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EDGE CONDITION FOR HAMILTONICITY IN BALANCED TRIPARTITE GRAPHS

Abstract. A well-known theorem of Entringer and Schmeichel asserts that a balanced bipartite graph of order $2n$ obtained from the complete balanced bipartite $K_{n,n}$ by removing at most $n - 2$ edges, is bipancyclic. We prove an analogous result for balanced tripartite graphs: If $G$ is a balanced tripartite graph of order $3n$ and size at least $3n^2 - 2n + 2$, then $G$ contains cycles of all lengths.

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1. INTRODUCTION AND MAIN RESULT

A well-known theorem of Entringer and Schmeichel [4] asserts that a balanced bipartite graph of order $2n$ and size at least $n^2 - n + 2$ is bipancyclic. The bound is best possible: A graph obtained from $K_{n,n-1}$ by adding a single vertex adjacent to precisely one vertex in the colour class of $n$ vertices, has size $n^2 - n + 1$ and contains no Hamilton cycle. One can consider an analogous problem for balanced tripartite graphs. It is readily seen that a balanced tripartite graph $G$ obtained from the complete balanced tripartite $K_3(n)$ by removing $2n - 1$ (that is, all but one) edges incident with a given vertex $v$ (see Fig. 1), contains no Hamilton cycle. As the size of such $G$ is $2n(n - 1) + n^2 + 1$, at least $3n^2 - 2n + 2$ edges are necessary to guarantee hamiltonicity of a balanced tripartite graph. The main result of this note asserts that this obvious necessary condition is, in fact, sufficient.
Let $f_3(n) := 3n^2 - 2n + 2$ for $n \geq 2$. We prove the following sufficient condition for a balanced tripartite graph to contain a Hamilton cycle:

**Theorem 1.1.** Let $G$ be a balanced tripartite graph of order $3n$, $n \geq 2$, and size at least $f_3(n)$. Then $G$ contains a Hamilton cycle.

**Remark 1.2.** The result is best possible, as seen in Figure 1. Paired with a theorem of Bondy [1] (stating that a hamiltonian graph $G$ satisfying $\|G\| \geq \frac{|G|^2}{4}$ is actually pancyclic), the condition $\|G\| \geq f_3(n)$ implies, in fact, that $G$ contains cycles of all lengths (see Corollary 3.1).

**Remark 1.3.** The hamiltonicity criteria for balanced tripartite graphs analogous to the classical ones for bipartite graphs have been sought for and studied over the last decade or so (see, e.g., [2] and [5]). Notice however that the edge-type conditions have not yet been accounted for and our bound does not follow from neither Dirac-type minimal degree nor Ore-type degree sum conditions on tripartite graphs. (For the sake of completeness, recall that a balanced tripartite graph $G$ with colour classes $V_1, V_2, V_3$ of cardinalities $n$ and minimal degree $\delta(G)$ is known to be hamiltonian if $\delta(G) > 5n/4$ (by [2]), or $|N_G(x) \cap V_j| + |N_G(y) \cap V_i| \geq n + 1$ for every pair of nonadjacent vertices $x \in V_i, y \in V_j$ ($i \neq j$) (by [5]).)

2. LEMMAS

Throughout the paper $\mathcal{G}_n$ will denote a family of balanced tripartite graphs $G$ with the vertex set $V(G)$ a disjoint union of three colour classes $V_1, V_2$ and $V_3$ of cardinalities $|V_i| = n$, $n \geq 2$, and such that $\|G\| \geq f_3(n)$, where $f_3(n) = 3n^2 - 2n + 2$. As usual, $|G|$ denotes the order of a graph $G$ and $\|G\|$ is the size of $G$. For a vertex $v$ of $G$, we denote by $N(v)$ the set of vertices adjacent to $v$; note that $N(v) \subset V(G) \setminus V_i$ if $v \in V_i$, so in particular $|N(v)| \leq 2n$.

We begin by showing the following three simple lemmas.

**Lemma 2.1.** Let $G \in \mathcal{G}_n$ ($n \geq 2$) and assume that the minimal degree of $G$ satisfies $\delta(G) \leq 2n - 2$. Then there exist $i \neq j$ and a pair of non-adjacent vertices $x \in V_i$, $y \in V_j$ such that both $x$ and $y$ have neighbours in the third colour class $V_k$. 

![Fig. 1](image-url)
Proof. Pick \( y \in V(G) \) with \( d(y) \leq 2n - 2 \), say \( y \in V_j \). There exists at least one pair \( x_1, x_2 \) of distinct non-neighbours of \( y \), with \( x_1, x_2 \in V(G) \setminus V_j \). For every such pair, we have \( d(x_1) + d(x_2) \geq 2n \). Indeed, as \( G \) is obtained from the complete tripartite graph \( K_3(n) \) by removing at least \((2n - 1) + (2n - 1) + 1 - d(x_1) - d(x_2)\) edges, then \( d(x_1) + d(x_2) \leq 2n - 1 \) implies \( ||G|| \leq 3n^2 - 2n < f_3(n) \); a contradiction.

Hence at least one of the \( x_1, x_2 \) has degree greater than \( n - 1 \). Consequently, we may choose \( x \in V_i \) (\( i \neq j \)) such that \( xy \notin E(G) \), \( yz \in E(G) \) for some \( z \) from the third colour class \( V_k \), and \( d(x) \geq n \). This last inequality together with \( xy \notin E(G) \) implies that \( x \) also has a neighbour in \( V_k \).

\[ \square \]

Lemma 2.2. Let \( G \in \mathcal{G}_n \) \((n \geq 2)\) and assume \( \delta(G) \leq 2n - 2 \). Then there exist \( i \neq j \) and a pair of non-adjacent vertices \( x \in V_i \), \( y \in V_j \) such that \( N(x) \cap N(y) \neq \emptyset \) (i.e., \( x \) and \( y \) have a common neighbour in the third class).

Proof. By Lemma 2.1, we may choose a pair of non-adjacent vertices \( x \in V_i \), \( y \in V_j \) such that both \( x \) and \( y \) have neighbours in the third colour class \( V_k \). Suppose that, for every \( z \) a neighbour of \( x \) in \( V_k \), \( z \) is not a neighbour of \( y \). Pick such \( z \in N(x) \cap V_k \). We may assume that \( z \) and \( y \) share no neighbour in \( V_i \); otherwise, if, say, \( x' \in N(z) \cap N(y) \), replace \( (x, y) \in V_i \times V_j \) with \( (z, y) \in V_k \times V_j \) and get \( zy \notin E(G) \), \( zx' \in E(G) \) and \( yx' \in E(G) \), as required.

Now, no vertex of \( V_k \) is a common neighbour of \( x \) and \( y \), no vertex of \( V_i \) is a common neighbour of \( z \) and \( y \), and both \( x \) and \( z \) have at most \( n - 1 \) neighbours in \( V_j \).

Counting the total number of neighbours of \( x, y \) and \( z \), we thus get

\[
d(x) + d(y) + d(z) \leq |V_i| + |V_k| + 2(|V_j| - 1) = 4n - 2,
\]

so that

\[
||G|| \leq ||G - \{x, y, z\}|| + d(x) + d(y) + d(z) \leq 3(n - 1)^2 + 4n - 2 < f_3(n);
\]

a contradiction. This shows that at least one neighbour of \( x \) in \( V_k \) is simultaneously adjacent to \( y \).

\[ \square \]

Let \( G^*_n \) denote a graph obtained from the complete tripartite \( K_3(n) \), with colour classes \( V_1, V_2, V_3 \), by removing a complete \( V_1 - V_2 \) matching; i.e., if \( V_1 = \{x_1, \ldots, x_n\} \), \( V_2 = \{y_1, \ldots, y_n\} \), then

\[
G^*_n = K_3(n) - \{x_1y_1, x_2y_2, \ldots, x_ny_n\}.
\]

Lemma 2.3. Let \( G \in \mathcal{G}_n \) be as in Lemma 2.2. Then either \( G \) contains (a copy of) \( G^*_n \) or else there is a triple of vertices \( x \in V_1 \), \( y \in V_2 \), \( z \in V_3 \) such that \( xy \notin E(G) \), \( xz \in E(G) \), \( yz \in E(G) \) and \( ||G - \{x, y, z\}|| \geq f_3(n - 1) \).

Proof. Let \( x \in V_1 \), \( y \in V_2 \), \( z \in V_3 \) be a triple guaranteed by Lemma 2.2. We have

\[
||G - \{x, y, z\}|| \geq f_3(n) - d(x) - d(y) - d(z) + 2,
\]

with the last summand arising from counting \( xz \) and \( yz \) twice in \( d(x) + d(y) + d(z) \). As \( xy \notin E(G) \), then \( d(x) \leq 2n - 1 \) and \( d(y) \leq 2n - 1 \), and the above inequality yields

\[
||G - \{x, y, z\}|| \geq f_3(n) - 6n + 4 = 3n^2 - 8n + 6,
\]

as required. \[ \square \]
whilst \( f_3(n-1) = 3n^2 - 8n + 7 \). It follows that \( \|G - \{x, y, z\}\| \geq f_3(n-1) \) unless 
\( d(x) = d(y) = 2n - 1 \) and \( d(z) = 2n \).

Suppose the latter holds. Then we may replace \( z \) by another \( z' \in V_3 \) and repeat the 
above argument with a triple \( \{x, y, z'\} \). We get again either \( \|G - \{x, y, z'\}\| \geq f_3(n-1) \) 
or else \( d(x) = d(y) = 2n - 1 \) and \( d(z') = 2n \).

Suppose then that \( d(z') = 2n \) for all \( z' \in V_k \). If there is no other pair of vertices 
\( x' \in V_1 \) and \( y' \in V_2 \) with \( x'y' \notin E(G) \), then \( G = K_3(n) - xy \) contains \( G^*_n \). Otherwise, 
pick \( x' \in V_1 \) and \( y' \in V_2 \) with \( x'y' \notin E(G) \) and repeat the argument with \( \{x', y', z\} \). 
If \( \|G - \{x', y', z\}\| < f_3(n-1) \), repeat the argument with a triple \( \{x', y', z'\} \) for some 
\( z' \in V_3 \setminus \{z\} \), and so on.

It is readily seen that in this way we find a triple \( \tilde{x} \in V_1 \), \( \tilde{y} \in V_2 \), \( \tilde{z} \in V_3 \) 
with \( \|G - \{\tilde{x}, \tilde{y}, \tilde{z}\}\| \geq f_3(n-1) \) unless there exist subsets \( \{x_1, \ldots, x_s\} \subset V_1 \) 
and \( \{y_1, \ldots, y_s\} \subset V_2 \), \( s \leq n \), such that \( G = K_3(n) - \{x_1y_1, x_2y_2, \ldots, x_sy_s\} \) contains \( G^*_n \).

3. PROOF OF THE MAIN RESULT

We are now ready to prove Theorem 1.1. Let \( G \) be a balanced tripartite graph of 
order \( 3n \), \( n \geq 2 \), and size at least \( f_3(n) = 3n^2 - 2n + 2 \). We proceed by induction on \( n \).

As \( f_3(2) = 10 \), a balanced tripartite graph \( G \) on 6 vertices with \( \|G\| \geq f_3(2) \) is 
obtained from \( K_3(2) \) by removing at most two edges. One easily verifies that every 
such a graph is hamiltonian.

Suppose then that \( n \geq 3 \) and the assertion of the theorem holds for \( n - 1 \). If 
\( \delta(G) \geq 2n - 1 \), then \( G \) is hamiltonian by Dirac’s theorem [3], as \( 2n - 1 \geq \frac{|G|}{2} \) for 
\( n \geq 2 \). We may thus assume that \( \delta(G) \leq 2n - 2 \), and hence Lemma 2.3 applies to \( G \).

Denote, as before, the colour classes of \( G \) by \( V_1, V_2 \) and \( V_3 \). Recall that by \( G^*_n \) 
we denote a graph obtained from \( K_3(n) \) by removing a complete \( V_1 - V_2 \) matching.

If \( G \) contains a subgraph isomorphic to \( G^*_n \), then we can define explicitly a Hamilton 
cycle as follows: Write \( V_1 = \{x_1, \ldots, x_n\} \), \( V_2 = \{y_1, \ldots, y_n\} \) and \( V_3 = \{z_1, \ldots, z_n\} \), 
where \( G \) contains all the \( x_iy_j, x_iz_k, y_jz_k \) edges except at most \( x_1y_1, \ldots, x_ny_n \). Then 
\( x_1y_1z_2x_2z_3y_2z_3 \ldots x_{n-1}y_{n-1}z_nx_ny_1z_1 \) is a required cycle in \( G \).

Assume then that \( G \) contains no \( G^*_n \), and hence by Lemma 2.3, there is a triple 
of vertices \( x \in V_1, y \in V_2 \) and \( z \in V_3 \) such that \( xy \notin E(G), xz \in E(G), yz \in E(G) \) 
and \( \|G - \{x, y, z\}\| \geq f_3(n-1) \). Put \( H := G - \{x, y, z\} \). By the inductive hypothesis, 
\( H \) contains a Hamilton cycle \( C \).

Observe that \( \delta(G) \geq 2 \), for otherwise \( G \) would have at least \( 2n - 1 \) edges less 
than \( K_3(n) \) and hence \( \|G\| \leq 3n^2 - 2n + 1 < f_3(n) \); a contradiction. Therefore, as 
\( xy \notin E(G) \), both \( x \) and \( y \) have a neighbour on \( C \), say \( w_x \) and \( w_y \) respectively.

Observe next that \( d(x) + d(y) \geq 2n + 1 \), for otherwise, as \( d(z) \leq 2n \), would have 
\( \|G\| = \|H\| + d(x) + d(y) + d(z) - 2 \leq 3(n-1)^2 + 2n + 2n - 2 < f_3(n) \); a contradiction. 
Hence at least one of the vertices \( x, y \) has more than one neighbour on \( C \) and we may 
assume that \( w_x \neq w_y \) (see Fig. 2). Now, taking \( C + xz + zy + yw_y \) and splitting \( C \) 
at \( w_y \), we obtain a Hamilton path \( xzyw_y \ldots v_x \) in \( G \), and by reversing the orientation 
of \( C \), another Hamilton path \( xzyw_y \ldots v'_x \). Similarly, \( G \) contains two Hamilton paths
starting at \( y \): \( yzxw \ldots v_y \) and \( yzxw \ldots v'_y \) (see Fig. 2). As \( n \geq 3 \), \( |C| \geq 6 \) and at least one of the pairs \((v_x, v_y), (v'_x, v'_y)\) is a pair of distinct vertices; say \( v_x \neq v_y \).

![Diagram](image)

**Fig. 2**

Suppose first that \( d_G(x) + d_G(y) + d_H(v_x) + d_H(v_y) > 6n - 4 \). Then at least one of \( d_G(x) + d_H(v_x) \) and \( d_G(y) + d_H(v_y) \) is greater than \( 3n - 2 \), say
\[
d_G(x) + d_H(v_x) \geq 3n - 1.
\]

Consider the Hamilton \( x-v_x \) path \( P \) in \( G \); write \( P = xzyv_1v_2 \ldots v_{3n-4}v_x \). We may assume that \( xv_x \notin E(G) \), for otherwise \( P + v_xx \) is a Hamilton cycle in \( G \). Define
\[
\tilde{N}_P(x) = \{v_i : xv_{i+1} \in E(G)\} \quad \text{and} \quad N_P(v_x) = \{v_i : v_i v_x \in E(G)\}.
\]

We have \( |\tilde{N}_P(x)| \geq d_G(x) - 2 \) and \( |N_P(v_x)| = d_H(v_x) \), hence \( |\tilde{N}_P(x)| + |N_P(v_x)| \geq 3n - 3 \). By the pigeonhole principle, there exists \( 1 \leq i \leq 3n - 5 \) such that \( v_i v_x \in E(G) \) and \( xv_{i+1} \in E(G) \), hence a Hamilton cycle \( xzyv_1 \ldots v_{i}v_xv_{3n-4} \ldots v_{i+1} \) in \( G \).

Suppose now that
\[
d_G(x) + d_G(y) + d_H(v_x) + d_H(v_y) \leq 6n - 4. \tag{3.1}
\]

As \( H \) is obtained from \( K_3(n-1) \) by removing at least \( 4n - 5 - d_H(v_x) - d_H(v_y) \) edges, we have
\[
\|H\| \leq 3(n - 1)^2 - 4n + 5 + d_H(v_x) + d_H(v_y). \tag{3.2}
\]

Then \( \|G\| = \|H\| + d_G(x) + d_G(y) + d_G(z) - 2 \), together with (3.1), (3.2) and \( d_G(z) \leq 2n \), yield
\[
3n^2 - 2n + 2 = f_3(n) \leq \|G\| \leq 3(n - 1)^2 - 4n + 5 + d_H(v_x) + d_H(v_y) + d_G(x) + d_G(y) + 2n - 2 \leq 3n^2 - 2n + 2.
\]

This is only possible if \( \|H\| \) actually equals \( 3(n - 1)^2 - 4n + 5 + d_H(v_x) + d_H(v_y) \); i.e., for every pair of distinct vertices \( v_1, v_2 \in V(H) \setminus \{v_x, v_y\} \), either \( v_1, v_2 \) belong to
the same colour class of \( G \) or else they are adjacent. Note that for any such pair, \( H \) is obtained from \( K_3(n-1) \) by removing at least \( 4n - 5 - d_H(v_1) - d_H(v_2) + 1 \) edges, so that
\[
\| H \| \leq 3(n-1)^2 - 4n + 4 + d_H(v_1) + d_H(v_2) \tag{3.3}
\]
Now, if \( v'_x \neq v'_y \), then we can repeat the above calculations with (3.3) in place of (3.2), to get
\[
\| G \| \leq 3(n-1)^2 - 4n + 4 + d_H(v'_x) + d_H(v'_y) + d_G(x) + d_G(y) + 2n - 2 \leq 3n^2 - 2n + 1,
\]
provided \( d_G(x) + d_G(y) + d_H(v'_x) + d_H(v'_y) \leq 6n - 4 \). This however contradicts \( \| G \| \geq f_3(n) \), hence without loss of generality \( d_G(x) + d_H(v'_x) \geq 3n - 1 \), and we produce a Hamilton cycle from the path \( xzyw \ldots v'_x \), as above.

It does remain to consider the case \( v'_x = v'_y \). Then the Hamilton \( x-v'_x \) path \( P' \) in \( G \) is as in Figure 2; i.e., of the form \( P' = xzyw_yv_x \ldots w_xv'_x \). Since \( d(x) + d(y) \geq 2n + 1 \), then without loss of generality \( d(y) \geq n + 1 \geq 4 \), and hence \( y \) has a neighbour in \( G \), say \( w'_y \), different from \( z \), \( w_y \) and \( w_x \). It follows that \( w'_y \) on \( P' \) has a neighbour \( v''_x \) different from \( v_y \), \( v'_y \) and \( v_x \). In particular, \( v'_y \) and \( v''_y \) are adjacent, else from the same colour class. We now repeat our calculations with the endvertices of the Hamilton paths \( y-v'_y \) and \( xzyw'_y-v''_x \), with \( v'_y \) and \( v''_y \) in place of \( v_1 \) and \( v_2 \) in (3.3), to get that \( d_G(x) + d_H(v'_y) \geq 3n - 1 \) or \( d_G(x) + d_H(v''_x) \geq 3n - 1 \). This again implies a Hamilton cycle, which completes the proof.

**Corollary 3.1.** Let \( G \) be a balanced tripartite graph of order \( 3n \) and size at least \( 3n^2 - 2n + 2 \). Then \( G \) is pancyclic.

**Proof.** By a theorem of Bondy [1], pancyclicity of \( G \) follows from its hamiltonicity, provided \( \| G \| \geq \frac{|G|^2}{4} \). But \( f_3(n) = 3n^2 - 2n + 2 \geq \frac{(3n)^2}{4} \) for all \( n \in \mathbb{N} \).

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