CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. In this paper, we give some necessary and sufficient conditions for an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings converging to a common fixed of the mappings in convex metric spaces. Our results extend and improve some recent results of Sun, Wittmann, Xu and Ori, and Zhou and Chang.

Keywords: implicit iteration process, finite family of asymptotically quasi-nonexpansive mappings, common fixed point, convex metric space.

Mathematics Subject Classification: 47H05, 47H09, 47H10.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that \( X \) is a metric space and let \( F(T_i) = \{ x \in X : T_i x = x \} \) be the set of all fixed points of the mappings \( T_i \) \((i = 1, 2, \ldots, N)\) respectively. The set of common fixed points of \( T_i \) \((i = 1, 2, \ldots, N)\) denoted by \( \mathcal{F} \), that is, \( \mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \).

Definition 1.1 ([4,5]). Let \( T : X \to X \) be a mapping.

(1) The mapping \( T \) is said to be nonexpansive if
\[
d(Tx, Ty) \leq d(x,y), \quad \forall x, y \in D(T).
\]

(2) The mapping \( T \) is said to be quasi-nonexpansive if \( F(T) \neq \emptyset \) and
\[
d(Tx, p) \leq d(x,p), \quad \forall x \in D(T), \forall p \in F(T).
\]
(3) The mapping $T$ is said to be asymptotically nonexpansive if there exists a sequence $k_n \in [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y), \quad \forall x, y \in D(T), \quad \forall n \in \mathbb{N}.$$ 

(4) The mapping $T$ is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $k_n \in [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$d(T^n x, p) \leq k_n d(x, p), \quad \forall x \in D(T), \quad \forall p \in F(T), \quad \forall n \in \mathbb{N}.$$ 

Remark 1.2. (i) From the definition 1.1, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold.

(ii) It is obvious that if $T$ is nonexpansive, then it is asymptotically nonexpansive with the constant sequence $\{1\}$.

In 2001, Xu and Ori [16] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space $H$. Let $C$ be a nonempty subset of $H$. Let $T_1, T_2, \ldots, T_N$ be self mappings of $C$ and suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i, i = 1, 2, \ldots, N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows: Let $\{t_n\}$ a real sequence in $(0, 1), x_0 \in C$:

$$x_1 = t_1 x_0 + (1 - t_1) T_1 x_1,$$
$$x_2 = t_2 x_1 + (1 - t_2) T_2 x_2,$$
$$\ldots = \ldots$$
$$x_N = t_N x_{N-1} + (1 - t_N) T_N x_N,$$
$$x_{N+1} = t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1},$$
$$\ldots = \ldots$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1, \quad (1.1)$$

where $T_k = T_{k(modN)}$. (Here the mod $N$ function takes values in the set $\{1, 2, \ldots, N\}$.)

In 2003, Sun [12] extend the process (1.1) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in $(0, 1)$ and an initial point $x_0 \in C$, which is defined as follows:

$$x_1 = \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1,$$
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\[ x_2 = \alpha_2 x_1 + (1 - \alpha_2)T_2 x_2, \]
\[ \ldots = \ldots \]
\[ x_N = \alpha_N x_{N-1} + (1 - \alpha_N)T_N x_N, \]
\[ x_{N+1} = \alpha_{N+1} x_N + (1 - \alpha_{N+1})T_1^2 x_{N+1}, \]
\[ \ldots = \ldots \]
\[ x_{2N} = \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N})T_N^2 x_{2N}, \]
\[ x_{2N+1} = \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1})T_1^3 x_{2N+1}, \]
\[ \ldots = \ldots \]

which can be written in the following compact form:

\[ x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_i^k x_n, \quad n \geq 1, \tag{1.2} \]

where \( n = (k - 1)N + i, \quad i \in \{1, 2, \ldots, N\} \).

Sun [12] proved the strong convergence of the process (1.2) to a common fixed point in real uniformly convex Banach spaces, requiring only one member \( T \) in the family \( \{T_i : i = 1, 2, \ldots, N\} \) to be semi compact. The result of Sun [12] generalized and extended the corresponding main results of Wittmann [15] and Xu and Ori [16].

The purpose of this paper is to study the convergence of an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. The results presented in this paper extend and improve the corresponding results of Sun [12], Wittmann [15], Xu and Ori [16] and Zhou and Chang [17] and many others.

For the sake of convenience, we also recall some definitions and notations.

In 1970, Takahashi [13] introduced the concept of convexity in a metric space and the properties of the space.

**Definition 1.3** ([13]). Let \((X, D)\) be a metric space and \(I = [0, 1]\). A mapping \(W : X \times X \times I \to X\) is said to be a convex structure on \(X\) if for each \((x, y, \lambda) \in X \times X \times I\) and \(u \in X\),

\[ d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y). \]

\(X\) together with a convex structure \(W\) is called a **convex metric space**, denoted by \((X, d, W)\). A nonempty subset \(K\) of \(X\) is said to be **convex** if \(W(x, y, \lambda) \in K\) for all \((x, y, \lambda) \in K \times K \times I\).

**Remark 1.4.** Every normed space is a convex metric space, where a convex structure \(W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z\), for all \(x, y, z \in E\) and \(\alpha, \beta, \gamma \in I\) with \(\alpha + \beta + \gamma = 1\). In fact,

\[ d(u, W(x, y, z; \alpha, \beta, \gamma)) = \|u - (\alpha x + \beta y + \gamma z)\| \leq \alpha \|u - x\| + \beta \|u - y\| + \gamma \|u - z\| = \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \quad \forall u \in X. \]

But there exists some convex metric spaces which can not be embedded into a normed space.
Example 1.5. Let $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$. For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, we define a mapping $W: \mathbb{R}^3 \to X$ by

$$W(x, y, z; \alpha, \beta, \gamma) = (\alpha x_1 + \beta y_1 + \gamma z_1, \alpha x_2 + \beta y_2 + \gamma z_2, \alpha x_3 + \beta y_3 + \gamma z_3)$$

and define a metric $d: X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|.$$ 

Then we can show that $(X, d, W)$ is a convex metric space, but it is not a normed space.

Example 1.6. Let $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For each $x = (x_1, x_2), y = (y_1, y_2) \in Y$ and $\lambda \in I$. We define a mapping $W: \mathbb{R}^2 \times I \to Y$ by

$$W(x, y; \lambda) = \left(\lambda x_1 + (1 - \lambda)y_1, \frac{\lambda x_2 + (1 - \lambda)y_1 y_2}{\lambda x_1 + (1 - \lambda)y_1}\right)$$

and define a metric $d: Y \times Y \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_2 y_2 - y_1 y_2|.$$ 

Then we can show that $(Y, d, W)$ is a convex metric space, but it is not a normed space.

Definition 1.7. Let $(X, d, W)$ be a convex metric space with a convex structure $W$ and let $T_1, T_2, \ldots, T_N: X \to X$ be $N$ asymptotically quasi-nonexpansive mappings. For any given $x_0 \in X$, the iteration process $\{x_n\}$ defined by

$$x_1 = W(x_0, T_1 x_1, u_1; \alpha_1, \beta_1, \gamma_1),$$
$$\ldots = \ldots$$

$$x_N = W(x_{N-1}, T_N x_N, u_N; \alpha_N, \beta_N, \gamma_N),$$
$$x_{N+1} = W(x_N, T_1^2 x_{N+1}, u_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}),$$
$$\ldots = \ldots$$

$$x_{2N} = W(x_{2N-1}, T_N^2 x_{2N}, u_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),$$
$$x_{2N+1} = W(x_{2N}, T_1^3 x_{2N+1}, u_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1}),$$
$$\ldots = \ldots$$

which can be written in the following compact form:

$$x_n = W(x_{n-1}, T_n^{n(\text{mod} N)} x_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 1, \quad (1.3)$$

where $\{u_n\}$ is a bounded sequence in $X$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for $n = 1, 2, \ldots$. Iteration process (1.3) is called the implicit iteration process with errors for a finite family of mappings $T_i$ ($i = 1, 2, \ldots, N$).
If \( u_n = 0 \) in (1.3) then,
\[
x_n = W(x_{n-1}, T_{n(modN)}^n x_n; \alpha_n, \beta_n), \quad n \geq 1,
\]
(1.4)
where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \([0,1]\) such that \( \alpha_n + \beta_n = 1 \) for \( n = 1, 2, \ldots \).

Iteration process (1.4) is called the implicit iteration process for a finite family of mappings \( T_i \ (i = 1, 2, \ldots, N) \).

**Proposition 1.8.** Let \( T_1, T_2, \ldots, T_N : X \to X \) be \( N \) asymptotically nonexpansive mappings. Then there exists a sequence \( \{k_n\} \subset [1, \infty) \) with \( k_n \to 1 \) as \( n \to \infty \) such that
\[
d(T_i^n x, T_i^n y) \leq k_n d(x, y), \quad \forall n \geq 1,
\]
(1.5)
for all \( x, y \in X \) and for each \( i = 1, 2, \ldots, N \).

**Proof.** Since for each \( i = 1, 2, \ldots, N \), \( T_i : X \to X \) is an asymptotically nonexpansive mapping, there exists a sequence \( \{k_n^{(i)}\} \subset [1, \infty) \) with \( k_n^{(i)} \to 1 \) as \( n \to \infty \) such that
\[
d(T_i^n x, T_i^n y) \leq k_n^{(i)} d(x, y), \quad \forall n \geq 1.
\]
Letting
\[
k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\},
\]
therefore we have \( \{k_n\} \subset [1, \infty) \) with \( k_n \to 1 \) as \( n \to \infty \) and
\[
d(T_i^n x, T_i^n y) \leq k_n d(x, y) \leq k_n^{(i)} d(x, y) \leq k_n d(x, y), \quad \forall n \geq 1,
\]
for all \( x, y \in X \) and for each \( i = 1, 2, \ldots, N \). \( \Box \)

The above theorem is also holds for asymptotically quasi-nonexpansive mappings since an asymptotically nonexpansive mapping with a nonempty fixed point set is called an asymptotically quasi-nonexpansive mapping.

**Remark 1.9.** We see, from the proof of the preceding proposition, that
\[
\sum_{n=1}^{\infty} (k_n - 1) < \infty \iff \sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty,
\]
for all \( i \in \{1, 2, \ldots, N\} \).

## 2. MAIN RESULTS

In order to prove our main result of this paper, we need the following lemma.

**Lemma 2.1** ([7]). Let \( \{a_n\} \), \( \{b_n\} \) and \( \{r_n\} \) be sequences of nonnegative real numbers satisfying the inequality
\[
a_{n+1} \leq (1 + r_n)a_n + b_n, \quad n \geq 1.
\]
If \( \sum_{n=1}^{\infty} b_n < \infty \) and \( \sum_{n=1}^{\infty} r_n < \infty \). Then:
(a) \( \lim_{n \to \infty} a_n \) exists.
(b) If \( \lim \inf_{n \to \infty} a_n = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

Now we state and prove our main theorems of this paper.

**Theorem 2.2.** Let \((X, d, W)\) be a complete convex metric space. Let \(T_1, T_2, \ldots, T_N : X \to X\) be \(N\) asymptotically quasi-nonexpansive mappings. Suppose \(\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset\). Let \(\{u_n\}\) be a bounded sequence in \(X\), \(\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}\) be three sequences in \([0, 1]\), \(\{\alpha_n\} \subset (s, 1 - s)\) for some \(s \in (0, 1)\) and \(\{k_n\}\) be the sequence defined by (1.5) satisfying the following conditions:

(i) \(\alpha_n + \beta_n + \gamma_n = 1, \ \forall n \geq 1\);
(ii) \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\);
(iii) \(\sum_{n=1}^{\infty} \gamma_n < \infty\).

Then the implicit iteration process with errors \(\{x_n\}\) generated by (1.3) converges to a common fixed point of \(\{T_1, T_2, \ldots, T_N\}\) if and only if

\[
\lim_{n \to \infty} \inf D_d(x_n, \mathcal{F}) = 0,
\]

where \(D_d(y, \mathcal{F})\) denotes the distance from \(y\) to the set \(\mathcal{F}\), that is, \(D_d(y, \mathcal{F}) = \inf_{z \in \mathcal{F}} d(y, z)\).

**Proof.** The necessity is obvious. Now, we will only prove the sufficient condition. Setting \(k_n = 1 + \lambda_n\) with \(\lim_{n \to \infty} \lambda_n = 0\). Since \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\), so \(\sum_{n=1}^{\infty} \lambda_n < \infty\). For any \(p \in \mathcal{F}\), from (1.3), it follows that

\[
d(x_n, p) = d(W(x_{n-1}, T_{n(modN)}^n x_n, u_n; \alpha_n, \beta_n, \gamma_n), p) \leq \alpha_n d(x_{n-1}, p) + \beta_n d(T_{n(modN)}^n x_n, p) + \gamma_n d(u_n, p) \leq \alpha_n d(x_{n-1}, p) + \beta_n d(x_{n-1}, p) + \gamma_n d(u_n, p) \leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n) (1 + \lambda_n) d(x_{n-1}, p) + \gamma_n d(u_n, p) \leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n + \lambda_n) d(x_{n-1}, p) + \gamma_n d(u_n, p) \leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n + \lambda_n) \frac{d(x_{n-1}, p)}{s} + \gamma_n d(u_n, p)
\]

for all \(p \in \mathcal{F}\).

Therefore we have

\[
d(x_n, p) \leq d(x_{n-1}, p) + \frac{\lambda_n}{\alpha_n} d(x_{n-1}, p) + \frac{\gamma_n}{\alpha_n} d(u_n, p). \tag{2.2}
\]

Since \(0 < s < \alpha_n < 1 - s\), it follows from (2.2) that

\[
d(x_n, p) \leq d(x_{n-1}, p) + \frac{\lambda_n}{s} d(x_{n-1}, p) + \frac{\gamma_n}{s} d(u_n, p). \tag{2.3}
\]

Since \(\sum_{n=1}^{\infty} \lambda_n < \infty\), there exists a positive integer \(n_0\) such that \(s - \lambda_n > 0\) and \(\lambda_n < \frac{s}{2}\) and for all \(n \geq n_0\).
Thus, we have
\[ d(x_n, p) \leq \left(1 + \frac{\lambda_n}{s - \lambda_n}\right)d(x_{n-1}, p) + \frac{\gamma_n}{s - \lambda_n}d(u_n, p). \quad (2.4) \]
It follows from (2.4) that, for each \( n = (n - 1)N + i \geq n_0 \), we have
\[ d(x_n, p) \leq \left(1 + \frac{2\lambda_n}{s}\right)d(x_{n-1}, p) + \frac{2\gamma_n}{s}d(u_n, p). \quad (2.5) \]
Setting \( b_n = \frac{2\lambda_n}{s} \), where \( n = (n - 1)N + i, i \in \{1, 2, \ldots, N\} \), then we obtain
\[ d(x_n, p) \leq (1 + b_n)d(x_{n-1}, p) + \frac{2M}{s}\gamma_n, \quad \forall p \in \mathcal{F}, \quad (2.6) \]
where, \( M = \sup_{n \geq 1} d(u_n, p) \). This implies that
\[ D_d(x_n, \mathcal{F}) \leq (1 + b_n)d(x_{n-1}, \mathcal{F}) + \frac{2M}{s}\gamma_n. \quad (2.7) \]
Since \( \sum_{n=1}^{\infty} \lambda_n < \infty \), it follows that \( \sum_{n=1}^{\infty} b_n < \infty \) and \( \sum_{n=1}^{\infty} \gamma_n < \infty \), thus from Lemma 2.1, we have
\[ \lim_{n \to \infty} D_d(x_n, \mathcal{F}) = 0. \]
Next, we will prove that \( \{x_n\} \) is a Cauchy sequence. Note that when \( x > 0, 1 + x \leq e^x \), from (2.6) we have
\[ d(x_{n+m}, p) \leq (1 + b_{n+m})d(x_{n+m-1}, p) + \frac{2M}{s}\gamma_{n+m} \leq \]
\[ \leq e^{b_{n+m}}d(x_{n+m-1}, p) + \frac{2M}{s}\gamma_{n+m} \leq \]
\[ \leq e^{b_{n+m}}\left[e^{b_{n+m-1}}d(x_{n+m-2}, p) + \frac{2M}{s}\gamma_{n+m-1}\right] + \frac{2M}{s}\gamma_{n+m} \leq \]
\[ \leq e^{(b_{n+m}+b_{n+m-1})}d(x_{n+m-2}, p) + \frac{2M}{s}e^{b_{n+m}}[\gamma_{n+m} + \gamma_{n+m-1}] \leq \]
\[ \leq \ldots \]
\[ \leq e^{(b_{n+m}+b_{n+m-1}+\ldots+b_{n+1})}d(x_n, p)+ \]
\[ + \frac{2M}{s}e^{(b_{n+m}+b_{n+m-1}+\ldots+b_{n+2})}[\gamma_{n+m} + \gamma_{n+m-1} + \ldots + \gamma_{n+1}] \leq (2.8) \]
\[ \leq e^{\sum_{k=n+1}^{n+m} b_k}d(x_n, p)+ \frac{2M}{s}e^{\sum_{k=n+1}^{n+m} b_k} \sum_{j=n+1}^{n+m} \gamma_j \leq \]
\[ \leq e^{\sum_{k=n+1}^{n+m} b_k}d(x_n, p)+ \frac{2M}{s}e^{\sum_{k=n+1}^{n+m} b_k} \sum_{j=n+1}^{n+m} \gamma_j \leq \]
\[ \leq e^{\sum_{k=n+1}^{n+m} b_k}\left\{d(x_n, p)+ \frac{2M}{s} \sum_{j=n+1}^{n+m} \gamma_j\right\} \leq \]
\[ \leq M'\left\{d(x_n, p)+ \frac{2M}{s} \sum_{j=n+1}^{n+m} \gamma_j\right\} < \infty, \]
for all \( p \in F \) and \( n, m \in \mathbb{N} \), where \( M' = e^{\sum_{k=n+1}^{n+m} b_k} < \infty \). Since \( \lim_{n \to \infty} D_d(x_n, F) = 0 \) and \( \sum_{n=1}^{\infty} \gamma_n < \infty \), there exists a natural number \( n_1 \) such that for all \( n \geq n_1 \),

\[
D_d(x_n, F) < \frac{\varepsilon}{4M'} \quad \text{and} \quad \sum_{j=n_1+1}^{\infty} \gamma_j < \frac{s \cdot \varepsilon}{8MM'}.
\]

Thus there exists a point \( p_1 \in F \) such that \( d(x_{n_1}, p_1) < \frac{\varepsilon}{4M'} \), by the definition of \( D_d(x_n, F) \). It follows from (2.8) that for all \( n \geq n_1 \) and \( m \geq 0 \),

\[
d(x_{n+m}, x_n) \leq d(x_{n+m}, p_1) + d(x_n, p_1) \leq M'd(x_{n_1}, p_1) + \frac{2MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j + M'd(x_n, p_1) + \frac{2MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\]

This implies that \( \{x_n\} \) is a Cauchy sequence. Since the space is complete, the sequence \( \{x_n\} \) is convergent. Let \( \lim_{n \to \infty} x_n = p \). Moreover, since the set of fixed points of an asymptotically quasi-nonexpansive mapping is closed, so is \( F \), thus \( p \in F \) from \( \lim_{n \to \infty} D_d(x_n, F) = 0 \), that is, \( p \) is a common fixed point of the mappings \( \{T_1, T_2, \ldots, T_N\} \). This completes the proof.

If \( u_n = 0 \), in Theorem 2.2, we can easily obtain the following theorem.

**Theorem 2.3.** Let \((X, d, W)\) be a complete convex metric space. Let \( T_1, T_2, \ldots, T_N: X \to X \) be \( N \) asymptotically quasi-nonexpansive mappings. Suppose \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). Let \( x_0 \in X \) and \( \{\alpha_n\}, \{\beta_n\} \) be two sequences in \([0, 1]\), \( \{\gamma_n\} \) the sequence defined by (1.5) and \( \{\alpha_n\} \subset (s, 1-s) \) for some \( s \in (0, 1) \) satisfying the following conditions:

(i) \( \alpha_n + \beta_n = 1, \ \forall n \geq 1; \)

(ii) \( \sum_{n=1}^{\infty} (\gamma_n - 1) < \infty. \)

Then the implicit iteration process \( \{x_n\} \) generated by (1.3) converges to a common fixed point of \( \{T_1, T_2, \ldots, T_N\} \) if and only if

\[
\lim_{n \to \infty} \inf D_d(x_n, F) = 0.
\]

From Theorem 2.2, we can easily obtain the following theorem.

**Theorem 2.4.** Let \((X, d, W)\) be a complete convex metric space. Let \( T_1, T_2, \ldots, T_N: X \to X \) be \( N \) quasi-nonexpansive mappings. Suppose \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) and \( x_0 \in X \). Let \( \{u_n\} \) be an arbitrary bounded sequence in \( X \), \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) three sequences in \([0, 1]\) satisfying the following conditions:

(i) \( \alpha_n + \beta_n + \gamma_n = 1, \ \forall n \geq 1; \)

(ii) \( \{\alpha_n\} \subset (s, 1-s) \) for some \( s \in (0, 1) \);

(iii) \( \sum_{n=1}^{\infty} \gamma_n < \infty. \)
Then the implicit iteration process with errors \( \{x_n\} \) generated by (1.3) converges to a common fixed point of \( \{T_1, T_2, \ldots, T_N\} \) if and only if

\[
\liminf_{n \to \infty} D_d(x_n, \mathcal{F}) = 0.
\]

**Remark 2.5.** Our results extend and improve the corresponding results of Wittmann [15] and Xu and Ori [16] to the case of a more general class of nonexpansive mappings and implicit iteration process with errors.

**Remark 2.6.** Our results also extend and improve the corresponding results of Sun [12] to the case of an implicit iteration process with errors.

**Remark 2.7.** The main result of this paper is also an extension and improvement of the well-known corresponding results in [1–11].

**Remark 2.8.** Our results also extend and improve the corresponding results of Zhou and Chang [17] to the case of a more general class of asymptotically nonexpansive mappings.

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