\nu_j} - \delta\sum_{j\in N}\beta_{k.}A_{.j}\delta^j_{\nu_j} + \alpha \\ &= P(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N}A_{.j}\delta^j_{\nu_j}).\end{aligned}$$

In addition, the replacement $c^{B_k}_{\mu(B_k)} \to c^{'B_k}_{\mu(B_k)}$ has an affect on the non-basic reduced costs:

$$c^j_{\nu_j} - \mathbf{c}^{'}_B\mathbf{B}^{-1}A_{.j} = c^j_{\nu_j} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} - \delta\beta_{k.}A_{.j} = \Delta^{'}_j - \delta\beta_{k.}A_{.j},$$

$$c^j_{\nu_j-1} - \mathbf{c}^{'}_B\mathbf{B}^{-1}A_{.j} = c^j_{\nu_j-1} - \mathbf{c}_B\mathbf{B}^{-1}A_{.j} - \delta\beta_{k.}A_{.j} = \bar{\Delta}^{'}_j - \delta\beta_{k.}A_{.j}.$$

Therefore, to satisfy the optimality condition, we can determine the new values η^+_j and η^-_j as

$$\begin{aligned}\tilde{\eta}^+_j(\mathbf{x}^*) &= \Delta^{'}_j - \delta\beta_{k.}A_{.j} - \frac{P(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N}A_{.j}\delta^j_{\nu_j})}{D(\mathbf{x}^*)}\Delta^{''}_j = \\ &= \frac{(\Delta^{'}_j - \delta\beta_{k.}A_{.j})D(\mathbf{x}^*) - \Big(P(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N}A_{.j}\delta^j_{\nu_j})\Big)\Delta^{''}_j}{D(\mathbf{x}^*)} \geq 0. \qquad (3.8)\end{aligned}$$

The relation (3.8) is satisfied if

$$(\Delta_j^{'} - \delta\beta_{k.}A_{.j})D(\mathbf{x}^*) - \Big(P(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\Big)\Delta_j^{''} \geq 0,$$

which implies

$$\delta\beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\Delta_j^{''}\Big) \leq D(\mathbf{x}^*)\eta_j^+(\mathbf{x}^*), \qquad j \in N.$$

Thus

$$\max_{j\in N}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\Delta_j^{''}\Big)} : \beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\Delta_j^{''}\Big) < 0\right\}$$

$$\leq \delta \leq \tag{3.9}$$

$$\min_{j\in N}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\Delta_j^{''}\Big)} : \beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\Delta_j^{''}\Big) > 0\right\}.$$

Similarly, if $\tilde{\eta}_j^-(\mathbf{x}^*) \leq 0$, we will get

$$\max_{j\in N}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}_j^{''}\Big)} : \beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}_j^{''}\Big) > 0\right\}$$

$$\leq \delta \leq \tag{3.10}$$

$$\min_{j\in N}\left\{\frac{D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}_j^{''}\Big)} : \beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta_{\nu_j}^j)\bar{\Delta}_j^{''}\Big) < 0\right\}.$$

Therefore, we have proved the following theorem.

Theorem 3.3. *If* δ *satisfies* (3.9), *(3.10) and the convexity condition for* $f_j(x_j)$ *holds then optimal solution* $\mathbf{x}^*$ *of the original* $PLFP$ *problem is also an optimal solution of the perturbed* $PLFP$ *problem (with* $c_{\mu(B_k)}^{B_k} \to c_{\mu(B_k)}^{'B_k}$*).*

Remark 3.4. Observe that the range obtained in Theorem 3.3 may be considered as a generalization of the corresponding range for the LFP, PLP and LP problems. Thus we have

1. If both $f_j(x_j)$ and $g_j(x_j)$, $j = 1, 2, \ldots n$, are linear functions then the $PLFP$ reduces to LFP. In this case, the restrictions (3.9) and (3.10) reduce to

$$\max_{j\in N}\left\{\frac{D(\mathbf{x}^*)\ \eta_j(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta_j^{''}\Big)} :\ \beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta_j^{''}\Big) < 0\right\} \leq \delta \leq$$

$$\leq \min_{j\in N}\left\{\frac{D(\mathbf{x}^*)\ \eta_j(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta_j^{''}\Big)} :\ \beta_{k.}\Big(A_{.j}D(\mathbf{x}^*) + \mathbf{b}\Delta_j^{''}\Big) > 0\right\},$$

where $\eta_j = \Delta_j^{'} - Z(\mathbf{x}^*)\Delta_j^{''}$.

2. If $\beta_0 = 1$ and $g_j(x_j) = 0, j = 1, 2, \ldots, n$, then the $PLFP$ reduces to PLP. Therefore the relations (3.9) and (3.10) exchange to

$$\max_{j\in N}\left\{\frac{\Delta_j^{'}}{\beta_{k.}A_{.j}} : \beta_{k.}A_{.j} < 0\right\} \le \delta \le \min_{j\in N}\left\{\frac{\Delta_j^{'}}{\beta_{k.}A_{.j}} : \beta_{k.}A_{.j} > 0\right\},$$

$$\max_{j\in N}\left\{\frac{\bar{\Delta}_j^{'}}{\beta_{k.}A_{.j}} : \beta_{k.}A_{.j} > 0\right\} \le \delta \le \min_{j\in N}\left\{\frac{\bar{\Delta}_j^{'}}{\beta_{k.}A_{.j}} : \beta_{k.}A_{.j} < 0\right\}.$$

Example 3.5. Consider Example 2.3. For the given optimal basis and solution we consider the following two cases:

Non-basic index: Let $c_1^3 \to c_1^3 + \delta$. In this case, $\gamma = 3$ and $N = \{3\}$. Since $\gamma \ne j \in N$, the relations (3.4) and (3.6) are not applicable. Hence, from (3.5), (3.7) and the convexity of $f_j(x_j)$ we have

$$D(\mathbf{x}^*) - \delta_1^3\Delta_3^{''} = 29 > 0 \;\Rightarrow \delta \ge \frac{\frac{-139}{5}(1.031)}{29} = -0.99,$$

$$D(\mathbf{x}^*) - \delta_1^3\bar{\Delta}_3^{''} = 27 > 0 \;\Rightarrow \delta \le \frac{\frac{-139}{5}(-0.854)}{27} = 0.88,$$

$$1 \le 2 + \delta \le 3 \;\Rightarrow -1 \le \delta \le 1.$$

Finally, we obtain the following bounds for δ:

$$-0.99 \le \delta \le 0.88.$$

Basic index: Let $c_1^1 \to c_1^1 + \delta$. In this case by using (3.9), (3.10) and the convexity of $f_j(x_j)$ we have

$$\beta_{3.}\big(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\Delta_3^{''}\big) = 9.2 > 0 \Rightarrow$$
$$\Rightarrow \delta \le \min_{j\in\{3\}}\Big\{\frac{D(\mathbf{x}^*)\eta_j^+(\mathbf{x}^*)}{\beta_{3.}\big(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\Delta_3^{''}\big)}\Big\} = \frac{\frac{139}{5}(1.031)}{9.2} = 3.115,$$

$$\beta_{3.}\big(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\bar{\Delta}_3^{''}\big) = \frac{62}{5} > 0 \Rightarrow$$
$$\Rightarrow \delta \ge \max_{j\in\{3\}}\Big\{\frac{D(\mathbf{x}^*)\eta_j^-(\mathbf{x}^*)}{\beta_{3.}\big(A_{.3}D(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta_{\nu_3}^3)\bar{\Delta}_3^{''}\big)}\Big\} = \frac{\frac{139}{5}(-0.854)}{\frac{62}{5}} = -1.915,$$
$$3 \le 4 + \delta \;\Rightarrow \delta \ge -1$$

Hence, we obtain the following bounds for δ:

$$-1 \le \delta \le 3.115.$$

4. CHANGES IN THE COEFFICIENTS OF THE DENOMINATOR OF THE OBJECTIVE

In this section, our goal is to determine the lower and upper bounds for δ, which guarantee that the replacement $d_i^j \to {d'}_i^j = d_i^j + \delta$ does not affect the optimal basis, and the original optimal solution $\mathbf{x}^*$ remains feasible and optimal.

By considering this replacement, we have to distinguish the following two cases:
Case 1. $d_i^j \in \left\{ d_{\nu_j}^j : \nu_j \in \{0, 1, \ldots, \tau_j\} \right\}$,
Case 2. $d_i^j \in \left\{ d_{\mu(B_1)}^{B_1}, \ldots, d_{\mu(B_m)}^{B_m} \right\}$.
Case 1. d_i^j is the coefficient of a non-basic variable. Thus, this change of the coefficient does not affect the feasibility of the vector $\mathbf{x}^*$. However, it may affect the optimal value of $Z(\mathbf{x})$ and hence, can change the reduced costs $\eta_j^+(\mathbf{x}^*)$ and $\eta_j^-(\mathbf{x}^*)$. So, by replacing $d_{\nu_\gamma}^\gamma \to d_{\nu_\gamma}^\gamma + \delta$ we will have

$$\tilde{Z}(\mathbf{x}^*) = \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}. \tag{4.1}$$

To preserve the strict positivity of the denominator $D(\mathbf{x})$, we need to have

$$D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma > 0 \Rightarrow \delta > \frac{-D(\mathbf{x}^*)}{\delta_{\nu_\gamma}^\gamma}. \tag{4.2}$$

Now, the optimal solution $\mathbf{x}^*$ of the original $PLFP$ problem remains optimal for the perturbed $PLFP$ problem if we have

$$\tilde{\eta}_j^+(\mathbf{x}^*) = \begin{cases} (c_{\nu_j}^j - \mathbf{c}_B\mathbf{B}^{-1}A_{.j}) - \dfrac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}\ (d_{\nu_j}^j - \mathbf{d}_B\mathbf{B}^{-1}A_{.j}) \geq 0,\ j \neq \gamma, j \in N, \\ (c_{\nu_\gamma}^\gamma - \mathbf{c}_B\mathbf{B}^{-1}A_{.\gamma}) - \dfrac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}\ (d_{\nu_\gamma}^\gamma + \delta - \mathbf{d}_B\mathbf{B}^{-1}A_{.\gamma}) \geq 0,\ j = \gamma, \end{cases}$$

or

$$\begin{cases} \Delta_j' - \dfrac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}\Delta_j'' \geq 0, & j \neq \gamma, \quad j \in N, \\ \Delta_\gamma' - \dfrac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma}(\Delta_\gamma'' + \delta) \geq 0, & j = \gamma. \end{cases} \tag{4.3}$$

From (4.2), the relation (4.3) is satisfied if

$$\begin{cases} \Delta_j'\big[D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma\big] - P(\mathbf{x}^*)\Delta_j'' \geq 0, & j \neq \gamma,\ j \in N, \\ \Delta_\gamma'\big[D(\mathbf{x}^*) + \delta\delta_{\nu_\gamma}^\gamma\big] - P(\mathbf{x}^*)\big[\Delta_\gamma'' + \delta\big] \geq 0, & j = \gamma. \end{cases} \tag{4.4}$$

Therefore, we will get

$$\max_{\substack{j\in N\\ j\neq\gamma}}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\Delta_j^{'}}:\Delta_j^{'}>0\right\}\leq\delta\leq\min_{\substack{j\in N\\ j\neq\gamma}}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\Delta_j^{'}}:\ \Delta_j^{'}<0\right\},\quad(4.5)$$

and

$$\delta\begin{cases}\geq\dfrac{-D(\mathbf{x}^*)\ \eta_\gamma^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\Delta_\gamma^{'}-P(\mathbf{x}^*)}, & \text{if}\quad \delta_{\nu_\gamma}^{\gamma}\Delta_\gamma^{'}-P(\mathbf{x}^*)>0,\\ \leq\dfrac{-D(\mathbf{x}^*)\ \eta_\gamma^+(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\Delta_\gamma^{'}-P(\mathbf{x}^*)}, & \text{if}\quad \delta_{\nu_\gamma}^{\gamma}\Delta_\gamma^{'}-P(\mathbf{x}^*)<0.\end{cases}\quad(4.6)$$

Similarly, if $\tilde{\eta}_j^-\leq 0$ we will get

$$\max_{\substack{j\in N\\ j\neq\gamma}}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\bar{\Delta}_j^{'}}:\ \bar{\Delta}_j^{'}<0\right\}\leq\delta\leq\min_{\substack{j\in N\\ j\neq\gamma}}\left\{\frac{-D(\mathbf{x}^*)\ \eta_j^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\bar{\Delta}_j^{'}}:\ \bar{\Delta}_j^{'}>0\right\},\quad(4.7)$$

and

$$\delta\begin{cases}\leq\dfrac{-D(\mathbf{x}^*)\ \eta_\gamma^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\bar{\Delta}_\gamma^{'}-P(\mathbf{x}^*)}, & \text{if}\quad \delta_{\nu_\gamma}^{\gamma}\bar{\Delta}_\gamma^{'}-P(\mathbf{x}^*)>0,\\ \geq\dfrac{-D(\mathbf{x}^*)\ \eta_\gamma^-(\mathbf{x}^*)}{\delta_{\nu_\gamma}^{\gamma}\bar{\Delta}_\gamma^{'}-P(\mathbf{x}^*)}, & \text{if}\quad \delta_{\nu_\gamma}^{\gamma}\bar{\Delta}_\gamma^{'}-P(\mathbf{x}^*)<0.\end{cases}\quad(4.8)$$

Therefore, we have proved the following theorem:

Theorem 4.1. *If* δ *satisfies* (4.2), (4.5), (4.6), (4.7), (4.8) *and* $g_j(x_j)$ *is concave, then* $\mathbf{x}^*$ *is an optimal solution of the perturbed* $PLFP$ *problem (where* $d_{\nu_\gamma}^{\gamma}\to d_{\nu_\gamma}^{\gamma}+\delta$*).*

Case 2. d_i^j is the coefficient of a basic variable. Thus the replacement $d_{\mu(B_k)}^{B_k}\to d_{\mu(B_k)}^{'B_k}=d_{\mu(B_k)}^{B_k}+\delta$ affects the optimal value of $D(\mathbf{x})$ as well as $Z(\mathbf{x})$

$$\begin{aligned}\hat{D}(\mathbf{x}^*)&=\mathbf{d}_B\mathbf{B}^{-1}\mathbf{b}+\delta\beta_{k.}\mathbf{b}+\sum_{j\in N}(d_{\nu_j}^{j}-\mathbf{d}_B\mathbf{B}^{-1}A_{.j})\delta_{\nu_j}^{j}-\delta\sum_{j\in N}\beta_{k.}A_{.j}\delta_{\nu_j}^{j}+\beta_0=\\&=D(\mathbf{x}^*)+\delta\beta_{k.}(\mathbf{b}-\sum_{j\in N}A_{.j}\delta_{\nu_j}^{j}).\end{aligned}$$

To preserve the strict positivity of the denominator $D(\mathbf{x})$, we need to have

$$D(\mathbf{x}^*)+\delta\beta_{k.}(\mathbf{b}-\sum_{j\in N}A_{.j}\delta_{\nu_j}^{j})>0.\quad(4.9)$$

Therefore, we will have

$$\delta \begin{cases} > \dfrac{-D(\mathbf{x}^*)}{\beta_{k.}(\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j})}, & \text{if} \quad \beta_{k.}(\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j}) > 0, \\ < \dfrac{-D(\mathbf{x}^*)}{\beta_{k.}(\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j})}, & \text{if} \quad \beta_{k.}(\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j}) < 0. \end{cases} \tag{4.10}$$

In addition, the replacement $d^{B_k}_{\mu(B_k)} \to d^{'B_k}_{\mu(B_k)}$ has an effect on the non-basic reduced costs:

$$d^j_{\nu_j} - \mathbf{d}^{'}_B\mathbf{B}^{-1}A_{.j} = d^j_{\nu_j} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j} - \delta\beta_{k.}A_{.j} = \Delta^{''}_j - \delta\beta_{k.}A_{.j},$$

$$d^j_{\nu_j-1} - \mathbf{d}^{'}_B\mathbf{B}^{-1}A_{.j} = d^j_{\nu_j-1} - \mathbf{d}_B\mathbf{B}^{-1}A_{.j} - \delta\beta_{k.}A_{.j} = \bar{\Delta}^{''}_j - \delta\beta_{k.}A_{.j}.$$

Therefore, to satisfy the optimality condition, we can determine the new values η^+_j and η^-_j as

$$\begin{aligned} \hat{\eta}^+_j(\mathbf{x}^*) &= \Delta^{'}_j - \frac{P(\mathbf{x}^*)}{D(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})}(\Delta^{''}_j - \delta\beta_{k.}A_{.j}) = \\ &= \frac{\Delta^{'}_j\big(D(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})\big) - P(\mathbf{x}^*)(\Delta^{''}_j - \delta\beta_{k.}A_{.j})}{D(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})} \geq 0. \end{aligned} \tag{4.11}$$

From (4.9), the relation (4.11) is satisfied if

$$\Delta^{'}_j\big(D(\mathbf{x}^*) + \delta\beta_{k.}(\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})\big) - P(\mathbf{x}^*)(\Delta^{''}_j - \delta\beta_{k.}A_{.j}) \geq 0.$$

Therefore, we have

$$\begin{aligned} &\max_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta^+_j(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j})\Delta^{'}_j\Big)} : \beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})\Delta^{'}_j\Big) > 0\right\} \\ &\leq \delta \leq \\ &\min_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta^+_j(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j})\Delta^{'}_j\Big)} : \beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})\Delta^{'}_j\Big) < 0\right\}. \end{aligned} \tag{4.12}$$

Similarly, if $\hat{\eta}^-_j(\mathbf{x}^*) \leq 0$, we will get

$$\begin{aligned} &\max_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta^-_j(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j})\bar{\Delta}^{'}_j\Big)} : \beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})\bar{\Delta}^{'}_j\Big) < 0\right\} \\ &\leq \delta \leq \\ &\min_{j\in N}\left\{\frac{-D(\mathbf{x}^*)\ \eta^-_j(\mathbf{x}^*)}{\beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum\limits_{j\in N} A_{.j}\delta^j_{\nu_j})\bar{\Delta}^{'}_j\Big)} : \beta_{k.}\Big(A_{.j}P(\mathbf{x}^*) + (\mathbf{b} - \sum_{j\in N} A_{.j}\delta^j_{\nu_j})\bar{\Delta}^{'}_j\Big) > 0\right\}. \end{aligned} \tag{4.13}$$

Thus, we have the following theorem:

Theorem 4.2. *If δ satisfies (4.10), (4.12), (4.13) and $g_j(x_j)$ is concave, then $\mathbf{x}^*$ is an optimal solution for the perturbed PLFP problem (with $d^{B_k}_{\mu(B_k)} \to d'^{B_k}_{\mu(B_k)}$).*

Example 4.3. Consider Example 2.3. For the given optimal basis and solution, we consider the following two cases:

Non-basic index: Let $d_1^3 \to d_1^3 + \delta$. In this case, $\gamma = 3$ and $N = \{3\}$. Since $\gamma \neq j \in N$, therefore the relations (4.5) and (4.7) are not applicable. Hence, from (4.2), (4.6), (4.8) and concavity of $g_j(x_j)$ we will have

$\delta > \frac{\frac{-139}{5}}{2} = -13.9,$

$\delta^{\gamma}_{\nu_\gamma}\Delta'_{\gamma} - P(\mathbf{x}^*) = -23 < 0 \;\Rightarrow \delta \leq \frac{\frac{-139}{5}(1.031)}{-23} = 1.246,$

$\delta^{\gamma}_{\nu_\gamma}\bar{\Delta}'_{\gamma} - P(\mathbf{x}^*) = -25 < 0 \;\Rightarrow \delta \geq \frac{\frac{-139}{5}(-0.854)}{-25} = -0.949,$

$1 \leq 2 + \delta \leq 3 \;\Rightarrow -1 \leq \delta \leq 1.$

Hence, we obtain the following range for δ:

$$-0.961 \leq \delta \leq 1.$$

Basic index: Let $d_1^1 \to d_1^1 + \delta$. In this case using (4.10), (4.12), (4.13) and the concavity of $g_j(x_j)$ we will have

$\beta_{3.}(\mathbf{b} - \mathbf{A}_{.3}\delta_1^3) = \frac{16}{5} > 0 \;\Rightarrow \delta > \frac{\frac{-139}{5}}{\frac{16}{5}} = -8.685,$

$\beta_{3.}\big(A_{.3}P(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta^3_{\nu_3})\Delta'_3\big) = \frac{62}{5} > 0 \;\Rightarrow \delta \leq \frac{\frac{-139}{5}(1.031)}{\frac{62}{5}} = -2.311,$

$\beta_{3.}\big(A_{.3}P(\mathbf{x}^*) + (\mathbf{b} - A_{.3}\delta^3_{\nu_3})\bar{\Delta}'_3\big) = \frac{46}{5} > 0 \;\Rightarrow \delta \leq \frac{\frac{-139}{5}(-0.854)}{\frac{46}{5}} = 2.581,$

$3 + \delta \leq 4 \;\Rightarrow \delta \leq 1.$

Finally, the following range is obtained for δ:

$$-2.311 \leq \delta \leq 1.$$

5. SUMMARY

The sensitivity analysis of optimal solutions has been presented in this paper. Three cases were considered: (i) changes in the right-hand-side vector, (ii) changes in the coefficients of the numerator of the objective function, (iii) changes in the coefficients of the denominator of the objective function. In each case the underlying theory for sensitivity analysis has been presented to obtain the bounds for each perturbation.

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