RESEARCH PROBLEMS FROM THE 18TH WORKSHOP '3IN1' 2009

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Abstract. A collection of open problems that were posed at the 18th Workshop '3in1', held on November 26-28, 2009 in Krakow, Poland. The problems are presented by Zdenek Ryjacek in "Does the Thomassen's conjecture imply N=NP?" and "Dominating cycles and hamiltonian prisms", and by Carol T. Zamfirescu in "Two problems on bihomogeneously traceable digraphs".

Keywords: Hamilton-connected graph, hamiltonian graph, dominating cycle, bihomogeneously traceble graph.

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1. DOES THE THOMASSEN'S CONJECTURE IMPLY N=NP?

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By a *graph* we mean a simple loopless finite undirected graph $G = (V(G), E(G))$. A graph G is *Hamilton-connected* if G has a hamiltonian (x, y)-path for any $x, y \in V(G)$, and, for an integer $k \geq 1$, G is *k-Hamilton-connected* if $G - X$ is Hamilton-connected for any $X \subset V(G)$ with $|X| = k$. Denote $E^+(G) = \{xy|\ x, y \in V(G)\}$, and for $X \subset E^+(G)$ set $G + X = (V(G), E(G) \cup X)$ (i.e., X is a set of "new" edges that are "added" to G; if $e_1 = \{x, y\} \in E(G)$ and $e_2 = \{x, y\} \in X$, we consider e_1 and e_2 as parallel edges of $G + X$). A graph G is said to be *k-edge-Hamilton-connected* if, for any $X \subset E^+(G)$ such that $|X| = k$ and the the edges of X determine a path system, the graph $G + X$ has a hamiltonian cycle containing all edges in X. The following facts are easy to observe.

1. A graph G is 1-edge-Hamilton-connected if and only if G is Hamilton-connected.

2. A graph G is 2-edge-Hamilton-connected if and only if:
   (i) G is 1-Hamilton-connected (i.e., $G - x$ is Hamilton-connected for any vertex $x \in V(G)$), and
   (ii) for any four distinct vertices $x_1, x_2, x_3, x_4 \in V(G)$, G has a path factor consisting of 2 paths P_1, P_2 such that both P_1 and P_2 have one endvertex in $\{x_1, x_2\}$ and one endvertex in $\{x_3, x_4\}$.
3. If G is 2-edge-Hamilton-connected, then G is 4-connected.

Consider the following two decision problems.

k-E-HC
Instance: *A graph G.*
Question: *Is G k-edge-Hamilton-connected?*

k-E-HCL
Instance: *A line graph G.*
Question: *Is G k-edge-Hamilton-connected?*

(i.e., k-E-HCL is k-E-HC restricted to line graphs).

Question 1: *Determine the complexity of 2-E-HCL.*

The following facts are known:

- **HAM**
  Instance: *A graph G.*
  Question: *Does G contain a hamiltonian cycle?*
  HAM $\in$ NPC, even if restricted to line graphs.
- **H-PATH**
  Instance: *A graph G and distinct vertices $u, v \in V(G)$.*
  Question: *Does G contain a hamiltonian (u, v)-path?*
  H-PATH $\in$ NPC, even if restricted to line graphs [1].
- **H-CONN**
  Instance: *A graph G.*
  Question: *Is G Hamilton-connected?*
  H-CONN $\in$ NPC [3].
- **1-H-CONN**
  Instance: *A graph G.*
  Question: *Is G 1-Hamilton-connected?*
  1-H-CONN $\in$ NPC [6].

Thus, a common guess would be that probably 2-E-HCL $\in$ NPC.

Question 2: *Why is Question 1 interesting?*

The following conjecture was posed in [5].

Conjecture [Thomassen**].** *Every 4-connected line graph is hamiltonian.*

There are many known equivalent versions of the Thomassen's conjecture; among others, we mention the following.

Theorem. *The following statements are equivalent:*

(i) *Every 4-connected line graph is hamiltonian.*

(ii) *Every 4-connected line graph is 2-edge-Hamilton-connected* [4].
(iii) *Every snark has a dominating cycle* [2].

Thus, if the Thomassen's conjecture is true, then a line graph G is 2-edge-Hamilton-connected if and only if G is 4-connected, implying that 2-E-HCL is polynomial. Consequently, proving the "common guess" 2-E-HCL $\in$ NPC would mean

- disproving the Thomassen's conjecture,
- proving the existence of a snark with no dominating cycle,

unless P=NP.

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2. DOMINATING CYCLES AND HAMILTONIAN PRISMS

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The *prism over a graph* G, denoted $G\square K_2$, is the Cartesian product of G and K_2. It consists of two disjoint copies of G and a perfect matching connecting a vertex in one copy of G to its "clone" in the other copy.

A graph G is *hamiltonian* if it has a hamiltonian cycle and *traceable* if it has a hamiltonian path. Define a *k-walk* in a graph to be a spanning closed walk in which every vertex is visited at most k times

The following implications are easy to verify:

G is hamiltonian $\Rightarrow G$ is traceable $\Rightarrow G\Box K_2$ is hamiltonian $\Rightarrow G$ has a 2-walk.

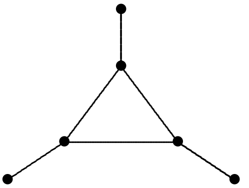
Thus the question whether G has a hamiltonian prism (i.e whether $G\Box K_2$ is hamiltonian) is "sandwiched" between hamiltonicity and having a 2-walk. Specifically, the property of having a hamiltonian prism can be considered as a "relaxation" of hamiltonicity. More information about prism-hamiltonicity of a graph can be found e.g. in [1] and [2].

A *dominating cycle* in a graph G is a cycle C such that every edge of G has at least one vertex on C, i.e. such that the graph $G - C$ is edgeless. Clearly, a hamiltonian cycle is dominating, and hence the property of having a dominating cycle can be considered as another relaxation of hamiltonicity.

There is a natural question whether there is any relation between these two properties.

Example 1. Let H be any 2-connected cubic nonhamiltonian graph, and let G be obtained from H by replacing every vertex of H with a triangle (such a G is sometimes called the *inflation* of H). Then G is a 2-connected line graph and these are known [2] to be prism-hamiltonian. On the other hand, since H is nonhamiltonian, any cycle in G has to miss at least one "new" triangle and hence G has no dominating cycle. Thus, there are "many" graphs showing that hamiltonian prism does not imply having a dominating cycle.

Example 2. The graph in the figure below shows that also the existence of a dominating cycle does not imply having hamiltonian prism.



However, all such known examples are of low toughness (recall that G is 1-tough if, for any $S \subset V(G)$, the graph $G - S$ has at most $|S|$ components). This motivates the following question.

Conjecture. *Let G be a 1-tough graph having a dominating cycle. Then G has hamiltonian prism.*

Comments. Suppose that G has a dominating cycle C of even length. Set $M = V(G) \setminus V(C)$ and $N = \{x \in V(C) | \ x \text{ has a neighbor in } M\}$. Then the graph induced by $M \cup N$ has a matching containing all vertices from M (this follows by the toughness assumption and by the Hall's theorem). Using this matching, it is easy to construct a hamiltonian cycle in $G\Box K_2$.

The difficult case is when all dominating cycles in G are of odd length.

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3. TWO PROBLEMS ON BIHOMOGENEOUSLY TRACEABLE DIGRAPHS

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We concern ourselves here exclusively with simple finite oriented graphs (i.e. digraphs with no multiple edges, a finite number of vertices, and without cycles of length 2), calling these simply *graphs*. A graph is called *homogeneously traceable*, if for every vertex v there exists a hamiltonian path starting at v. If, additionally, the graph has the property that in every vertex a hamiltonian path ends, we call it *bihomogeneously traceable*. In this setting, and in a graph on n vertices, *arc-minimality* (or 2-diregularity) means that the graph has precisely $2n$ edges (i.e. every vertex has in-degree 2 and out-degree 2). We remark that bihomogeneous traceability does not imply hamiltonicity, for instance hypohamiltonian graphs are non-hamiltonian and bihomogeneously traceable.

Z. Skupień [3] presented in 1981 an infinite family of arc-minimal non-hamiltonian bihomogeneously traceable graphs, featuring graphs of all orders greater or equal to 7. Another such infinite family of graphs (but not arc-minimal) was provided independently by S. Hahn and T. Zamfirescu [1] in the same year.

In 1983, L. E. Penn and D. Witte [2] proved that the cartesian product of two oriented cycles of length a and b is hypohamiltonian (whence, non-hamiltonian and bihomogeneously traceable) if and only if there exist relatively prime numbers $m, n \in \mathbb{N}$ such that $am + nb = ab - 1$. We note that these graphs are also arc-minimal.

In their 1981 paper, Hahn and Zamfirescu presented two planar non-hamiltonian bihomogeneously traceable graphs, one of which is arc-minimal, and asked the natural question whether infinitely many such graphs do exist. Very recently it was proven that this is indeed the case, see [4].

The following problems, however, are still open.

Problem 1. Is there an infinite family of planar arc-minimal non-hamiltonian bihomogeneously traceable oriented graphs?

Problem 2. Are there such graphs on all orders greater than some integer? Even if one removes the condition of arc-minimality, this problem is still open.

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