Opuscula Mathematica • Vol. 30 • No. 2 • 2010

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FRÉCHET DIFFERENTIAL OF A POWER SERIES IN BANACH ALGEBRAS

Abstract. We present two new forms in which the Fréchet differential of a power series in a unitary Banach algebra can be expressed in terms of absolutely convergent series involving the commutant $C(T) : A \mapsto [A, T]$. Then we apply the results to study series of vector-valued functions on domains in Banach spaces and to the analytic functional calculus in a complex Banach space.

Keywords: Fréchet differentiation in Banach algebras, functional calculus.

Mathematics Subject Classification: 58C20, 46H, 47A60.

1. INTRODUCTION

In this work we consider a general formula for the Fréchet differential of a power series in a unitary Banach algebra $\mathcal{A}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In Theorem 2.11 it is proved that the Fréchet differential of the map $g : \mathcal{A} \ni T \mapsto \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{A}$, can be expressed in terms of absolutely convergent series involving the commutant $C(T) : \mathcal{A} \ni h \mapsto hT - Th \in \mathcal{A}$ with $T \in \mathcal{A}$, in three different forms containing $C(T)$, $C(T^n)$ or $C(T)^n$.

The two forms of the Fréchet differential of a power series containing $C(T)$ and $C(T^n)$ given in statements (1) and (2) of Theorem 2.11 are new. While we give a different proof with respect to [13], of the known formula in statement (3), containing the form $C(T)^n$.

These results are then applied to study series of vector-valued functions on domains in Banach spaces and also applied to analytic functional calculus in complex Banach spaces.

The commutant $C(T)$-forms in the differential of a power series g in a noncommutative Banach algebra $\mathcal{A}$, allows us to strongly simplify the formula of the derivative of a function of the type $\mathbb{R} \supseteq D \ni t \mapsto g(\mathcal{T}(t))$, whenever $C(\mathcal{T}(t))^n \left(\frac{d\mathcal{T}}{dt}(t)\right) = 0$ for some $n \in \mathbb{N} - \{0\}$, (see (2.37) and (2.38)). Here $\mathcal{T}$ is a derivable map defined on an open subset D of $\mathbb{R}$ and with values in $\mathcal{A}$, and $D \ni t \mapsto \frac{d\mathcal{T}}{dt}(t) \in \mathcal{A}$ is the derivative of $\mathcal{T}$. In a similar way we obtain simplification also for the more general case of differential maps (see Remark 2.13). To obtain these type of simplification procedures in calculating the derivative or the differential maps of functions valued in a noncommutative Banach algebra, represents one of the main motivations of this work.

Let us start fixing some notations. Let $\mathcal{A}$ be a unitary Banach algebra over $\mathbb{K}$ and denote by $B(\mathcal{A})$ the unitary Banach algebra of all bounded linear operators on $\mathcal{A}$ with the standard sup-norm. Define the linear maps $\mathcal{R}(T) : \mathcal{A} \ni h \mapsto Th \in \mathcal{A}$ and $\mathcal{L}(T) : \mathcal{A} \ni h \mapsto hT \in \mathcal{A}$ for all $T \in \mathcal{A}$, then it results that $\mathcal{R}, \mathcal{L} \in B(\mathcal{A}, B(\mathcal{A}))$ such that $\|\mathcal{R}\|_{B(\mathcal{A},B(\mathcal{A}))} \leq 1$ and $\|\mathcal{L}\|_{B(\mathcal{A},B(\mathcal{A}))} \leq 1$. Let $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$, with $\lambda \in \mathbb{K}$, the coefficients α_n in $\mathbb{K}$ and $R > 0$ its radius of convergence. In order to simplify the notations, we convein to denote by the same symbol g, both the functions: the numerical map $g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n \in \mathbb{K}$, with $\lambda \in \mathbb{K}$ such that $|\lambda| < R$, and the $\mathcal{A}$-valued map $g(T) = \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{A}$, with $T \in \mathcal{A}$ such that $\|T\|_{\mathcal{A}} < R$, where $\|\cdot\|_{\mathcal{A}}$ is the norm on $\mathcal{A}$, (see Def. 2.6).

Hence if we denote by $g^{(p)}$ the p-derivative of the numerical map g, we have $g^{(p)}(\lambda) = \sum_{n=p}^{\infty} p! \binom{n}{p} \alpha_n \lambda^{n-p} \in \mathbb{K}$, with $\lambda \in \mathbb{K}$ such that $|\lambda| < R$, while by considering $g^{(p)}$ as a $B(\mathcal{A})$-valued map we obtain $g^{(p)}(Q) = \sum_{n=0}^{\infty} p! \binom{n}{p} \alpha_n Q^{n-p} \in B(\mathcal{A})$, with $Q \in B(\mathcal{A})$ such that $\|Q\|_{B(\mathcal{A})} < R$. Thus we have for all $T \in \mathcal{A}$ such that $\|T\|_{\mathcal{A}} < R$

$$g^{(p)}(\mathcal{R}(T)) = \sum_{n=p}^{\infty} p! \binom{n}{p} \alpha_n \mathcal{R}(T)^{n-p} \in B(\mathcal{A}). \tag{1.1}$$

Denote by $B_r(\mathbf{0})$ a ball of radius $r > 0$ in $\mathcal{A}$ and let g be considered as an $\mathcal{A}$-valued map, so $g : B_R(\mathbf{0}) \ni T \mapsto \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{A}$, then $g^{[1]} : B_R(\mathbf{0}) \to B(\mathcal{A})$ denotes the Fréchet differential map of g. Therefore for $T \in B_R(\mathbf{0})$ the element $g^{[1]}(T) \in B(\mathcal{A})$ is uniquely determined by the following

$$\lim_{\substack{h \to \mathbf{0} \\ h \neq \mathbf{0}}} \frac{\|g(T+h) - g(T) - g^{[1]}(T)(h)\|_{\mathcal{A}}}{\|h\|_{\mathcal{A}}} = 0.$$

Finally given a series $N = \sum_{n=0}^{\infty} P_n$, where $P_n : \mathcal{A} \to B(\mathcal{A})$ for all $n \in \mathbb{N}$, we say that it converges absolutely uniformly on $B_r(\mathbf{0})$, or absolutely uniformly for $T \in B_r(\mathbf{0})$, if

$$\sum_{n=0}^{\infty} \sup_{T \in B_r(\mathbf{0})} \|P_n(T)\|_{B(\mathcal{A})} < \infty.$$

For a more general definition see Def. 2.8.

It is a well-known result that a power series $g(T) \doteqdot \sum_{n=0}^{\infty} \alpha_n T^n$ in a Banach algebra $\mathcal{A}$ is Fréchet differentiable term by term, the corresponding power series of its Fréchet differential $g^{[1]}$ is absolutely uniformly convergent on $B_r(\mathbf{0})$ in the norm topology of $B(\mathcal{A})$ for all $0 < r < R$, and finally that $g^{[1]}$ is continuous, where the radius of convergence R of $\sum_{n=0}^{\infty} \alpha_n \lambda^n$ is different to zero.

The Fréchet differentiability of g can be seen as a particular case of the Fréchet differentiability of a power series of polynomials between two Banach spaces over $\mathbb{K}$, whose proof for $\mathbb{K} = \mathbb{C}$, was given for the first time in [10]; while the one for $\mathbb{K} = \mathbb{R}$, was given for the first time in [11], used a weak form of Markoff's inequality for the derivative of a polynomial, see [14].

Our proof in Lemma 2.9 of the Fréchet differentiability term by term of g has the advantage of giving for the particular case of Banach algebras a unified approach for both the cases real and complex.

We are now able to state the results of the main **Theorem 2.11** of this work. We give for the first time the Fréchet differential $g^{[1]}$ of the $\mathcal{A}$-valued function $g(T) = \sum_{n=0}^{\infty} \alpha_n T^n$, in a $C(T)$-*depending absolutely uniformly convergent series on* $B_r(\mathbf{0})$, *for all* $0 < r < R$, *in* (1.2) *and in a* $C(T^k)$-*depending absolutely uniformly convergent series on* $B_r(\mathbf{0})$, *for all* $0 < r < R$ *and with* $k \geq 1$, *in* (1.3). This allows us to give immediately a simplified formula for the value $g^{[1]}(T)(h)$ in the case of the commutativity $[T, h] = \mathbf{0}$, with $T \in B_R(\mathbf{0})$ and $h \in \mathcal{A}$ (see Remark 2.13).

Finally we give a different proof with respect to [13] and in such a way generalizing that in [5], of the known formula in (1.4), in case $0 < r < \frac{R}{3}$.

1. For all $T \in B_R(\mathbf{0})$

$$g^{[1]}(T) = \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(T)^{n-1} - \left\{ \sum_{p=0}^{\infty} \left\{ \sum_{n=p+2}^{\infty} (n-p-1)\alpha_n \mathcal{L}(T)^{n-(2+p)} \right\} \mathcal{R}(T)^p \right\} C(T) \tag{1.2}$$

(here all the series converge absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$).

2. For all $T \in B_R(\mathbf{0})$

$$g^{[1]}(T) = \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(T)^{n-1} - \sum_{k=2}^{\infty} \left\{ \sum_{n=k}^{\infty} \alpha_n \mathcal{L}(T)^{n-k} \right\} C(T^{k-1}). \tag{1.3}$$

(here all the series converge absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$).

3. For all $T \in B_{\frac{R}{3}}(\mathbf{0})$

$$g^{[1]}(T) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(\mathcal{R}(T)) C(T)^{p-1}. \tag{1.4}$$

(Here the series converges absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < \frac{R}{3}$, $g^{(p)} : \mathbb{K} \to \mathbb{K}$ is the p-th derivative of the function g and $g^{(p)}(\mathcal{R}(T))$ is given in (1.1).) Finally we applied these results in Corollary 2.16, Remarks. 2.17 and 2.18, for describing the differential map of a series of vector-valued functions differentiable on domains in Banach spaces, and in Cor. 3.1 to study the differential map of the function $X \supseteq D \ni x \mapsto g(\mathcal{T}(x)) \in B(G)$. Here G and X are Banach spaces, D is an open set of X, g is the operator-valued map coming from the analytic functional calculus on G and $\mathcal{T} : D \to B(G)$ is a differential map, where $B(G)$ is the unitary Banach algebra of all bounded linear operators on G.

2. FRÉCHET DIFFERENTIAL OF A POWER SERIES OF DIFFERENTIABLE FUNCTIONS

Notations 2.1. We denote by $\mathbb{N}$ the set of all natural numbers $\{0, 1, 2, \ldots\}$. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\langle G, \|\cdot\|_G\rangle$, or simply G, be a Banach space over $\mathbb{K}$, then for all $a \in G$ and $r > 0$ we define the open ball centered in a of radius r, to be the following set $B_r(a) \doteqdot \{v \in G \mid \|v - a\|_G < r\}$, hence its closure in G is $\overline{B}_r(a) \doteqdot \overline{B_r(a)} = \{v \in G \mid \|v - a\|_G \leq r\}$.

Let F, G be two Banach spaces over $\mathbb{K}$, briefly $\mathbb{K}$-Banach spaces, then $\left\langle B(F, G), \|\cdot\|_{B(F,G)}\right\rangle$, will denote the $\mathbb{K}$-Banach space of all linear continuous mappings of F to G and $\|U\|_{B(F,G)} \doteqdot \sup_{\|v\|_F \leq 1} \|U(v)\|_G$, we also set $\left\langle B(G), \|\cdot\|_{B(G)}\right\rangle \doteqdot \left\langle B(G, G), \|\cdot\|_{B(G,G)}\right\rangle$.

Let $\{G_1, \ldots, G_n\}$ be a finite set of $\mathbb{K}$-Banach spaces, then $\left\langle \prod_{k=1}^n G_k, \|\cdot\|_{\prod_{k=1}^n G_k}\right\rangle$ is the Banach space, where $\prod_{k=1}^n G_k$ is the product of the vector spaces $\{G_1, \ldots, G_n\}$, and $\|(v_1, \ldots, v_n)\|_{\prod_{k=1}^n G_k} \doteqdot \max_{k \in \{1,\ldots,n\}} \|v_k\|_{G_k}$.

If $G_k = G$ for all $k \in \{1, \ldots, n\}$, then we will use the following notation $\langle G^n, \|\cdot\|_{G^n}\rangle \doteqdot \left\langle \prod_{k=1}^n G_k, \|\cdot\|_{\prod_{k=1}^n G_k}\right\rangle$. Let $\{F_1, \ldots, F_n, G\}$ be a finite set of $\mathbb{K}$-Banach spaces, then $B_n(\prod_{k=1}^n F_k; G)$ is the $\mathbb{K}$-vector space of all n-multilinear continuous mappings defined on $\prod_{k=1}^n F_k$ with values in G. If $F_k = F$ for all $k \in \{1, \ldots, n\}$, then we set $B_n(F^n; G) \doteqdot B_n(\prod_{k=1}^n F_k; G)$.

In the sequel we shall deal with Fréchet differentiable functions

$$f : U \subseteq F \to G$$

defined on an open set U of a $\mathbb{K}$-Banach space F and with values in a $\mathbb{K}$-Banach space G. Its Fréchet differential function will be denoted by

$$f^{[1]} : U \subseteq F \to B(F, G).$$

Recall that a map $f : U \subseteq F \to G$ is Fréchet differentiable at $x_0 \in U$ if there exists a $T \in B(F, G)$ such that

$$\lim_{\substack{h \to \mathbf{0} \\ h \neq \mathbf{0}}} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|_G}{\|h\|_F} = 0.$$

T is called the Fréchet differential of f at x_0 and is denoted by $f^{[1]}(x_0)$. f is Fréchet differentiable on U if f is Fréchet differentiable at each $x \in U$, and in this case the map $f^{[1]} : U \to B(F, G)$ is called the Fréchet differential function of f. For the properties of Fréchet differentials see Ch. 8 of the Dieudonne book [7].

Let $\mathcal{A}$ be an associative algebra over $\mathbb{K}$ (or briefly an associative algebra) then the standard Lie product on $\mathcal{A}$ is the following map

$$[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \ni (A, B) \mapsto [A, B] \doteqdot AB - BA \in \mathcal{A}$$

the commutator of A, B, and for all $T \in \mathcal{A}$ the adjoint linear map of T is so defined $ad(T) : \mathcal{A} \ni h \mapsto [T, h] \in \mathcal{A}$. We denote by $\mathcal{A}^{\mathcal{A}}$ the set of all maps from $\mathcal{A}$ to $\mathcal{A}$, let $\mathcal{R} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ and $\mathcal{L} : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ be defined by

$$\begin{cases} \mathcal{R}(T) : \mathcal{A} \ni h \mapsto Th \in \mathcal{A}, \\ \mathcal{L}(T) : \mathcal{A} \ni h \mapsto hT \in \mathcal{A}, \end{cases} \tag{2.1}$$

for all $T \in \mathcal{A}$. We also define the map $C : \mathcal{A} \to \mathcal{A}^{\mathcal{A}}$ by

$$C \doteqdot -ad = \mathcal{L} - \mathcal{R}.$$

We consider for any $n \in \mathbb{N}$ the following mapping

$$u_n : \mathcal{A} \ni T \mapsto T^n \in \mathcal{A}.$$

A Banach algebra over $\mathbb{K}$ (or briefly Banach algebra), see for example [6] or [12], is an associative algebra $\mathcal{A}$ over $\mathbb{K}$ with a norm $\|\cdot\|$ on it such that $\langle \mathcal{A}, \|\cdot\| \rangle$ is a Banach space and for all $A, B \in \mathcal{A}$ we have

$$\|AB\| \leq \|A\| \|B\|.$$

If $\mathcal{A}$ contains the unit element then it is called a unitary Banach algebra. We assume for any unitary Banach algebra with unit $\mathbf{1}$ that $\|\mathbf{1}\| = 1$.

It is easy to verify directly that for all $T_1, T_2 \in \mathcal{A}$

$$[\mathcal{R}(T_1), \mathcal{L}(T_2)] = \mathbf{0}. \tag{2.2}$$

By recalling definition (2.1) we have for all $T, h \in \mathcal{A}$ that $\|\mathcal{R}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}} \|h\|_{\mathcal{A}}$, and $\|\mathcal{L}(T)(h)\|_{\mathcal{A}} \leq \|T\|_{\mathcal{A}} \|h\|_{\mathcal{A}}$, hence

$$\mathcal{R}(T), \mathcal{L}(T) \in B(\mathcal{A}) \tag{2.3}$$

with

$$\|\mathcal{R}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}, \|\mathcal{L}(T)\|_{B(\mathcal{A})} \leq \|T\|_{\mathcal{A}}, \|C(T)\|_{B(\mathcal{A})} \leq 2\|T\|_{\mathcal{A}}. \tag{2.4}$$

Since $\mathcal{L}$ and $\mathcal{R}$ are linear mappings we can conclude that

$$\begin{cases} \mathcal{L}, \mathcal{R} \in B(\mathcal{A}, B(\mathcal{A})), \\ \|\mathcal{R}\|_{B(\mathcal{A}, B(\mathcal{A}))}, \|\mathcal{L}\|_{B(\mathcal{A}, B(\mathcal{A}))} \leq 1. \end{cases} \tag{2.5}$$

Finally for $l, k \in \mathbb{N}$ and $\{A_j\}_{j=0}^{l} \subset \mathcal{A}$ we use the following conventions $\prod_{j=k}^{l} A_j = \mathbf{1}$ and $\sum_{j=k}^{l} A_j = \mathbf{0}$ for $l < k$.

We now present a simple formula which will be used later to decompose the commutator $C(T^n)$ in terms of $C(T)$.

Lemma 2.2. *Let $\mathcal{A}$ be an associative algebra, then for all $n \in \mathbb{N}$ and $A_1, \ldots, A_{n+1}, B \in \mathcal{A}$ we have*

$$\left[\prod_{k=1}^{n+1} A_k, B\right] = \sum_{s=0}^{n} \left(\prod_{k=1}^{s} A_k\right) [A_{s+1}, B] \prod_{j=s+2}^{n+1} A_j. \tag{2.6}$$

Proof. We shall prove the statement by induction. For $n = 0$ (2.6) is trivial. Let (2.6) be true for $n-1$, then for each $A_1, \ldots, A_{n+1}, B \in \mathcal{A}$ we have

$$\left[\prod_{k=1}^{n+1} A_k, B\right] = \left[\left(\prod_{k=1}^{n} A_k\right) A_{n+1}, B\right] =$$

and since $[A_1 A_2, B] = A_1[A_2, B] + [A_1, B]A_2$ it follows

$$= \left(\prod_{k=1}^{n} A_k\right) [A_{n+1}, B] + \left[\prod_{k=1}^{n} A_k, B\right] A_{n+1} =$$

and by hypothesis of the induction we conclude

$$= \left(\prod_{k=1}^{n} A_k\right) [A_{n+1}, B] + \sum_{s=0}^{n-1} \left(\prod_{k=1}^{s} A_k\right) [A_{s+1}, B] \left(\prod_{j=s+2}^{n} A_j\right) A_{n+1} =$$

$$= \sum_{s=0}^{n} \left(\prod_{k=1}^{s} A_k\right) [A_{s+1}, B] \prod_{j=s+2}^{n+1} A_j.$$

□

Corollary 2.3. *Let $\mathcal{A}$ be a unitary associative algebra, then for all $n \in \mathbb{N}$ and $T \in \mathcal{A}$ we have*

$$C(T^{n+1}) = \sum_{s=0}^{n} \mathcal{R}(T)^s C(T) \mathcal{L}(T)^{n-s} = \sum_{s=0}^{n} \mathcal{R}(T)^s \mathcal{L}(T)^{n-s} C(T).$$

Proof. The second equality follows by Lemma 2.2, where $A_1 = A_2 = \ldots = A_{n+1} = T$, the first one by the second and (2.2). □

The following equality is stated without proof in the exercise 19, §1, Ch. 1 of [3]. For the sake of completeness we give a proof.

Lemma 2.4. *Let $\mathcal{A}$ be a unitary associative algebra, then we have for all $T \in \mathcal{A}$ and $n \in \mathbb{N}$ that*

$$C(T)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{R}(T)^k \mathcal{L}(T)^{n-k}.$$

Proof. Since by (2.2) $\mathcal{L}(T)$ and $\mathcal{R}(T)$ commute the statement follows. □

Lemma 2.5. *Let $\mathcal{A}$ be a unitary associative algebra. Then for all $T \in \mathcal{A}$ and $n \in \mathbb{N}$ we have*

$$\sum_{p=1}^{n} \binom{n}{p} \mathcal{R}(T)^{n-p} C(T)^{p-1} = \sum_{s=1}^{n} \mathcal{R}(T)^{n-s} \mathcal{L}(T)^{s-1}. \tag{2.7}$$

Proof. Since $\mathcal{L} = C + \mathcal{R}$ and since $C(T)$ and $\mathcal{R}(T)$ commute (cf.(2.2)) we have

$$\begin{aligned}
\sum_{p=1}^{n} \mathcal{R}(T)^{n-p} \mathcal{L}(T)^{p-1} &= \sum_{p=1}^{n} \mathcal{R}(T)^{n-p} (C(T) + \mathcal{R}(T))^{p-1} = \\
&= \sum_{p=1}^{n} \mathcal{R}(T)^{n-p} \sum_{k=0}^{p-1} \binom{p-1}{k} \mathcal{R}(T)^{p-1-k} C(T)^{k} = \\
&= \sum_{k=0}^{n-1} \left(\sum_{p=k+1}^{n} \binom{p-1}{k} \right) \mathcal{R}(T)^{n-1-k} C(T)^{k} = \\
&= \sum_{s=1}^{n} \left(\sum_{p=s}^{n} \binom{p-1}{s-1} \right) \mathcal{R}(T)^{n-s} C(T)^{s-1} = \\
&= \sum_{s=1}^{n} \binom{n}{s} \mathcal{R}(T)^{n-s} C(T)^{s-1}.
\end{aligned}$$

□

Definition 2.6. Let $\mathcal{A}$ be a unitary Banach algebra, $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$, where the coefficients $\alpha_n \in \mathbb{K}$ and has the radius of convergence $R > 0$. Then for all $T \in \mathcal{A}$ such that $\|T\|_{\mathcal{A}} < R$ we can define

$$f(T) \doteqdot \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{A}.$$

It is well-known that the map u_n is Fréchet differentiable. For the sake of completeness we give a direct proof of the Fréchet differential function of u_n in several forms which will be used in the sequel.

Lemma 2.7. *Let $\mathcal{A}$ be a unitary Banach algebra. Then for all $n \in \mathbb{N}$ the map $u_n : \mathcal{A} \ni T \mapsto T^n \in \mathcal{A}$ is Fréchet differentiable and its Fréchet differential map $u_n^{[1]} : \mathcal{A} \to B(\mathcal{A})$ is such that for all $T \in \mathcal{A}$ and $n \in \mathbb{N}$*

$$\begin{aligned}
u_n^{[1]}(T) &= \sum_{p=1}^{n} \mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1} = \\
&= n\mathcal{L}(T)^{n-1} - \sum_{k=2}^{n} \mathcal{L}(T)^{n-k} C(T^{k-1}) = \\
&= \sum_{p=1}^{n} \binom{n}{p} \mathcal{R}(T)^{n-p} C(T)^{p-1} = \\
&= n\mathcal{L}(T)^{n-1} - \sum_{s=0}^{n-2} (n-s-1)\mathcal{L}(T)^{n-(s+2)}\mathcal{R}(T)^{s} C(T),
\end{aligned} \tag{2.8}$$

and

$$\|u_n^{[1]}(T)\|_{B(\mathcal{A})} \leq n\|T\|_{\mathcal{A}}^{n-1}. \tag{2.9}$$

Proof. For brevity in this proof we write $\|\cdot\|$ for $\|\cdot\|_{\mathcal{A}}$. The cases $n = 0, 1$ are trivial. Assume that $n \in \mathbb{N} - \{0, 1\}$ and $T, h \in \mathcal{A}$

$$\sum_{p=1}^{n} \mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}(h) = hT^{n-1} + ThT^{n-2} + \ldots + T^{k-1}hT^{n-k} + \ldots + T^{n-1}h$$

so

$$(T+h)^n = T^n + \sum_{p=1}^{n} \mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}(h) + \mathfrak{T}(h; T; 2).$$

Here $\mathfrak{T}(h; T; 2)$ is a polynomial in the two variables T and h each monomial of which is at least of degree 2 with respect to the variable h. Hence

$$\begin{aligned}
&\lim_{h\to\mathbf{0}} \frac{\|(T+h)^n - T^n - \sum_{p=1}^{n} \mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1}(h)\|}{\|h\|} = \\
&\qquad = \lim_{h\to\mathbf{0}} \frac{\|\mathfrak{T}(h; T; 2)\|}{\|h\|} \leq \lim_{h\to\mathbf{0}} \frac{\mathfrak{T}(\|h\|; \|T\|; 2)}{\|h\|} = 0.
\end{aligned} \tag{2.10}$$

Here $\mathfrak{T}(\|h\|; \|T\|; 2)$ is the polynomial in the variables $\|h\|$ and $\|T\|$ obtained by replacing in $\mathfrak{T}(h; T; 2)$ the variable h with $\|h\|$ and T with $\|T\|$. Hence

$$u_n^{[1]}(T) = \sum_{p=1}^{n} \mathcal{R}(T)^{n-p}\mathcal{L}(T)^{p-1} \tag{2.11}$$

and (2.9) and the first of equalities (2.8) follow. Therefore we have for all $T \in \mathcal{A}$ and $h \in \mathcal{A}$

$$\begin{aligned}
u_n^{[1]}(T)(h) &= hT^{n-1} + ThT^{n-2} + \ldots + T^{k-1}hT^{n-k} + \ldots + T^{n-1}h = \\
&= hT^{n-1} + [T, h]T^{n-2} + hT^{n-1} + \ldots + [T^{k-1}, h]T^{n-k} + hT^{n-1} + \ldots + \\
&\quad + [T^{n-1}, h] + hT^{n-1},
\end{aligned}$$

hence

$$u_n^{[1]}(T)(h) = nhT^{n-1} + \sum_{k=2}^{n} [T^{k-1}, h]T^{n-k}. \tag{2.12}$$

This is the second equality in (2.8). The fourth equality in (2.8) follows by the second one, by the commutativity property in (2.2) and by Corollary 2.3. By the first equality in (2.8) and Lemma 2.5 we obtain the third equality in (2.8). □

Definition 2.8. Let S be a no empty set and X be a Banach space over $\mathbb{K}$, then we define

$$\mathcal{B}(S, X) \doteqdot \left\{ F : S \to X \mid \|F\|_{\mathcal{B}(S,X)} \doteqdot \sup_{u \in S} \|F(u)\|_X < \infty \right\}. \tag{2.13}$$

Then $\langle \mathcal{B}(S, X), \|\cdot\|_{\mathcal{B}(S,X)} \rangle$ is a Banach space over $\mathbb{K}$ and the convergence in it is called the *uniform convergence on S in $\|\cdot\|_X$-topology* , or simply when this does not cause confusion, the *uniform convergence on S*, (see Ch. 10 of [1]).

Let $\{f_\alpha\}_{\alpha \in D} \subset \mathcal{B}(S, X)$ then the sum $\sum_{\alpha \in D} f_\alpha$ *converges uniformly on S*[1)] if the net of all finite partial sums converges in $\mathcal{B}(S, X)$, i.e. by denoting with $\mathcal{P}_\omega(D)$ the direct ordered set of all finite subsets of D ordered by inclusion, there exists $W \in \mathcal{B}(S, X)$ such that

$$\lim_{J \in \mathcal{P}_\omega(D)} \sup_{u \in S} \left\| W(u) - \sum_{\alpha \in J} f_\alpha(u) \right\|_X = 0. \tag{2.14}$$

The sum $\sum_{\alpha \in D} f_\alpha$ *converges absolutely uniformly on S* or *converges absolutely uniformly for $u \in S$* if

$$\sum_{\alpha \in D} \sup_{u \in S} \|f_\alpha(u)\|_X \doteq \lim_{J \in \mathcal{P}_\omega(D)} \sum_{\alpha \in J} \sup_{u \in S} \|f_\alpha(u)\|_X < \infty.$$

Since $\mathcal{B}(S, X)$ is a Banach space, the absolute uniform convergence implies uniform convergence. Similar definitions for sequences and in particular for series $\sum_{n=0}^{\infty} f_n$, by replacing D with $\mathbb{N}$, while $\sum_{\alpha \in J}$ and $\lim_{\alpha \in \mathcal{P}_\omega(D)}$ with resp. $\sum_{n=0}^{N}$ and $\lim_{N \to \infty}$, finally $\sum_{\alpha \in D}$ with $\sum_{n=0}^{\infty}$.

Now we shall show that a power series $g(T) \doteqdot \sum_{n=0}^{\infty} \alpha_n T^n$ in a Banach algebra $\mathcal{A}$ is Fréchet differentiable term by term, the corresponding power series of its Fréchet differential $g^{[1]}$ is uniformly convergent on $B_r(\mathbf{0})$ in the norm topology of $B(\mathcal{A})$ for all $0 < r < R$, and finally that $g^{[1]}$ is continuous, where the radius of convergence R of $\sum_{n=0}^{\infty} \alpha_n \lambda^n$ is different to zero. The proofs are based on the well-known results stating that uniform convergence in Banach spaces, preserves Fréchet differentiability and continuity, see Theorem 8.6.3. of the [7] for the first and Theorem (2), §1.6., Ch. 10 of the [1] for the second one.

The Fréchet differentiability of g can be seen as a particular case of the Fréchet differentiability of a power series of polynomials between two Banach spaces over

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1) For the general definition of a summable family in a Hausdorff commutative topological group G see Ch. 3, §5.1. of [1], in our case $G = \mathcal{B}(S, X)$.

$\mathbb{K}$, whose proof for $\mathbb{K} = \mathbb{C}$, was given for the first time in [10]; while the one for $\mathbb{K} = \mathbb{R}$, given for the first time in [11], used a weak form of Markoff's inequality for the derivative of a polynomial, see [14].

If E_1 and E_2 are Banach spaces, a homogeneous polynomial of degree n on E_1 to E_2 is a function $p_n(x)$ with values in E_2, defined for all elements x in E_1 and having the properties (a) $p_n(tx) = t^n p_n(x)$, (b) $p_n(x + ty)$ is a polynomial of degree not greater than n in the numerical variable t, with coefficients in E_2, (c) $\|p_n(x)\| \leq m\|x\|^n$ for some constant m and every x; the smallest m satisfying (c) is called the modulus $m(p_n)$ of the homogeneous polynomial. A series of the form $f(x) = \sum_0^\infty p_n(x)$ is called a power series.

Michal in [11], considers real Banach spaces only, and defines the radius of analyticity r of the power series as the radius of convergence of the ordinary power series $\sum_0^\infty m(p_n)t^n$. He proves that if $r > 0$ the function $f(x)$ has Fréchet differentials of all orders when $\|x\| < r$ and that these differentials are given by successive term-by-term differentiation of the series for $f(x)$. For complex Banach spaces this result is well known. It was first proved in [10].

Our proof in Lemma 2.9 has the advantage of giving for the particular case of Banach algebras a unified approach for both the cases real and complex.

Lemma 2.9 (Fréchet differentiability of a power series in a Banach algebra)**.** *Let $\mathcal{A}$ be a unitary Banach algebra, $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ be such that the radius of convergence of the series $g(\lambda) \doteqdot \sum_{n=0}^\infty \alpha_n \lambda^n$ is $R > 0$.*

1. *The series*

$$\sum_{n=0}^\infty \alpha_n u_n$$

converges absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$.[2)]
Hence we can define the map $g : B_R(\mathbf{0}) \to \mathcal{A}$ as $g(T) \doteqdot \sum_{n=0}^\infty \alpha_n u_n(T)$.

2. *g is Fréchet differentiable on $B_R(\mathbf{0})$ and*

$$g^{[1]} = \sum_{n=1}^\infty \alpha_n u_n^{[1]}. \tag{2.15}$$

Here the series converges absolutely uniformly on $B_r(\mathbf{0})$, for all $0 < r < R$[3)] *and $g^{[1]}$ is continuous.*

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2) By Def. 2.8,

$$\sum_{n=0}^\infty \sup_{T\in B_r(\mathbf{0})} \|\alpha_n T^n\|_{\mathcal{A}} < \infty$$

for all $0 < r < R$.

3) By Def. 2.8

$$\sum_{n=1}^\infty \sup_{T\in B_r(\mathbf{0})} \|\alpha_n u_n^{[1]}(T)\|_{B(\mathcal{A})} < \infty$$

for all $0 < r < R$.

Proof. For all $r \in (0, R)$, $T \in B_r(\mathbf{0})$ and $n \in \mathbb{N}$ we have $\|\alpha_n T^n\|_{\mathcal{A}} \leq |\alpha_n|\|T\|_{\mathcal{A}}^n \leq |\alpha_n| r^n$, so

$$\sum_{n=0}^{\infty} \sup_{T \in B_r(\mathbf{0})} \|\alpha_n T^n\|_{\mathcal{A}} \leq \sum_{n=0}^{\infty} |\alpha_n| r^n < \infty.$$

Which is statement (1).

By (2.9) for all $0 < r < R$

$$\sum_{n=0}^{\infty} \sup_{T \in B_r(\mathbf{0})} \|\alpha_n u_n^{[1]}(T)\|_{B(\mathcal{A})} \leq \sum_{n=0}^{\infty} |\alpha_n| n r^{n-1} < \infty.$$

Hence the series $\sum_{n=0}^{\infty} \alpha_n u_n^{[1]}$ converges absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$. Thus the mapping

$$T \ni B_R(\mathbf{0}) \subset \mathcal{A} \mapsto \sum_{n=0}^{\infty} \alpha_n u_n^{[1]}(T) \in B(\mathcal{A}) \tag{2.16}$$

is well defined on $B_R(\mathbf{0})$ and the series converges uniformly for $T \in B_r(\mathbf{0})$ for all $0 < r < R$. Hence we can apply Theorem 8.6.3. of the [7] and then deduce (2.15).

Now it remains to show the last part of the statement (2), i.e. the continuity of the differential function $g^{[1]}$. By the first part of Lemma 2.7 applied to the unitary Banach algebra $B(\mathcal{A})$ and by (2.5) for all $n \in \mathbb{N}$ the maps

$$\mathcal{A} \ni T \mapsto \mathcal{L}(T)^n \in B(\mathcal{A}),\ \mathcal{A} \ni T \mapsto \mathcal{R}(T)^n \in B(\mathcal{A}) \tag{2.17}$$

and the product on $B(\mathcal{A}) \times B(\mathcal{A})$ are continuous in the norm topology of $B(\mathcal{A})$, so by the first equality in (2.8) for all $n \in \mathbb{N}$

$$u_n^{[1]} : \mathcal{A} \to B(\mathcal{A}) \text{ is continuous.} \tag{2.18}$$

By (2.18), the uniform convergence of which in the first part of statement (2), and finally by the fact that the set of all continuous maps is closed with respect to the topology of uniform convergence, see for example Theorem (2), §1.6., Ch. 10 of the [1], we conclude that for all $0 < r < R$ the mapping $g^{[1]} \upharpoonright B_r(\mathbf{0}) : B_r(\mathbf{0}) \subset \mathcal{A} \to B(\mathcal{A})$ is continuous. This ends the proof of statement (2). $\square$

Remark 2.10. By statement (2) of Lemma 2.9 we have

$$g^{[1]}(T)(h) = \sum_{n=1}^{\infty} \alpha_n u_n^{[1]}(T)(h).$$

Here the series converges absolutedly uniformly for $(T, h) \in B_r(\mathbf{0}) \times B_L(\mathbf{0})$, for all $L > 0$ and $0 < r < R$, i.e.

$$\sum_{n=1}^{\infty} \sup_{(T,h) \in B_r(\mathbf{0}) \times B_L(\mathbf{0})} |\alpha_n| \|u_n^{[1]}(T)(h)\|_{\mathcal{A}} < \infty.$$

Theorem 2.11 (**Fréchet differential of a power series**). *Let $\mathcal{A}$ be a unitary Banach algebra, $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ be such that the radius of convergence of the series $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is $R > 0$. Then:*

1. *For all $T \in B_R(\mathbf{0})$*

$$g^{[1]}(T) = \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(T)^{n-1} - \left\{ \sum_{p=0}^{\infty} \left\{ \sum_{n=p+2}^{\infty} (n-p-1)\alpha_n \mathcal{L}(T)^{n-(2+p)} \right\} \mathcal{R}(T)^p \right\} C(T). \tag{2.19}$$

Here all the series converge absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$.

2. *For all $T \in B_R(\mathbf{0})$*

$$g^{[1]}(T) = \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(T)^{n-1} - \sum_{k=2}^{\infty} \left\{ \sum_{n=k}^{\infty} \alpha_n \mathcal{L}(T)^{n-k} \right\} C(T^{k-1}). \tag{2.20}$$

Here all the series converge absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$.

3. *For all $T \in B_{\frac{R}{3}}(\mathbf{0})$*

$$g^{[1]}(T) = \sum_{p=1}^{\infty} \frac{1}{p!} (g)^{(p)}(\mathcal{R}(T)) C(T)^{p-1}. \tag{2.21}$$

Here the series converges absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < \frac{R}{3}$ and $g^{(p)} : \mathbb{K} \to \mathbb{K}$ is the p-th derivative of the function g.

Remark 2.12. If $R/3 \leq r < R$ then in general the series in (2.21) may not converge, see for a counterexample the [5].

Remark 2.13. By using Def. 2.6 we have for all $T \in B_R(\mathbf{0})$

$$\frac{1}{p!} g^{(p)}(\mathcal{R}(T)) = \sum_{n=p}^{\infty} \binom{n}{p} \alpha_n \mathcal{R}(T)^{n-p} \in B(\mathcal{A}).$$

Clearly both (2.19) and (2.20) immediately imply that if $T, h \in \mathcal{A}$ are such that $[T, h] = \mathbf{0}$, then

$$g^{[1]}(T)(h) = \sum_{n=1}^{\infty} n\alpha_n h T^{n-1}.$$

Proof of Theorem 2.11. By Lemmes 2.7 and 2.9

$$\begin{aligned} g^{[1]}(T) &= \sum_{n=1}^{\infty} \alpha_n u_n^{[1]}(T) = \\ &= \alpha_1 \mathbf{1} + \sum_{n=2}^{\infty} \alpha_n \left(n\mathcal{L}(T)^{n-1} - \sum_{s=0}^{n-2} (n-s-1) \mathcal{L}(T)^{n-(s+2)} \mathcal{R}(T)^s C(T) \right). \end{aligned}$$

By (2.4) for all $0 < r < R$

$$\sum_{n=2}^{\infty} \sup_{T \in B_r(\mathbf{0})} \|\alpha_n n \mathcal{L}(T)^{n-1}\|_{B(\mathcal{A})} \leq \sum_{n=2}^{\infty} n|\alpha_n| r^{n-1} < \infty$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty} \sum_{s=0}^{n-2} \sup_{T \in B_r(\mathbf{0})} \|\alpha_n (n-s-1) \mathcal{L}(T)^{n-(s+2)} \mathcal{R}(T)^s C(T)\|_{B(\mathcal{A})} \leq \\
&\leq \sum_{n=2}^{\infty} |\alpha_n| \sum_{s=0}^{n-2} (n-s-1) r^{n-2}(2r) = \sum_{n=2}^{\infty} |\alpha_n|(n-1) n r^{n-1} < \infty.
\end{aligned} \tag{2.22}$$

Therefore

$$g^{[1]}(T) = \sum_{n=1}^{\infty} \alpha_n n \mathcal{L}(T)^{n-1} - \sum_{n=2}^{\infty} \alpha_n \sum_{s=0}^{n-2} (n-s-1) \mathcal{L}(T)^{n-(s+2)} \mathcal{R}(T)^s C(T). \tag{2.23}$$

Here each series converges absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$. Inequality (2.22) also implies that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \alpha_n \sum_{s=0}^{n-2} (n-s-1) \mathcal{L}(T)^{n-(s+2)} \mathcal{R}(T)^s C(T) = \\
&= \sum_{s=0}^{\infty} \sum_{n=s+2}^{\infty} (n-s-1) \alpha_n \mathcal{L}(T)^{n-(2+s)} \mathcal{R}(T)^s C(T) = \\
&= \left\{ \sum_{s=0}^{\infty} \left\{ \sum_{n=s+2}^{\infty} (n-s-1) \alpha_n \mathcal{L}(T)^{n-(2+s)} \right\} \mathcal{R}(T)^s \right\} C(T).
\end{aligned}$$

Here each series converging absolutely uniformly on $B_r(\mathbf{0})$ for all $0 < r < R$ and statement (1) follows. Using (2.4) we can estimate

$$\begin{aligned}
&\sum_{n=2}^{\infty} \sum_{k=2}^{n} \sup_{T \in B_r(\mathbf{0})} \left\| \sum_{s=0}^{k-2} \alpha_n \mathcal{L}(T)^{n-k} \mathcal{R}(T)^s \mathcal{L}(T)^{k-(2+s)} C(T) \right\|_{B(\mathcal{A})} = \\
&= \sum_{n=2}^{\infty} \sum_{k=2}^{n} \sup_{T \in B_r(\mathbf{0})} \left\| \sum_{s=0}^{k-2} \alpha_n \mathcal{R}(T)^s \mathcal{L}(T)^{n-(2+s)} C(T) \right\|_{B(\mathcal{A})} \leq \\
&\leq \sum_{n=2}^{\infty} \sum_{k=2}^{n} \sum_{s=0}^{k-2} |\alpha_n| \sup_{T \in B_r(\mathbf{0})} \|\mathcal{R}(T)\|_{B(\mathcal{A})}^{s} \|\mathcal{L}(T)\|_{B(\mathcal{A})}^{n-(2+s)} \|C(T)\|_{B(\mathcal{A})} \leq \\
&\leq 2 \sum_{n=2}^{\infty} \sum_{k=2}^{n} \sum_{s=0}^{k-2} |\alpha_n| \sup_{T \in B_r(\mathbf{0})} \|T\|_{\mathcal{A}}^{n-1} = \\
&= \sum_{n=2}^{\infty} n(n-1)|\alpha_n| \sup_{T \in B_r(\mathbf{0})} \|T\|_{\mathcal{A}}^{n-1} = \sum_{n=2}^{\infty} n(n-1)|\alpha_n| r^{n-1} < \infty,
\end{aligned} \tag{2.24}$$

and therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sum_{s=0}^{k-2}\alpha_n\mathcal{L}(T)^{n-k}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)=\\
&=\sum_{k=2}^{\infty}\sum_{n=k}^{\infty}\sum_{s=0}^{k-2}\alpha_n\mathcal{L}(T)^{n-k}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)=\\
&=\sum_{k=2}^{\infty}\sum_{n=k}^{\infty}\alpha_n\mathcal{L}(T)^{n-k}\sum_{s=0}^{k-2}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)=\\
&=\sum_{k=2}^{\infty}\left\{\sum_{n=k}^{\infty}\alpha_n\mathcal{L}(T)^{n-k}\right\}C(T^{k-1}).
\end{aligned}\tag{2.25}$$

All the series uniformly converge for $T \in B_r(\mathbf{0})$. Here in the last equality we used Corollary 2.3 and the fact that $\mathcal{L}(C(T^{k-1})) \in B(B(\mathcal{A}))$. Moreover by (2.2)

$$\begin{aligned}
&\sum_{n=2}^{\infty}\sum_{k=2}^{n}\sum_{s=0}^{k-2}\alpha_n\mathcal{L}(T)^{n-k}\mathcal{R}(T)^s\mathcal{L}(T)^{k-(2+s)}C(T)=\\
&=\sum_{n=2}^{\infty}\sum_{s=0}^{n-2}(n-s-1)\alpha_n\mathcal{L}(T)^{n-(2+s)}\mathcal{R}(T)^sC(T)
\end{aligned}$$

hence by (2.25) and (2.23) we obtain statement (2). Finally we have for all $r < \frac{R}{3}$

$$\begin{aligned}
\mathfrak{A} &\doteqdot \sum_{n=1}^{\infty}\sum_{p=1}^{n}\sup_{T\in B_r(\mathbf{0})}\left\|\alpha_n\binom{n}{p}\mathcal{R}(T)^{n-p}C(T)^{p-1}\right\|_{B(\mathcal{A})}\leq\\
&\leq\sum_{n=1}^{\infty}\sum_{p=1}^{n}\binom{n}{p}|\alpha_n|\sup_{T\in B_r(\mathbf{0})}\|\mathcal{R}(T)\|_{B(\mathcal{A})}^{n-p}\|C(T)\|_{B(\mathcal{A})}^{p-1}\leq\\
&\leq\sum_{n=1}^{\infty}\sum_{p=1}^{n}\binom{n}{p}|\alpha_n|\sup_{T\in B_r(\mathbf{0})}\|T\|_{\mathcal{A}}^{n-p}2^{p-1}\|T\|_{\mathcal{A}}^{p-1}=\sum_{n=1}^{\infty}\sum_{p=1}^{n}\binom{n}{p}|\alpha_n|r^{n-1}2^{p-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathfrak{A} &\leq \sum_{n=1}^{\infty}|\alpha_n|r^{n-1}\sum_{p=1}^{n}\binom{n}{p}2^{p-1}<\sum_{n=1}^{\infty}|\alpha_n|r^{n-1}\sum_{p=0}^{n}\binom{n}{p}2^{p}=\\
&=\sum_{n=1}^{\infty}|\alpha_n|r^{n-1}3^n=r^{-1}\sum_{n=1}^{\infty}|\alpha_n|(3r)^n<\infty.
\end{aligned}$$

Thus by the third equality in Lemma 2.7 and Lemma 2.9 we obtain statement (3). □

The previous Theorem 2.11 is the main result of the present work. Let $\mathcal{A}$ be a unitary $\mathbb{K}$-Banach algebra and $\sum_{n=0}^{\infty}\alpha_n\lambda^n$ a series at coefficients in $\mathbb{K}$ having radius of convergence $R>0$. We give for the first time the Fréchet differential $g^{[1]}$ of the $\mathcal{A}$-valued function $g(T)=\sum_{n=0}^{\infty}\alpha_n T^n$, in a *$C(T)$-depending absolutely uniformly convergent series on $B_r(\mathbf{0})$, for all $0<r<R$, in statement* (1)*; and in a $C(T^k)$-depending absolutely uniformly convergent series on $B_r(\mathbf{0})$, for all $0<r<R$ and with $k\geq 1$, in statement* (2). This allows us to give immediately a simplified formula for the value $g^{[1]}(T)(h)$ in case of the commutativity $[T,h]=\mathbf{0}$, with $T\in B_R(\mathbf{0})$ and $h\in\mathcal{A}$, (see Remark 2.13).

Finally we give a different proof with respect to [13] and in such a way generalizing that in [5], of the known formula in statement (3), in case $0<r<\frac{R}{3}$, see Remark 2.15 and Remark 2.20.

Remark 2.14. We note that the formula (2.19) explicitly contains $C(T)$ as a factor, formula (2.20) gives an expansion in terms of $C(T^k)$ and finally formula (2.21) gives an expansion in terms of $C(T)^k$.

Remark 2.15. For all T such that $\|T\|<\frac{R}{3}$ and for all $h\in\mathcal{A}$ we have

$$g^{[1]}(T)(h)=\sum_{p=1}^{\infty}\frac{1}{p!}g^{(p)}(T)C(T)^{p-1}(h). \tag{2.26}$$

Here the series is uniformly convergent for $(T,h)\in B_r(\mathbf{0})\times B_L(\mathbf{0})$ for all $0<r<\frac{R}{3}$ and $L>0$, i.e.

$$\sum_{p=1}^{\infty}\frac{1}{p!}\sup_{(T,h)\in B_r(\mathbf{0})\times B_L(\mathbf{0})}\|g^{(p)}(T)C(T)^{p-1}(h)\|_{\mathcal{A}}<\infty.$$

Corollary 2.16 (Fréchet differential of a power series of differentiable functions defined on an open set of a $\mathbb{K}$-Banach Space and at values in a $\mathbb{K}$-Banach algebra $\mathcal{A}$)**.** *Let $\mathcal{A}$ be a unitary Banach algebra, and $\{\alpha_n\}_{n\in\mathbb{N}}\subset\mathbb{K}$ be such that the radius of convergence of the series $g(\lambda)\doteqdot\sum_{n=0}^{\infty}\alpha_n\lambda^n$ is $R>0$ and $0<r<R$. Finally let X be a Banach space over $\mathbb{K}$, $D\subseteq X$ an open set in X and $\mathcal{T}:D\to\mathcal{A}$ a Fréchet differentiable mapping such that $\mathcal{T}(D)\subseteq B_r(\mathbf{0})$ or alternatively D is convex and bounded and $\sup_{x\in D}\|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})}<\infty$. If we set $\widetilde{r}\doteqdot\sup_{x\in D}\|\mathcal{T}(x)\|_{\mathcal{A}}$, then:*

1. *$\widetilde{r}<\infty$ and if $\widetilde{r}<R$ then*
$$g\circ\mathcal{T}=\sum_{n=0}^{\infty}\alpha_n\mathcal{T}^n.$$
Here the series is uniformly convergent on D, while $\mathcal{T}^n:D\ni x\mapsto\mathcal{T}(x)^n$.
2. *If $0<\widetilde{r}<R$ then the function $g\circ\mathcal{T}$ is Fréchet differentiable and*
$$[g\circ\mathcal{T}]^{[1]}(x)=\sum_{n=0}^{\infty}\alpha_n u_n^{[1]}(\mathcal{T}(x))\mathcal{T}^{[1]}(x),\quad\forall x\in D. \tag{2.27}$$
Here the series converges in $B(X,\mathcal{A})$. Moreover:

a) *If* $\mathcal{T}^{[1]} : D \to B(X, \mathcal{A})$ *is continuous then the function* $[g \circ \mathcal{T}]^{[1]} : D \to B(X, \mathcal{A})$, *is also continuous.*
b) *If* $\sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$*, then the series in* (2.27) *absolutely uniformly converges on* D.

Proof. Let us consider the case in which D is convex and bounded, and $\sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$. Let $a, b \in D$ and $S_{a,b}$ the segment jointing a, b. D is convex so $S_{a,b} \subset D$. By an application of the Mean Value Theorem, see Theorem 8.6.2 of [7], we have for any $x_0 \in D$

$$\|\mathcal{T}(b) - \mathcal{T}(a) - \mathcal{T}^{[1]}(x_0)(b - a)\|_{\mathcal{A}} \leq \|b - a\|_X \cdot \sup_{x \in S_{a,b}} \|\mathcal{T}^{[1]}(x) - \mathcal{T}^{[1]}(x_0)\|_{B(X,\mathcal{A})}.$$

Thus by $\|A\| - \|B\| \leq \|A - B\|$ in any normed space,

$$\begin{aligned} \sup_{a \in D} \|\mathcal{T}(a)\|_{\mathcal{A}} &\leq \sup_{a \in D} \|\mathcal{T}(b) - \mathcal{T}^{[1]}(x_0)(b - a)\|_{\mathcal{A}} + \\ &\quad + \sup_{a \in D} \|b - a\|_X \cdot \sup_{a \in D} \sup_{x \in S_{a,b}} \|\mathcal{T}^{[1]}(x) - \mathcal{T}^{[1]}(x_0)\|_{B(X,\mathcal{A})} \leq \\ &\leq \|\mathcal{T}(b)\|_{\mathcal{A}} + \|\mathcal{T}^{[1]}(x_0)\|_{B(X,\mathcal{A})} \sup_{a \in D} \|b - a\|_X + \\ &\quad + \sup_{a \in D} \|b - a\|_X \cdot \sup_{x \in D} \|\mathcal{T}^{[1]}(x) - \mathcal{T}^{[1]}(x_0)\|_{B(X,\mathcal{A})} \leq \\ &\leq \|\mathcal{T}(b)\|_{\mathcal{A}} + \\ &\quad + \sup_{a \in D} \|b - a\|_X \cdot \left(2\|\mathcal{T}^{[1]}(x_0)\|_{B(X,\mathcal{A})} + \sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} \right) < \infty. \end{aligned} \tag{2.28}$$

Here D is considered bounded and $\sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$ by hypothesis.

So by (2.28) $\widetilde{r} < \infty$ which is the first part of statement (1.).

Let $D \subseteq X$ be the open set of which in the hypotheses.

By $\widetilde{r} < \infty$ we can assume that $0 < \widetilde{r} < R$, then the second part of statement (1.) follows by statement (1.) of Lemma 2.9.

In the sequel of the proof we assume that $0 < \widetilde{r} < R$.

By statement (2.) of Lemma 2.9 and by the Chain Theorem, see 8.2.1. of the [7], $g \circ \mathcal{T}$ is Fréchet differentiable and its differential map is

$$[g \circ \mathcal{T}]^{[1]} : D \ni x \mapsto g^{[1]}(\mathcal{T}(x)) \circ \mathcal{T}^{[1]}(x) = \left\{ \sum_{n=0}^{\infty} \alpha_n u_n^{[1]}(\mathcal{T}(x)) \right\} \circ \mathcal{T}^{[1]}(x) \in B(X, \mathcal{A}), \tag{2.29}$$

here it was used the fact that uniform convergence implies puntual convergence. By statement (2.) of Lemma 2.9 and $\widetilde{r} < R$ the previous series converges in $B(\mathcal{A})$, moreover $\mathfrak{b} : B(\mathcal{A}) \times B(X, \mathcal{A}) \ni (\phi, \psi) \mapsto \phi \circ \psi \in B(X, \mathcal{A})$ is a bilinear and continuous map i.e.

$$\mathfrak{b} \in B_2(B(\mathcal{A}) \times B(X, \mathcal{A}); B(X, \mathcal{A})), \tag{2.30}$$

since $\|\phi \circ \psi\|_{B(X,\mathcal{A})} \leq \|\phi\|_{B(\mathcal{A})} \cdot \|\psi\|_{B(X,\mathcal{A})}$. Thus (2.29) implies (2.27).

Set
$$\Gamma : D \ni x \mapsto (g^{[1]} \circ \mathcal{T}(x), \mathcal{T}^{[1]}(x)) \in B(\mathcal{A}) \times B(X, \mathcal{A}).$$
According to (2.29)
$$[g \circ \mathcal{T}]^{[1]} = \mathfrak{b} \circ \Gamma. \tag{2.31}$$
$\mathcal{T}^{[1]}$ is continuous by hypothesis, and $g^{[1]}$ is continuous by statement (2.) of Lemma 2.9, while $\mathcal{T}$ is continuous being differentiable by hypothesis, so $g^{[1]} \circ \mathcal{T}$ is continuous. Therefore by Proposition 1, §4.1., Ch. 1, of the [1] the map Γ is continuous. Thus by (2.31) and (2.30) $[g \circ \mathcal{T}]^{[1]}$ is continuous and statement (a)) follows.

By (2.9)

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sup_{x \in D} \|\alpha_n u_n^{[1]}(\mathcal{T}(x)) \circ \mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} \leq \\
&\leq \sum_{n=0}^{\infty} \sup_{x \in D} |\alpha_n| \|u_n^{[1]}(\mathcal{T}(x))\|_{B(\mathcal{A})} \cdot \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} \leq \\
&\leq \sum_{n=0}^{\infty} \sup_{x \in D} |\alpha_n| n \|\mathcal{T}(x)\|_{\mathcal{A}}^{n-1} \cdot \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} \leq \\
&\leq \sum_{n=0}^{\infty} \sup_{x \in D} |\alpha_n| n \|\mathcal{T}(x)\|_{\mathcal{A}}^{n-1} \cdot \sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} \leq \\
&\leq M \sum_{n=0}^{\infty} |\alpha_n| n \widetilde{r}^{n-1} < \infty,
\end{aligned}$$

where $M \doteqdot \sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})}$ and statement (b)) follows. □

Remark 2.17. By (2.29), statement (3) of Theorem 2.11 and (2.30), if $0 < \widetilde{r} < \frac{R}{3}$, we have for all $x \in D$

$$[g \circ \mathcal{T}]^{[1]}(x) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(\mathcal{R}(\mathcal{T}(x))) C(\mathcal{T}(x))^{p-1} \mathcal{T}^{[1]}(x). \tag{2.32}$$

In addition if $\sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$, then the series in the (2.32) is absolutely uniformly convergent on D.

If $0 < \widetilde{r} < \frac{R}{3}$ by (2.32) we have for all $h \in X$

$$[g \circ \mathcal{T}]^{[1]}(x)(h) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(\mathcal{T}(x)) C(\mathcal{T}(x))^{p-1} (\mathcal{T}^{[1]}(x)(h)). \tag{2.33}$$

In addition if $\sup_{x \in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$, then the series in the (2.33) is absolutely uniformly convergent for $(x, h) \in D \times B_L(\mathbf{0})$, for all $L > 0$, i.e.

$$\sum_{p=1}^{\infty} \frac{1}{p!} \sup_{(x,h) \in D \times B_L(\mathbf{0})} \|g^{(p)}(\mathcal{T}(x)) C(\mathcal{T}(x))^{p-1} (\mathcal{T}^{[1]}(x)(h))\|_{\mathcal{A}} < \infty.$$

Remark 2.18. In a similar way as in the proof of statement b) of Corollary 2.16 we have:

1. if $0 < \widetilde{r} < R$

$$\begin{aligned}
\left[g \circ \mathcal{T}\right]^{[1]}(x)(h) &= \sum_{n=1}^{\infty} n\alpha_n \mathcal{T}^{[1]}(x)(h)\mathcal{T}(x)^{n-1} + \\
&+ \sum_{n=2}^{\infty}\sum_{p=0}^{n-2}(n-p-1)\alpha_n \mathcal{T}(x)^p[\mathcal{T}(x), \mathcal{T}^{[1]}(x)(h)]\mathcal{T}(x)^{n-(2+p)} = \\
&= \sum_{n=1}^{\infty} n\alpha_n \mathcal{T}^{[1]}(x)(h)\mathcal{T}(x)^{n-1} + \\
&+ \sum_{p=0}^{\infty}\sum_{n=p+2}^{\infty}(n-p-1)\alpha_n \mathcal{T}(x)^p[\mathcal{T}(x), \mathcal{T}^{[1]}(x)(h)]\mathcal{T}(x)^{n-(2+p)}.
\end{aligned} \tag{2.34}$$

   If in addition $\sup_{x\in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$, then all the series in (2.34) are absolutely uniformly convergent for $(x, h) \in D \times B_L(\mathbf{0})$, for all $L > 0$.
2. If $0 < \widetilde{r} < R$ we have

$$\begin{aligned}
&\left[g \circ \mathcal{T}\right]^{[1]}(x) = \\
&= \sum_{n=1}^{\infty} n\alpha_n \mathcal{L}(\mathcal{T}(x))^{n-1}\mathcal{T}^{[1]}(x) - \sum_{k=2}^{\infty}\left\{\sum_{n=k}^{\infty}\alpha_n \mathcal{L}(\mathcal{T}(x))^{n-k}\right\} C(\mathcal{T}(x)^{k-1})\mathcal{T}^{[1]}(x).
\end{aligned} \tag{2.35}$$

   If in addition $\sup_{x\in D} \|\mathcal{T}^{[1]}(x)\|_{B(X,\mathcal{A})} < \infty$, then all the series in (2.35) are absolutely uniformly convergent on D.

Definition 2.19. Let $\langle G, \|\cdot\|_G\rangle$ be a $\mathbb{C}$-Banach space, then we denote by $G_{\mathbb{R}}$ the vector space G over $\mathbb{R}$ whose operation of summation is the same as that of the $\mathbb{C}$-vector space G, and whose multiplication by scalars is the restriction to $\mathbb{R}\times G$ of the multiplication by scalars on $\mathbb{C}\times G$, finally we set $\|\cdot\|_{G_{\mathbb{R}}} \doteqdot \|\cdot\|_G$. Then $\langle G_{\mathbb{R}}, \|\cdot\|_{G_{\mathbb{R}}}\rangle$ is a Banach space over $\mathbb{R}$ and will be called the $\mathbb{R}$-Banach space associated to $\langle G, \|\cdot\|_G\rangle$.

Let F, G be two $\mathbb{C}$-Banach spaces then of course $B(F, G) \subset B(F_{\mathbb{R}}, G_{\mathbb{R}})$, where the inclusion is to be intended only as a set inclusion. Let $A \subseteq F$ then if A is open in F it is open also in $F_{\mathbb{R}}$. For a mapping $f : A \subseteq F \to G$, we will denote with the symbol $f^{\mathbb{R}} : A \subseteq F_{\mathbb{R}} \to G_{\mathbb{R}}$ the same mapping but considered defined in the subset A of the $\mathbb{R}$-Banach space associated to F and at values in the $\mathbb{R}$-Banach space associated to G.

Remark 2.20. Let Y, Z be two $\mathbb{C}$-Banach spaces, then by considering that $B(Y, Z) \subset B(Y_{\mathbb{R}}, Z_{\mathbb{R}})$, we have that for each Fréchet differential function $f : A \subseteq Y \to Z$ the same function $f^{\mathbb{R}} : A \subseteq Y_{\mathbb{R}} \to Z_{\mathbb{R}}$ considered in the corresponding real Banach spaces, is differentiable, in addition $f^{[1]} = (f^{\mathbb{R}})^{[1]}$. Therefore if we get a real Banach space X, we shall obtain the same statements of Corollary 2.16, Remark 2.17 and Remark 2.18 by replacing $\mathcal{A}$ with $\mathcal{A}_{\mathbb{R}}$.

In particular take $X \doteqdot \mathbb{R}$, and recall that for every differential map $H : D \subseteq \mathbb{R} \to \mathcal{A}_{\mathbb{R}}$ we have $H^{[1]}(t)(1) = \frac{dH}{dt}(t)$ for all $t \in D$, where $\frac{dH}{dt} : D \to \mathcal{A}_{\mathbb{R}}$ is the derivative of H. Hence if we denote $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ and assume $0 < \widetilde{r} < \frac{R}{3}$, then we obtain by (2.33) that for all $t \in D \subseteq \mathbb{R}$

$$\frac{d\, g^{\mathbb{R}} \circ \mathcal{T}}{dt}(t) = \sum_{p=1}^{\infty} \frac{1}{p!} g^{(p)}(\mathcal{T}(t)) C(\mathcal{T}(t))^{p-1} \left(\frac{d\mathcal{T}}{dt}(t) \right). \tag{2.36}$$

In addition if $\sup_{t \in D} \| \frac{d\mathcal{T}}{dt}(t) \|_{\mathcal{A}} < \infty$, then the series in the (2.36) is absolutely uniformly convergent on D.

This formula has been shown for the first time by Victor I. Burenkov in [5].

Notice that $C(\mathcal{T}(t))^0 = \mathbf{1}$ and for all $n \in \mathbb{N} - \{0\}$

$$C(\mathcal{T}(t))^n \left(\frac{d\mathcal{T}}{dt}(t) \right) = \overbrace{\left[\cdots \left[\left[\right.\right.\right.}^{n} \frac{d\mathcal{T}}{dt}(t), \mathcal{T}(t) \overbrace{\left.\left.\left. \right], \mathcal{T}(t) \right], \cdots \right]}^{n}.$$

In particular if $\left[\frac{d\mathcal{T}}{dt}(t), \mathcal{T}(t) \right] = \mathbf{0}$, then

$$\frac{d\, g^{\mathbb{R}} \circ \mathcal{T}}{dt}(t) = g^{(1)}(\mathcal{T}(t)) \frac{d\mathcal{T}}{dt}(t). \tag{2.37}$$

If $\left[\left[\frac{d\mathcal{T}}{dt}(t), \mathcal{T}(t) \right], \mathcal{T}(t) \right] = \mathbf{0}$ then

$$\frac{d\, g^{\mathbb{R}} \circ \mathcal{T}}{dt}(t) = g^{(1)}(\mathcal{T}(t)) \frac{d\mathcal{T}}{dt}(t) + \frac{1}{2} g^{(2)}(\mathcal{T}(t)) \left[\frac{d\mathcal{T}}{dt}(t), \mathcal{T}(t) \right] \tag{2.38}$$

and so on.

Corollary 2.21. *Let $\mathcal{A}$ be a unitary Banach algebra, $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{K}$ be such that the radius of convergence of the series $g(\lambda) \doteqdot \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is $R > 0$. Finally let $W \in \mathcal{A}$, $0 < r < R$, $D_{(r,W)} \doteqdot \left] -\frac{r}{\|W\|}, \frac{r}{\|W\|} \right[$ with the convention $\frac{r}{0} \doteqdot \infty$ and*

$$\mathcal{T}(t) = tW$$

for all $t \in D_{(r,W)}$. Then with the notations adopted in the statements of Lemma 2.9, we have:

1. $$g^{\mathbb{R}} \circ \mathcal{T}(t) = \sum_{n=0}^{\infty} \alpha_n t^n W^n$$

   and the series is absolutely uniformly convergent for $t \in D_{(r,W)}$.
2. *$g^{\mathbb{R}} \circ \mathcal{T}$ is derivable, the following map*

$$\frac{d\, g^{\mathbb{R}} \circ \mathcal{T}}{dt}(t) = W \sum_{n=1}^{\infty} \alpha_n n t^{n-1} W^{n-1} = W \frac{d\, g}{d\lambda} \circ \mathcal{T}(t) \tag{2.39}$$

   for all $t \in D_{(r,W)}$ is the derivative function of $g^{\mathbb{R}} \circ \mathcal{T}$, is continuous and the series in the (2.39) is absolutely uniformly convergent.

Proof. Statement (1) is trivial. The map $\mathcal{T}$ is derivable with constant derivative equal to $W \in \mathcal{A}$, hence we have statement (2) by Remark 2.20 and (2.34). □

3. APPLICATION TO THE ANALYTIC FUNCTIONAL CALCULUS IN A $\mathbb{C}$-BANACH SPACE

In this section G is a complex Banach space and $Open(\mathbb{C})$ is the set of all open subsets of $\mathbb{C}$. We denote by $\sigma(T)$ the spectrum of T for all $T \in B(G)$, and for all $U \in Open(\mathbb{C})$ such that $\sigma(T) \subset U$ and $g : U \to \mathbb{C}$ analytic, by $g(T)$ the operator belonging to $B(G)$ as defined in the analytic functional calculus framework given in Definition 7.3.9. of the [8], that is

$$g(T) \doteqdot \frac{1}{2\pi i} \int_B g(\lambda) R(\lambda; T) d\lambda.$$

Here

$R(\lambda; T) \doteqdot (\lambda \mathbf{1} - T)^{-1}$ is the resolvent of T, while $B \subset U$ is the boundary of an open set containing $\sigma(T)$ and consisting of a finite number of rectifiable Jordan curves. If U is an open neighborhood of 0 and $g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ for all $\lambda \in U$, then by Theorem 7.3.10. of the [8] $g(T) = \sum_{n=0}^{\infty} \alpha_n T^n$ converging in $B(G)$. Therefore for this case we can apply all the results in Section 2.

Corollary 3.1 (Fréchet differential of an operator valued analytic function defined on an open set of a $\mathbb{R}$-Banach Space)**.** *Let U_0 be an open neighborhood of $0 \in \mathbb{C}$, $g : U_0 \to \mathbb{C}$ an analytic function such that $g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$, for all $\lambda \in U_0$. Let $R > 0$ be the radius of convergence of the series $\sum_{n=0}^{\infty} \alpha_n \lambda^n$. Finally let X be a Banach space over $\mathbb{R}$, $D \subseteq X$ an open set of X and $\mathcal{T} : D \to B(G)_{\mathbb{R}}$ a Fréchet differentiable mapping such that there exists $r \in \mathbb{R}^+ \mid 0 < r < R$ such that:*

1. $\mathcal{T}(D) \subseteq B_r(\mathbf{0})$
2. $\sigma(\mathcal{T}(x)) \subseteq U_0$*, for all* $x \in D$.

Then:

1. $$g^{\mathbb{R}} \circ \mathcal{T} = \sum_{n=0}^{\infty} \alpha_n \mathcal{T}^n.$$

   Here the series absolutely uniformly converges on D.
2. *The statements of Corollary 2.16, Remark 2.17 and Remark 2.18 hold with $\mathcal{A}$ replaced by $B(G)_{\mathbb{R}}$, while Remark 2.20 holds with $\mathcal{A}$ replaced by $B(G)$.*

Proof. The map $g^{\mathbb{R}} \circ \mathcal{T}$ is well defined by the condition $\sigma(\mathcal{T}(x)) \subseteq U_0$ for all $x \in D$, while the power series expansion follows by Theorem 7.3.10. of the [8]. Therefore statement 1. follows by the hypothesis $\mathcal{T}(D) \subseteq B_r(\mathbf{0})$, with $0 < r < R$ and Remark 2.20. Statement 2. is by Corollary 2.16 and Remark 2.20. □

Remark 3.2. If we assume that G is a complex Hilbert space and $\mathcal{T}(x)$ is a normal operator for all $x \in D$, then the condition $\mathcal{T}(D) \subseteq B_r(\mathbf{0})$ is equivalent to the following one $\sigma(\mathcal{T}(x)) \subseteq B_r(\mathbf{0})$ for all $x \in D$.

Although the following is a well-known result, for the sake of completeness we shall give a proof by using Corollary 2.21.

Corollary 3.3. *Let $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be such that the radius of convergence of the series $g(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is $R > 0$. Moreover let $W \in B(G)$, $0 < r < R$, and $D_{(r,W)} \doteqdot \left]-\frac{r}{\|W\|}, \frac{r}{\|W\|}\right[$ with the convention $\frac{r}{0} \doteqdot \infty$. Then the operator $g(tW)$ is well defined for all $t \in D_{(r,W)}$ and*

$$g(tW) = \sum_{n=0}^{\infty} \alpha_n t^n W^n.$$

Here the series converges absolutely uniformly on $D_{(r,W)}$. Moreover the map $D_{(r,W)} \ni t \mapsto \frac{dg}{d\lambda}(tW)$ is Lebesgue integrable in $B(G)$ in the sense defined in [4]*, Definition* 2 *Ch. IV, §3, n°4, and for all $u_1, u_2 \in D_{(r,W)}$*

$$W \int_{u_1}^{u_2} \frac{dg}{d\lambda}(tW)\, dt = g(u_2 W) - g(u_1 W).$$

Here $\int_{u_1}^{u_2} \frac{dg}{d\lambda}(tW)\, dt$ is the Lebesgue integral of the map $D_{(r,W)} \ni t \mapsto \frac{dg}{d\lambda}(tW)$ as defined in Definition 1 *Ch. IV, §4, n°1 of* [4].

Proof. By (2.3)

$$\mathcal{R}(W) \in B(B(G)). \tag{3.1}$$

Set $\mathcal{T}(t) \doteqdot tW$ for all $t \in D_{(r,W)}$. Then $\frac{dg}{d\lambda} \circ \mathcal{T}(t) = \sum_{n=1}^{\infty} \alpha_n n t^{n-1} W^{n-1}$ and the map $D_{(r,W)} \ni t \mapsto \frac{dg}{d\lambda} \circ \mathcal{T}(t)$ is continuous in $B(G)$, as a corollary of (2.39), by replacing the map g with $\frac{dg}{d\lambda}$, hence it is Lebesgue-measurable in $B(G)$. Finally let $u_1, u_2 \in D_{(r,W)}$

$$\begin{aligned}\int^{*}_{[u_1,u_2]} \|\frac{dg}{d\lambda}(tW)\| dt &= \int^{*}_{[u_1,u_2]} \|\sum_{n=1}^{\infty} \alpha_n n t^{n-1} W^{n-1}\| dt \leq \\ &\leq \int^{*}_{[u_1,u_2]} \sum_{n=1}^{\infty} |\alpha_n| n r^{n-1} dt = |u_2 - u_1| \sum_{n=1}^{\infty} |\alpha_n| n r^{n-1} < \infty.\end{aligned}$$

Here $\int^{*}_{[u_1,u_2]}$ is the upper integral of the Lebesgue measure on $[u_1, u_2]$. By this boundedness and by its Lebesgue- measurability we conclude by Theorem 5, *IV*.71 of [4] that $[u_1, u_2] \ni t \mapsto \frac{dg}{d\lambda} \circ \mathcal{T}(t) \in B(G)$ is Lebesgue-integrable in $B(G)$, so in particular by Definition 1, *IV*.33 of [4]

$$\int_{u_1}^{u_2} \frac{dg}{d\lambda} \circ \mathcal{T}(t) dt \in B(G). \tag{3.2}$$

Therefore by (3.1), (3.2), Theorem 1, IV.35 of [4] and (2.39)

$$W \int_{u_1}^{u_2} \frac{d\,g}{d\lambda} \circ \mathcal{T}(t) dt = \int_{u_1}^{u_2} W \frac{d\,g}{d\lambda} \circ \mathcal{T}(t) dt = \int_{u_1}^{u_2} \frac{d\,g^{\mathbb{R}}(\mathcal{T}(t))}{dt} dt. \tag{3.3}$$

Furthermore by the continuity of the map $D_{(r,W)} \ni t \mapsto \frac{d\,g}{d\lambda} \circ \mathcal{T}(t)$ in $B(G)$ and by (2.39), $D_{(r,W)} \ni t \mapsto \frac{d\,g^{\mathbb{R}}(\mathcal{T}(t))}{dt}$ is continuous in $B(G)$ and it is the derivative of the map $D_{(r,W)} \ni t \mapsto g^{\mathbb{R}} \circ \mathcal{T}(t)$. Therefore it is Lebesgue integrable in $B(G)$, where the integral has to be understood as defined in Ch. II of [2], see Proposition 3, $n°3$, §1, Ch. II of [2]. Finally the Lebesgue integral for functions with values in a Banach space as defined in Ch. II of [2], turns out to be the integral with respect to the Lebesgue measure as defined in Ch. IV, §4, $n°1$ of [4] (see Ch. III, §3, $n°3$ and example in Ch. IV, §4, $n°4$ of [4]). Thus the statement follows by (3.3). □

Finally we want to remark that one of the main aims of our work [16] is proving this formula for a certain class of unbounded operators in a Banach space and by considering the integral in weaker topologies than that induced by the norm in $B(G)$.

Acknowledgments
The Author would like to thank Prof. Victor Burenkov and Prof. Massimo Lanza de Cristoforis for their helpful comments in reading the manuscript. He also acknowledges the anonymous referee for his comments which helped to improve the presentation of the work. Reasearch supported by the Engineering and Physical Sciences Research Council (EPRSC).

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Received: November 3, 2009.
Revised: January 8, 2010.
Accepted: January 10, 2010.