Opuscula Mathematica • Vol. 30 • No. 4 • 2010

MINIMAL AND CO-MINIMAL PROJECTIONS IN SPACES OF CONTINUOUS FUNCTIONS

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Abstract. Minimal and co-minimal projections in the space $C[0,1]$ are studied. We construct a minimal and co-minimal projection from $C[0,1]$ onto a subspace Y defined in the introduction.

Keywords: minimal projection, co-minimal projection.

Mathematics Subject Classification: 46A22, 47A30, 47A58.

1. INTRODUCTION

Let X be a normed space over $\mathbb{R}$ and let Y be a linear subspace of X. A bounded linear operator $P\colon X \to Y$ is called a *projection* if $P_{|_Y} = Id_{|_Y}$. The set of all projections from X onto Y will be denoted by $\mathcal{P}(X,\ Y)$. A projection P_0 is called *minimal* if

$$\|P_0\| = \inf\Big\{\|P\| \ : \ P \in \mathcal{P}(X,\ Y)\Big\}.$$

A projection P_0 is called *co-minimal* if

$$\|P_0 - Id\| = \inf\Big\{\|P - Id\| \ : \ P \in \mathcal{P}(X,\ Y)\Big\}.$$

The constant

$$\lambda(X,\ Y) = \inf\Big\{\|P\| \ : \ P \in \mathcal{P}(X,\ Y)\Big\}$$

is called the *relative projection constant.*

Minimal and co-minimal projections are important for two main reasons. The first of them is the following Lebesgue inequality:

$$\|x - Px\| \leq \|Id - P\| \operatorname{dist}(x, Y) \leq (1 + \|P\|) \operatorname{dist}(x, Y).$$

The above inequality gives us a “good” linear approximation of elements from X by elements of Y if $\|P\|$ or $\|Id - P\|$ is small. The second reason is connected with the Hahn-Banach theorem; having a minimal projection we can linearly extend any functional $y^* \in Y^*$ to X^* by setting $x^* = y^* \circ P$ or equivalently we can speak of a linear extension of the operator $Id\colon Y \to Y$ to X of the smallest possible norm.

One of the most difficult problems in the theory of projections is to find formulas for minimal projections. The research concerning this problem has its origin in the famous paper [9], where the minimality of the classical Fourier projection F_n (defined on $C_0(2\pi)$) onto the subspace of trigonometric polynomials of degree $\leq n$ was proved. Since then many results concerning the minimality of projections have been obtained (see e.g. [7, 8, 10, 11, 13–17, 24–27]); the interested reader is also referred to [1, 2, 4–6, 9, 12–15, 17–21, 23] for further information on the subject).

Throughout the paper, we regard

$$X = \{f\colon [0,1] \to \mathbb{R} \ : f \text{ is continuous } \}$$

as a normed space equipped with the standard supremum norm. Suppose that a sequence $\{x_n\}_{n=1}^{\infty} \subset [0,1]$ satisfies the following conditions:

(1) $\{x_n\}_{n=1}^{\infty}$ is decreasing,
(2) $\lim_{n\to\infty} x_n = 1$,
(3) $x_1 = 0$.

For $n \in \mathbb{N}$, we define a functional $f_n \in X^*$ by

$$f_n(g) = g(x_n), \quad g \in X.$$

We set

$$Y = \bigcap_{i=1}^{\infty} \ker f_i, \quad X_1 = \{f \in X \ : \ f(1) = 0\}.$$

We also define a sequence $\{g_n\}_{n=1}^{\infty} \subset X_1$ by

$$g_1(x) = \begin{cases} -\frac{2}{x_2}x + 1 & \text{if } x \in [0, \frac{x_2}{2}], \\ 0 & \text{for the remaining } x, \end{cases}$$

$$g_n(x) = \begin{cases} \frac{2}{x_n - x_{n-1}}x - \frac{x_n + x_{n-1}}{x_n - x_{n-1}} & \text{if} \quad x \in [\frac{x_n + x_{n-1}}{2}, x_n], \\ \frac{-2}{x_{n+1} - x_n}x + 1 + \frac{2x_n}{x_{n+1} - x_n} & \text{if} \quad x \in [x_n, \frac{x_n + x_{n+1}}{2}], \\ 0 & \text{for the remaining} \quad x. \end{cases}$$

It is easy to see that

$$f_n(g_m) = g_m(x_n) = \delta_{nm}$$

for each $n, m \in \mathbb{N}$.

In this paper we will prove formulas for minimal and co-minimal projections in $\mathcal{P}(X,\ Y)$. More precisely, we will show that a projection $Q_s \in \mathcal{P}(X, Y)$ given by the formula

$$Q_s(f) = f - f(1) - \sum_{i=1}^{\infty} (f(x_i) - f(1))g_i, \quad f \in X,$$

is minimal and co-minimal.

2. MAIN RESULTS

For $n \in \mathbb{N}$, we define an operator $S_n \colon X_1 \to X_1$ by

$$S_n(h)(\cdot) = \sum_{i=1}^{n} f_i(h)g_i(\cdot) = \sum_{i=1}^{n} h(x_i)g_i(\cdot), \quad h \in X_1.$$

It is plain that

$$|S_n(h)(x)| \leq \max_{i\,\in\,\mathbb{N}} |h(x_i)| \sum_{i=1}^{n} g_i(x) \leq \max_{i\,\in\mathbb{N}} |h(x_i)|$$

for each $n \in \mathbb{N}$, $x \in [0,1]$ and $h \in X_1$.

We start our considerations with the ensuing lemma.

Lemma 2.1. *$\{S_n(h)\}_{n=1}^{\infty}$ is a Cauchy sequence in X_1 for each $h \in X_1$.*

Proof. Fix $h \in X_1$ and $n, m \in \mathbb{N}$ such that $n < m$. Note that

$$\begin{aligned}\|S_n(h) - S_m(h)\| &= \sup_{x\,\in\,[0,1]} \left| \sum_{i=n+1}^{m} h(x_i)g_i(x) \right| \leq \\ &\leq \sup_{x\,\in\,[0,1]} \left(\max_{i>n} |h(x_i)| \sum_{i=n+1}^{m} g_i(x) \right) \leq \max_{i>n} |h(x_i)|.\end{aligned}$$

Since $h(1) = 0$, it follows that $\lim_{n\to\infty} \left(\max_{i>n} |h(x_i)|\right) = 0$. This together with the above inequalities implies that $\{S_n(h)\}_{n=1}^{\infty}$ is a Cauchy sequence. □

Remark 2.2. The reader may easily convince himself that

$$\lim_{k\to\infty} S_k(h)(x) = \begin{cases} h(x_n)g_n(x), & \text{where } n \text{ is such that } x \in \left[\frac{x_{n-1}+x_n}{2}, \frac{x_n+x_{n+1}}{2}\right], \\ 0, & \text{if } x = 1. \end{cases}$$

Now we will prove the following theorem.

Theorem 2.3. *The set $\mathcal{P}(X_1,\, Y)$ is not empty. For any projection $P \in \mathcal{P}(X_1,\, Y)$ there exists a sequence of functions $\{y_n\}_{n=1}^{\infty} \subset X_1$, which satisfies the following conditions:*

(1) *a sequence $\sum_{i=1}^{\infty} f_i(h)y_i$ is convergent in X_1 for each $h \in X_1$,*
(2) *for each $i, j \in \mathbb{N}$ we get $f_i(y_j) = \delta_{ij}$,*
(3) *the operator P has the form*

$$P(\cdot) = Id(\cdot) - \sum_{i=1}^{\infty} f_i(\cdot)y_i.$$

Proof. By the Banach-Steinhaus theorem, there exists a constant $M > 0$ such that

$$\Big\| \sum_{i=1}^{\infty} f_i(h) y_i \Big\| \leq M, \tag{2.1}$$

for each $h \in X_1,\ \|h\| = 1$. Let

$$P_s(h) = h - \sum_{i=1}^{\infty} f_i(h) g_i, \quad h \in X_1.$$

The operator P_s is well-defined in X_1 because of Lemma 2.1 and Remark 2.2. From (2.1) we deduce that P_s is bounded. It is easy to see that $P_s \in \mathcal{P}(X_1,\ Y)$. Therefore, $\mathcal{P}(X_1, Y) \neq \emptyset$. Now fix $Q \in \mathcal{P}(X_1, Y)$. Since Q is a projection, we have

$$Q(P_s(h)) = Q\Big(h - \sum_{i=1}^{\infty} f_i(h) g_i\Big) = h - \sum_{i=1}^{\infty} f_i(h) g_i, \quad h \in X_1. \tag{2.2}$$

Condition (2.2) implies that

$$Q(h) = h - \sum_{i=1}^{\infty} f_i(h)(g_i - Q(g_i)), \quad h \in X_1.$$

We complete the proof by setting $y_i = g_i - Q(g_i)$ for each $i \in \mathbb{N}$. □

Theorem 2.4. *The set $\mathcal{P}(X,\ Y)$ is not empty. Any projection $Q \in \mathcal{P}(X,\ Y)$ has the form*

$$Q(f) = f(1)g + P_1(f - f(1)),$$

where $g \in Y$, $f \in X$ and $P_1 \in \mathcal{P}(X_1,\ Y)$.

Proof. Let us define an operator $T\colon X \to X_1$ by

$$T(f)(x) = f(x) - f(1), \quad f \in X,\, x \in [0,1]. \tag{2.3}$$

Fix $P \in \mathcal{P}(X_1, Y)$. It is easy to see that $P \circ T$ is a projection from X onto Y. Consequently, $\mathcal{P}(X,\ Y) \neq \emptyset$. Next, fix $Q \in \mathcal{P}(X, Y)$. For any $f \in X$, we have

$$Q(f) = Q(f - f(1) + f(1)) = Q(f(1)) + Q(f - f(1)) = f(1)Q(1) + Q(f - f(1)).$$

Clearly, $P_1 = Q_{|X_1} \in \mathcal{P}(X_1, Y)$ and $g = Q(1) \in Y$. The reader may easily convince himself that for each $g \in Y$ and $P_1 \in \mathcal{P}(X_1, Y)$ an operator Q given by the formula

$$Q(f) = f(1)g + P_1(f - f(1)), \quad f \in X,$$

is a projection from X onto Y. The proof is complete. □

Remark 2.5. We have proved that each projection $P \in \mathcal{P}(X,Y)$ has the form

$$P(f) = f(1)g + P_1(f - f(1)) = f(1)g + P_1(T(f)),$$

where $f \in X,\ g \in Y$, $P_1 \in \mathcal{P}(X_1, Y)$ and T is defined by (2.3). In view of Theorem 2.3, we have

$$P(f) = f(1)g + T(f) - \sum_{i=1}^{\infty} f_i(T(f))y_i = f(1)g + T(f) - \sum_{i=1}^{\infty}(f(x_i) - f(1))y_i.$$

Theorem 2.6. *A projection $P_s \in \mathcal{P}(X_1, Y)$ given by the formula*

$$P_s(\cdot) = Id(\cdot) - \sum_{i=1}^{\infty} f_i(\cdot)g_i,$$

is minimal.

Proof. Note that for each $x \in [0,1)$ and $f \in X_1$ we have

$$P_s(f)(x) = f(x) - f(x_n)g_n(x),$$

where $n \in \mathbb{N}$ is such that $x \in \left[\frac{x_{n-1}+x_n}{2}, \frac{x_n+x_{n+1}}{2}\right]$. Consequently, $\|P_s\| \le 2$. Fix $Q \in \mathcal{P}(X_1, Y)$. Now we will show that for each $\varepsilon > 0$ there exists $f \in X_1$, $\|f\| = 1$ such that $\|Q(f)\| > 2 - \varepsilon$. By Theorem 2.3,

$$Q(\cdot) = Id(\cdot) - \sum_{i=1}^{\infty} f_i(\cdot)y_i,$$

where $f_i(y_j) = \delta_{ij}$ for $i, j \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $y_n(x_n) = 1$, there exists $x_0 < x_n$ such that $0 < y_n(x_0) < 1$ and $1 - y_n(x_0) < \varepsilon$. Suppose that $f \in X_1$ satisfies the following conditions:

$$f(x_0) = 1, \quad f(x_n) = -1, \quad f(x_k) = 0 \text{ if } k \neq n, \quad \|f\| = 1.$$

Since $Q(f) = f + y_n$, we deduce that

$$Q(f)(x_0) = f(x_0) + y_n(x_0) > 1 + 1 - \varepsilon = 2 - \varepsilon,$$

and finally $\|Q\| \ge 2$. The proof is complete. □

Fix $Q \in \mathcal{P}(X, Y)$. By Theorem 2.4,

$$Q(f) = f(1)g + P_1(T(f)),$$

where $g \in Y,\ f \in X$ and $P_1 \in \mathcal{P}(X_1, Y)$. Hence,

$$\|Q\| \ge \|Q_{|X_1}\| = \|P_1\| \ge 2. \tag{2.4}$$

Now we will state and prove the principal result of this paper.

Theorem 2.7. *A projection* $Q_s \in \mathcal{P}(X,Y)$ *given by the formula*

$$Q_s(f) = P_s(f - f(1)), \quad f \in X,$$

is minimal.

Proof. For the purpose of the proof, we set

$$Q_n(f)(\cdot) = f(\cdot) - f(1) - \sum_{i=1}^{n}(f(x_i) - f(1))g_i(\cdot), \quad n \in \mathbb{N},\ f \in X.$$

From Theorem 2.3 we infer that $\lim_{n\to\infty} Q_n(f) = Q_s(f)$ for each $f \in X$. We will show that $\|Q_n(f)\| \le 2$ for each $f \in X$ such that $\|f\| = 1$. Observe that

$$\|Q_n(f)\| = \left\| f - f(1) - \sum_{i=1}^{n}(f(x_i) - f(1))g_i \right\| \le 1 + \left\| f(1) + \sum_{i=1}^{n}(f(x_i) - f(1))g_i \right\|.$$

In order to finish the proof, it suffices to show that

$$\left\| f(1) + \sum_{i=1}^{n}(f(x_i) - f(1))g_i \right\| \le 1.$$

Since the function

$$f(1) + \sum_{i=1}^{n}(f(x_i) - f(1))g_i$$

is piecewise linear, it follows that

$$\left\| f(1) + \sum_{i=1}^{n}(f(x_i) - f(1))g_i \right\| \le \max\Big\{|f(1)|,\ |f(x_i)|\ :\ i = 1, \ldots, n\Big\} \le 1. \qquad (2.5)$$

The above arguments show that $\|Q_n(f)\| \le 2$. This in turn yields $\|Q_s(f)\| = \lim_{n\to\infty} \|Q_n(f)\| \le 2$. The proof is complete. □

Theorem 2.8. *A projection* $Q_s \in \mathcal{P}(X,Y)$ *given by the formula*

$$Q_s(f) = P_s(f - f(1)), \quad f \in X,$$

is co-minimal.

Proof. Fix $Q \in \mathcal{P}(X,Y)$. By equation (2.4), we obtain

$$\|Id - Q\| \ge \|Q\| - \|Id\| \ge 2 - \|Id\| = 1.$$

In order to finish the proof, it suffices to show that $\|Id - Q_s\| = 1$. Observe that

$$\|f - Q_s(f)\| = \left\| f(1) + \sum_{i=1}^{\infty}(f(x_i) - f(1))g_i \right\| = \lim_{n\to\infty} \left\| f(1) + \sum_{i=1}^{n}(f(x_i) - f(1))g_i \right\|,$$

where $f \in X$. By equation (2.5), we obtain

$$\left\| f(1) + \sum_{i=1}^{n}(f(x_i) - f(1))g_i \right\| \le 1$$

for each $f \in X$ such that $\|f\| = 1$. This completes the proof. □

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Received: September 30, 2009.
Revised: February 26, 2010.
Accepted: March 3, 2010.