ON THE RELATIVE EQUILIBRIUM CONFIGURATIONS IN THE PLANAR FIVE-BODY PROBLEM

Agnieszka Siluszyk

Abstract. The number of central configurations in the Grebenicov-Elmabsout model of the planar five-body problem is estimated. An appropriate rational parameterization is used to reduce the equations defining such configurations to some polynomial ones. For the restricted five-body problem a sharp estimation is given by using the Sturm separation theorem.

Keywords: planar five-body problem, relative equilibrium, central configuration, Grebenicov-Elmabsout model.

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1. INTRODUCTION

I. The Newtonian N-body problem in celestial mechanics consists of studying the dynamics of N bodies with positions $q_i \in R^d$, $i = 1, \dots, N$ and masses $m_i \in R^+$ (the classical Newtonian problem), attracted to each other according to the law of universal gravitation [1]:

$$m_i \cdot \ddot{q}_i = \frac{\partial U}{\partial q_i}, \tag{1.1}$$

where U is the potential

$$U = \sum_{1 \le i < j \le N} \frac{m_i \cdot m_j}{|q_i - q_j|}. \tag{1.2}$$

In the N-body problem, the simplest possible motions are such initial positions $q_i, i = 1 \dots N$ that this configuration remains constant up to rotations and scaling, and each body describes a Keplerian orbit. Only some special configurations of particles are allowed in such motions. A. Wintner called them *central configurations* [2]. For a given instant $t = t_0$ the configuration of N bodies is central if the gravitational acceleration $\ddot{q}_i$ acting on every mass point m_i is proportional to its position q_i, that is

$\ddot{q}_i = \lambda \cdot q_i$ with $\lambda \neq 0$ for all $i = 1, \ldots, N$. So the configuration of N bodies is central in a barycentric frame of reference if there exists $\lambda \in R$ such that:

$$\lambda \cdot q_i = \sum_{j=1}^{N} m_j \frac{q_i - q_j}{|q_i - q_j|^3} \tag{1.3}$$

for all $i \neq j$. Usually, by an inertial barycentric system of coordinates we understand a frame, relative to which the center of mass (barycenter) given by $R = \frac{1}{M}\sum_{j=1}^{N} m_j q_j$, with $M = \sum_{j=1}^{N} m_j$, is considered at the origin of this system. Note, that if the q_i's are coplanar, a central configuration is called a *relative equilibrium* because they become equilibrium solutions in a rotating coordinate system [1]. For the 3-body problem, L.Euler has found a collinear relative equilibrium and J.L. Lagrange has found central configurations as two equilateral triangles (see [2]). In the collinear case, the exact number of central configurations of N bodies has been stated by F.R. Moulton [3]. He has established that there exists exactly $\frac{N!}{2}$ collinear relative equilibria. The number of planar central configurations of an N-body problem for an arbitrary number of positive masses has been established only for $N = 3$, i.e. there are always five relative equilibria. Already in the four body problem there is sufficient complexity to prevent a complete classification of non-collinear relative equilibria. Even the problem of finiteness of the number of central configurations is a very difficult one. This conjecture known also as the Wintner-Chazy-Smale conjecture was listed by S. Smale as 6^{th} problem on his list of problems for this century [4]. It is known that in the N-body problem, for $N = 4$ (see [5]) and for $N = 5$ (see [6]), a position of relative equilibrium where the bodies are at the vertices of a regular polygon with N sides, exists only if the masses are equal. A complete classification for the case of four equal masses has been purposed by A. Albouy [7]. The finiteness problem for the general four body problem was settled by M. Hampton and R. Moeckel [8]. Beyond these fundamental results not much more is known in terms of classification of planar central configurations for $N = 4$, whilst for $N \geq 5$ the problem becomes much more complicated. B. Elmabsout [9] and E.A. Grebenicov [10] have proved that if the masses are equal, there exists a configuration of relative equilibrium, where the bodies are located at the vertices of a regular N-gon for all $N \geq 4$. A new class of central configurations in the 5-body problem consisting of three bodies, located at the vertices of an equilateral triangle, and other two ones located on the perpendicular bisector, has been shown by J. Llibre and L.M. Mello [11]. Earlier, a case of central configurations in the 5-body problem with three bodies at the vertices of an equilateral triangle and with the other two situated symmetrically with respect to a perpendicular bisector has been studied by M. Hampton [12]. In 2009, J. Llibre and L.F. Mello [13] have considered a central con figuration of 7 masses in the form of five masses situated at the vertices of a regular pentagon and two other masses located in the interior of the pentagon symmetrically with respect to a perpendicular bisector according to one side of this pentagon.

II. In 90-ies B. Elmabsout [9] and E.A. Grebenicov [10] have proved that besides the class of gravitational models in the inertial barycentric system (the so-called gravitational model of Lagrange-Wintner), there exists a new class of gravitational models, i.e. the class of gravitational models in non inertial frames (the so-called gravitational model of Grebenicov-Elmabsout, we denote it by $GE(m_0, N)$). They have proved that there exists a relative equilibrium configuration in the $(N+1)$-body problem, with N bodies M_1, M_2,..., M_N having the same mass m, and situated at the vertices of an N-sided regular polygon, while the body M_0, of nonzero mass m_0, lies in the center of the polygon [9]. A geometric configuration of N bodies given by positions q_i, $i = \overline{1, N}$ is called a *central configuration* in a non inertial coordinate system, if there exists $\lambda \in R$ such that ([9, 14]):

$$\lambda \cdot q_i = \sum_{j=1}^{N} m_j \Big(\frac{q_j - q_i}{|q_j - q_i|^3} - \frac{q_j}{|q_j - q_0|^3} \Big) - \frac{m_0 + m_i}{|q_i - q_0|^3} q_i \tag{1.4}$$

for all $i \neq j$. Necessary and sufficient conditions for the existence of relative equilibrium configurations in the $(N+1)$-body problem for $N = 4, 5$ have been found numerically by E.A. Grebenicov, D. Kozak-Skoworodkin and M. Jakubiak [15].

In 1991 B. Elmabsout has stated the existence of a relative equilibrium configuration of the N-body problem ($N = p \cdot n$) for the Grebenicov-Elmabsout models ($GE(m_0, p, n)$, when N material particles are located at the vertices of p regular n-gons centered at a given mass m_0, with the bodies on the same n-gon having equal masses [16]. The existence of central configurations in an asymmetric N-body problem for $N = 7$ has been shown by A. Siluszyk [17, 18].

The main goal of this paper is to evaluate the number of relative equilibrium configurations in the non inertial Grebenicov-Elmabsout's model of the five-body problem.

2. MAIN RESULTS

Let q_0, q_1, q_2, q_3 and q_4 be some bodies with positive masses in the planar five-body problem in $GE(m_0, N)$. Let the bodies q_0, q_1 and q_3 lie on the X-axis with the mutual distance between q_1 and q_3 equal to $|q_1 - q_3| = 2r_0$ and with both m_1 and m_3 equal to m. Denote by $|q_2 - q_0|$ and by $|q_4 - q_0|$ the distances $|q_2|$ and $|q_4|$, respectively (we assume that q_0 is in the origin of coordinate). The existence of such model have been proved by B. Elmabsout [16]. If the bodies q_2 and q_4 also lie on the X-axis, then the number of relative equilibria is equal to sixty [3]. We are concerned with the case when q_2 and q_4 lie on the perpendicular OY. In the non inertial Cartesian coordinate system M_0xy, obtained from the initial barycentric system after a translation the potential U takes the form W, also called the perturbation function [15]:

$$W = \sum_{1 \le i < j \le N} m_i m_j \Big(\frac{1}{|q_i - q_j|} + \langle q_i, q_j \rangle \Big(\frac{1}{|q_i|^3} + \frac{1}{|q_j|^3} \Big) \Big) + \sum_{i=1}^{N} \frac{m_0 + m_i}{|q_i|} m_i. \tag{2.1}$$

Taking into account that $\lambda = \frac{W}{I}$, where $I = \sum_{i=1}^{N} m_i \left|q_i\right|^2$ is the moment of inertia, necessary and sufficient conditions for the existence of central configurations in the five-body problem in P_0xy system take the form [15]:

$$\sum_{j=1}^{4} m_j \Big(\frac{q_j - q_i}{|q_i - q_j|^3} - \frac{q_j}{|q_j|^3} \Big) - \frac{m_0 + m_i}{|q_i|^3} q_i = -\frac{W}{I} q_i, \tag{2.2}$$

where W and I are as above and $N = 4$. The equations describing our problem are very complicated; they contain irrationalities, therefore applying the method of parameterization from algebraic geometry, if applicable, could be a powerfull method to write up these equations in polynomial forms with rational coefficients. Following [19] we say that a system of N point masses is rationally parameterizable if its configuration $q_1, q_2, \dots q_N$ and the mutual distances $|q_i - q_j|$, $i, j = \overline{1, N}$, $i \neq j$ can be described by rational functions of independent parameters, i.e., parameters subject to no relations.

Theorem 2.1. *The $GE(m_0, 4)$ model with $m_1 = m_3$ is rationally parameterizable for all $m_i > 0$, $i = 0, 1, 2, 3, 4$.*

Proof. We start by noting that $\left|q_1 - q_2\right|^2 = \left|q_3 - q_2\right|^2 = r_0^2 + \left|q_2\right|^2$ and $\left|q_1 - q_4\right|^2 = \left|q_3 - q_4\right|^2 = r_0^2 + \left|q_4\right|^2$. Using the functions $g(\eta) = \frac{\eta^2 - 1}{2\eta}$ and $h(\eta) = \frac{\eta^2 + 1}{2\eta}$, both for $\eta > 0$, we can rewrite these equalities as $|q_2| = r_0 g(\eta_1)$ and $|q_4| = r_0 g(\eta_2)$. These transformations yield the following relations

$$\begin{cases} |q_1 - q_2| = r_0 h(\eta_1), \\ |q_1 - q_4| = r_0 h(\eta_2), \end{cases}$$

which in turn, leads to rational coefficients in the equations (2.2). □

Recall, that we do not distinct relative equilibrium configurations, which are obtained one from another by a rotation or dilatation; so we speak about the classes of relative equilibrium configurations.

Theorem 2.2. *In the general Grebenicov-Elmabsout model of the planar 5-body problem with two equal masses the number of classes of relative equilibrium configurations is at most 432.*

In contrast with the general $GE(m_0, 4)$ model, treated in this theorem the following result is concerned with the restricted 5-body problem, when the masses m_2 and m_4 are infinitesimally small. Recall, that the restricted N-body problem represents a generalization of the *classical restricted three-body problem* for the first time formulated by K.Jacobi, whose stability was profoundly studied by H. Poincare. In [20] the restricted three body problem has been considered and a new computation of the Birkhoff normal form of the Hamiltonian near the Lagrangian points has been provided which permitted the authors to obtain a new proof of Lyapunov stability of these particular solutions.

Theorem 2.3. *There are at most 12 classes of relative equilibrium configurations in the Grebenicov-Elmabsout model of the restricted planar 5-body problem with two equal masses.*

To prove finiteness of the number of central configurations we use an appropriate rational parameterization to reduce the equations of the corresponding relative equilibrium configurations to some equations of the polynomial types. In this way the problem becomes a geometric one and we are looking for the number of intersection points of the curves described by rational functions $f_i(x_1, x_2, \ldots, x_N)$, which in turn, can be reduced to some polynomial equations. The next step consists in the evaluation of the number of solutions of polynomial systems of equations by using the *Bezout Theorem*. For this we make use of the resultant. Let the polynomials $P, Q \in \mathbf{k}[x_1, x_2, \ldots, x_n]$ have positive degrees in x_1, i.e.

$$\begin{cases} P = a_r x_1^r + a_{r-1} x_1^{r-1} + \ldots + a_0, \ a_r \neq 0, \\ Q = b_s x_1^s + b_{s-1} x_1^{s-1} + \ldots + b_0, \ b_s \neq 0, \end{cases} \tag{2.3}$$

where $a_i, b_i \in \mathbf{k}[x_2, x_3, \ldots, x_n]$. Recall that the resultant of P and Q with respect to x_1 is the *Sylvester determinant*

$$Res(P, Q, x_1) = \begin{vmatrix} a_r & a_{r-1} & \ldots & a_0 & 0 & \ldots\ldots & 0 \\ 0 & a_r & a_{r-1} & \ldots & a_0 & 0 \ \ldots & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & \ldots & 0 & a_r & a_{r-1} & \ldots\ldots & a_0 \\ b_s & b_{s-1} & \ldots\ldots & & b_0 & 0 \ \ldots & 0 \\ 0 & b_s & b_{s-1} & \ldots\ldots & & b_0 \ \ldots & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & \ldots & 0 & b_s & b_{s-1} & \ldots\ldots & b_0 \end{vmatrix},$$

where there are exactly s rows of a_i's and r rows of b_i's.

The most important property of resultants for our purposes is the following one (see, e.g. [21]): if $P, Q \in \mathbf{k}[x_1, x_2, \ldots, x_n]$ have positive degrees in x_1, then $Res(P, Q, x_1)$ is in the elimination ideal defined by

$$\langle P, Q \rangle = \bigcap (\mathbf{k}[x_2, \ldots, x_n]).$$

It follows that if $(x_1^0, x_2^0, \ldots, x_n^0)$ is a solution of the system $P = Q = 0$, then

$$Res(P, Q, x_1)(x_2^0, \ldots, x_n^0) = 0.$$

Proof of Theorem 2.2. In our case the functions W and I take the form:

$$\begin{aligned} W = {} & \frac{m^2}{2r_0} + \frac{m_2^2}{|q_2|} + \frac{2mm_2}{\sqrt{r_0^2 + |q_2|^2}} + m_0\Big(\frac{2m}{r_0} + \frac{m_2}{|q_2|} + \frac{m_4}{|q_4|}\Big) + \frac{m_4^2}{|q_4|} - \\ & - m_2 m_4 \,|q_2|\,|q_4| \left(\frac{1}{|q_2|^3} + \frac{1}{|q_4|^3}\right) + \frac{m_2 m_4}{|q_2| + |q_4|} + \frac{2mm_4}{\sqrt{r_0^2 + |q_4|^2}}, \end{aligned} \tag{2.4}$$

$$I = 2mr_0^2 + m_2 \left|q_2\right|^2 + m_4 \left|q_4\right|^2 . \tag{2.5}$$

Hence the equation (2.2) for the planar five-body problem in the $GE(m_0, 4)$ model with $m_1 = m_3 = m$ is a set of four equations:

$$\begin{cases} \frac{W}{I}x_1 = m_2(\frac{x_2 - x_1}{|q_2 - q_1|^3} - \frac{x_2}{|q_2|^3}) + m_3(\frac{x_3 - x_1}{|q_3 - q_1|^3} - \frac{x_3}{|q_3|^3}) + \\ \qquad + m_4(\frac{x_4 - x_1}{|q_4 - q_1|^3} - \frac{x_4}{|q_4|^3}) - \frac{m_0 + m_1}{|q_1|^3}x_1, \\ \frac{W}{I}y_1 = m_2(\frac{y_2 - y_1}{|q_2 - q_1|^3} - \frac{y_2}{|q_2|^3}) + m_3(\frac{y_3 - y_1}{|q_3 - q_1|^3} - \frac{y_3}{|q_3|^3}) + \\ \qquad + m_4(\frac{y_4 - y_1}{|q_4 - q_1|^3} - \frac{y_4}{|q_4|^3}) - \frac{m_0 + m_1}{|q_1|^3}y_1, \\ \frac{W}{I}y_2 = m_1(\frac{y_1 - y_2}{|q_2 - q_1|^3} - \frac{y_1}{|q_1|^3}) + m_3(\frac{y_3 - y_2}{|q_3 - q_2|^3} - \frac{y_3}{|q_3|^3}) + \\ \qquad + m_4(\frac{y_4 - y_2}{|q_4 - q_2|^3} - \frac{y_4}{|q_4|^3}) - \frac{m_0 + m_2}{|q_2|^3}y_2, \\ \frac{W}{I}y_4 = m_1(\frac{y_1 - y_4}{|q_4 - q_1|^3} - \frac{y_1}{|q_1|^3}) + m_2(\frac{y_2 - y_4}{|q_4 - q_2|^3} - \frac{y_2}{|q_2|^3}) + \\ \qquad + m_3(\frac{y_3 - y_4}{|q_4 - q_3|^3} - \frac{y_3}{|q_3|^3}) - \frac{m_0 + m_4}{|q_4|^3}y_4. \end{cases} \tag{2.6}$$

It is easily seen, that the fourth equation can be obtained as a linear combination of the first three ones, so in what follows we omit it. We start by using a parameterization similar to that, used in the previous theorem, with the same functions g and h, and consider the following transformations (here T denote transposition):

$$\begin{aligned} [x_1, y_1]^T &= r_0[1, 0]^T + [x_0, y_0]^T, \\ [x_2, y_2]^T &= r_0[0, g(\eta)]^T + [x_0, y_0]^T, \\ [x_3, y_3]^T &= r_0[-1, 0]^T + [x_0, y_0]^T, \\ [x_4, y_4]^T &= r_0[0, g(\eta)]^T + [x_0, y_0]^T. \end{aligned} \tag{2.7}$$

The mutual distances between the bodies q_1, q_2, q_3 and q_4 can be expressed as rational functions of the variables η_1 and η_2, in the same way as in the proof of Theorem 2.1 with $|q_1 - q_2| = |q_3 - q_2|$, $|q_1 - q_4| = |q_3 - q_4|$, according to our model. Assuming $m_2 \neq m_4$, with $m_2, m_4 \neq 0$, and applying the transformations (2.7) to the system (2.6) we obtain a polynomial system of three equations $f_i(\eta_1, \eta_2; m, m_0, m_2, m_4) = \sum_{l_1=0}^{n_1} \sum_{l_2=0}^{n_2} C_s(m, m_0, m_2, m_4)\eta_1^{l_1}\eta_2^{l_2} = 0$ for $i = 1, 2, 3$, with $C_s(m, m_0, m_2, m_4)$ for $s = 1, 2, \ldots$ as constants. The concrete form of these polynomials has been established by using CAS "Mathematica" [22]. We write down only the first terms of these polynomials because of technical reasons:

$$\begin{aligned} f_1 &= -m_2(4m_0+m)\eta_1^{16}\eta_2^{13} + m_2(4m_0+m)\eta_1^{15}\eta_2^{14} - \\ &\quad - m_4(4m_0+m)\eta_1^{14}\eta_2^{15} + m_4(4m_0+m)\eta_1^{13}\eta_2^{16} + \ldots, \\ f_2 &= 3m_4\eta_1^{10}\eta_2^{6} + 3m_2\eta_1^{6}\eta_2^{10} + 3m_4\eta_1^{8}\eta_2^{6} + 3m_2\eta_1^{6}\eta_2^{8} + \ldots, \\ f_3 &= (2m+m_0+m_4)\eta_1^{16}\eta_2^{11} - (4m+2m_0+3m_4)\eta_1^{15}\eta_2^{12} - \\ &\quad - (2m+m_0+3m_2+2m_4)\eta_1^{13}\eta_2^{14} + (4m+2m_0+ \\ &\quad + 3m_2)\eta_1^{12}\eta_2^{15} - (2m+m_0+m_2)\eta_1^{11}\eta_2^{16} + \ldots \end{aligned} \tag{2.8}$$

The number of solutions of (2.8) will be estimated by the least number of solutions of the following systems:

$$(i): \begin{cases} f_1 = 0, \\ f_2 = 0, \end{cases} \quad (ii): \begin{cases} f_1 = 0, \\ f_3 = 0, \end{cases} \quad (iii): \begin{cases} f_2 = 0, \\ f_3 = 0. \end{cases}$$

Let us introduce the notation $K_{ij}(\eta_1) = Res(f_i, f_j, \eta_2)$, $L_{i'j'}(\eta_2) = Res(f_{i'}, f_{j'}, \eta_1)$ for $i, i', j, j' = 1, 2, 3$. Then the element (η_1, η_2) corresponds to a central configuration if $K_{ij}(\eta_1)$, $L_{i'j'}(\eta_2)$ do not vanish identically for some fixed $i, j, i' j'$ and $K_{ij}(\eta_1) = L_{i'j'}(\eta_2) = 0$. We proceed to show that the polynomials $K_{ij}(\eta_1)$, $L_{i'j'}(\eta_2)$ do not vanish for a fixed value of the corresponding variable. We take $\eta_1 = 1$, $\eta_2 = 1$ and, by using CAS "Mathematica" we obtain that $K_{12}(1) \neq 0$, i.e.:

$$\begin{cases} f_1(\eta_2) = a_{15}\eta_2^{15} + a_{14}\eta_2^{14} + a_{13}\eta_2^{13} + \ldots + a_0, \\ f_2(\eta_2) = b_{10}\eta_2^{10} + b_9\eta_2^{9} + b_8\eta_2^{8} + \ldots + b_0, \end{cases} \tag{2.9}$$

where $a_r, b_s \in Q[m, m_0, m_2, m_4, r_0]$, $r = 0, \ldots, 15$, $s = 0, \ldots, 10$ and for all $m_i, r_0 > 0, i = 2, 4$

$$a_{15} = -256 m_2 m_4 r_0^2 \neq 0, \qquad b_{10} = 4m_2 \neq 0.$$

Taking $\eta_2 = 1$ we conclude that $Res(f_i, f_j, \eta_2) = 0$. Repeating the procedure for the other cases and using the Bezout theorem for all pairs of polynomials f_i, f_j, $i, j = 1, 2, 3$, we obtain an upper bound, and, as a consequence, we obtain that the number of solutions of (i), (ii) or (iii) is ≤ 432. □

By using CAS "Mathematica" we demonstrate that the set of solutions of (i), (ii) and (iii) is not empty for some open subsets of the parameter domain (see Fig. 1 and Fig. 2). On these figures black dots mean material points q_i, $i = 0, 1, \ldots, 4$. For a relative equilibrium to exist, the configuration of masses must satisfy the following system of equations $\omega^2 q - \frac{q}{|q-q_0|^3} m_0 + \frac{\partial W}{\partial q} = 0$, where $q = (x, y)$ and $\sum_{j=1}^{N} m_j\left(\frac{1}{|q-q_j|} - \frac{\langle q, q_j\rangle}{|q_j|^3}\right)$ $(i \neq j)$ represents the perturbation function [15]. The rotation velocity of the N bodies around M_0xy must fullfill the conditions $\omega = \omega_1 = \omega_2 = \ldots = \omega_N$ (see [23]). The continuous and dot curves in the figures 1 and 2 represent the 0-levels of the function $\kappa_1(x, y) = \omega^2 x - \frac{m_0 x}{|q|^3} + \frac{\partial W}{\partial x}$ and $\kappa_2(x, y) = \omega^2 y - \frac{m_0 y}{|q|^3} + \frac{\partial W}{\partial y}$, respectively.

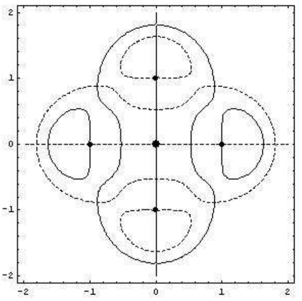

Fig. 1. Model of five bodies with two equal masses

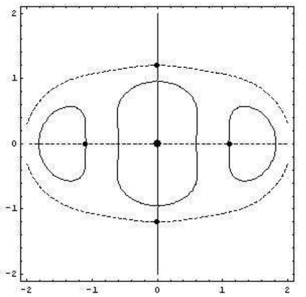

Fig. 2. Model of five bodies with two equal masses and two infinitesimal small ones

Proof of Theorem 2.3. In the restricted planar 5-body problem the masses m_2 and m_4 are infinitesimally small in comparison with m_0 and m, so in the system (2.2) we obtain only two equations. Similary as in the proof of theorem 2 we apply the transformations (2.7) to the system (2.2) and, after reductions, we obtain rational functions of the variable η with coefficients depending on the parameters m_0 and m. The rational function $F(\eta; m_0, m)$ is a sum of monomials; their degrees rise from 0 to 12

$$\begin{aligned} F(\eta; m_0, m) = {} & (4m_0 + m)\eta^{12} - 32(m_0 + 2m)\eta^9 - \\ & - 3(4m_0 + m)\eta^8 - 96(m_0 - 2m)\eta^7 - \\ & - 96(m_0 + 2m)\eta^5 + 3(4m_0 + m)\eta^4 - \\ & - 32(m_0 - 2m)\eta^3 - 4m_0 + m. \end{aligned} \tag{2.10}$$

Similarly as in the proof of Theorem 2.3 the polynomial equation $F(\eta; m_0, m) = 0$ possesses at least one solution from $\mathbb{R}[\eta]$. The number of solutions of a polynomial equation $F(\eta; m_0, m) = 0$ do not exceed 12. □

3. NUMERICAL CALCULATIONS

In the previous section we have proved that the number of roots in the ring $R[m_0, m]$ of the polynomial $F(\eta; m_0, m)$ for the restricted $GE(m_0, 4)$ model is at most 12. In this part, by using Sturm's method [24], we will prove that the number of real solutions of the equation $F(\eta; m_0, m) = 0$ is at most 3. Recall that a Sturm sequence is a finite sequence of polynomials $p_0, p_1, \ldots, p_m$ of decreasing degree with the following properties:

- $p_0 = p$ is square free,
- if $p(\alpha) = 0$, then $sign(p_1(\alpha)) = sign(p'(\alpha))$,
- if $p_i(\alpha) = 0$ for $0 < i < m$ then $sign(p_{i-1}(\alpha)) = -sign(p_{i+1}(\alpha))$,
- p_m does not change its sign.

To obtain a Sturm chain, Sturm himself proposed to choose the intermediary results when applying Euclid's algorithm to p and its derivative:

$$\begin{aligned} p_0(x) &:= p(x), \\ p_1(x) &:= p'(x), \\ p_2(x) &:= -rem(p_0, p_1) = p_1(x)q_0(x) - p_0(x), \\ p_3(x) &:= -rem(p_1, p_2) = p_2(x)q_1(x) - p_1(x), \\ &\dots, \\ 0 &:= -rem(p_{m-1}, p_m). \end{aligned} \tag{3.1}$$

That is, successively take the remainder with polynomial division and change their signs. Since $deg(p_{i+1}) < deg(p_i)$ for $0 \le i < m$, the algorithm terminates. The final polynomial, p_m, is the greatest common divisor of p and its derivative. Since p is square free, it has only simple roots and shares no roots with its derivative, so p_m will be a non-zero constant. The Sturm chain then is $p_0, p_1, \dots, p_m$.

Theorem 3.1 ([24]). *Let $K(\alpha)$ be the number of sign changes (zeroes are not counted) in the sequence $p(\alpha), p_1(\alpha), \dots, p_m(\alpha)$, where p is a square-free polynomial. Then for two real numbers $a < b$, the number of distinct roots in the half-open interval $(a, b]$ is $K(a) - K(b)$.*

Theorem 3.2. *Let Δ be the number of the real roots of the polynomial $F(\eta, m_0, m)$ with coefficients from the ring $R[m_0, m]$ of the $GE(m_0, 4)$ model of the restricted planar 5-body problem. Then $\max_{\eta\in R^+} \Delta = 3$ and $\min_{\eta\in R^+} \Delta = 1$ for each $m_0, m \in R^+$.*

Proof of Theorem 3.2. Let the functions $F(\eta, m_0, m)$, $F_1(\eta; m_0, m)$, $F_2(\eta; m_0, m)$,..., F_s = constant form the Sturm's sequence, where $F_1(\eta; m_0, m)$ is the remainder of division of $F(\eta; m_0, m)$ by $F'(\eta; m_0, m)$ taken with the opposite sign, $F_2(\eta; m_0, m)$ is the remainder of division of $F'(\eta; m_0, m)$ by $F_1(\eta; m_0, m)$ taken with the opposite sign and so on, with F_s = const. $\neq 0$ as the last term in this sequence. In our case this sequence is the following one:

$$\begin{aligned} F_1(\eta; m_0, m) &= -12(4m_0 + m_1)\eta^{11} + 288(m_0 + 2m_1)\eta^8 + \\ &+ 24(4m_0 + m_1)\eta^7 + 672(m_0 - 2m_1)\eta^6 + \\ &+ 480(m_0 + 2m_1)\eta^4 - 12(4m_0 + m_1)\eta^3 + \\ &+ 96(m_0 - 2m_1)\eta^2, \\ F_2(\eta; m_0, m) &= 8(m_0 + 2m_1)\eta^9 + (4m_0 + m_1)\eta^8 + \\ &+ 40(m_0 - 2m_1)\eta^7 + 56(m_0 + 2m_1)\eta^5 - \\ &- 2(4m_0 + m_1)\eta^4 + 24(m_0 - 2m_1)\eta^3 + \\ &+ (4m_0 - m_1), \\ &\dots, \\ F_{11}(\eta; m_0, m) &= \text{constant}, \end{aligned} \tag{3.2}$$

where the concrete form of the polynomials $F_i(\eta; m_0, m)$ for $i = 3, \ldots, 11$ are much more complicated and we omit them by technical reasons. In the next step for each element of this sequence in $(0, +\infty)$ we fix a sign. The coefficient of the highest monomial of $F_i(\eta; m_0, m)$ is listed bellow for $i = 0, 1, 2, 3, \ldots$:

$$A_0 = 4m_0 + m, \quad A_1 = -12(4m_0 + m), \quad A_2 = 8(m_0 + 2m),$$
$$A_3 = 3\Big(\frac{2304m_0^4 + 93440m_0^3m + 335328m_0^2m^2 + 413712m_0m^3}{128(m_0 + 2m^3)} + \frac{199169m^4}{128(m_0 + 2m^3)}\Big), \ldots$$

whilst the corresponding free terms of $F_i(\eta; m_0, m), i = 0, 1, 2, 3, \ldots$, are as follows:

$$B_0 = -(4m_0 - m), \quad B_1 = 0, \quad B_2 = 4m_0 - m,$$
$$B_3 = 3\Big(\frac{4864m_0^4 - 128m_0^3m - 20800m_0^2m^2 + 8m_0m^3}{128(m_0 + 2m)^3} + \frac{1281m^4}{128(m_0 + 2m)^3}\Big), \ldots$$

Let $V(\infty)$ be the number of sign changes in the Sturm sequence in $+\infty$, whilst $V(0)$ assigns the number of sign changes in the Sturm sequence in 0. Then, $(\max V(\infty) - \min V(0))$ is the maximal number of real solutions of the equation $F(\eta; m_0, m) = 0$. Moreover, there is a minimal number of the real solutions of the equation $F(\eta; m_0, m) = 0$ in the set

$$\{\min V(\infty) - \min V(0), \min V(\infty) - \max V(0)\}.$$

In our case we get that $V(\infty) = \{8, 9\}$, whilst $V(0) = \{6, 7\}$. In this way we have obtained that $\max_{\eta \in R^+} \Delta = 3$ and $\min_{\eta \in R^+} \Delta = 1$. □

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Agnieszka Siluszyk
sil_a@ap.siedlce.pl

University of Podlasie
Konarskiego 2, 08-110 Siedlce, Poland

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