RESEARCH PROBLEMS FROM THE 18TH WORKSHOP ‘3in1’ 2009

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Abstract. A collection of open problems that were posed at the 18th Workshop ‘3in1’, held on November 26-28, 2009 in Krakow, Poland. The problems are presented by Zdenek Ryjacek in “Does the Thomassen’s conjecture imply N=NP?” and “Dominating cycles and hamiltonian prisms”, and by Carol T. Zamfirescu in “Two problems on bihomogeneously traceable digraphs”.

Keywords: Hamilton-connected graph, hamiltonian graph, dominating cycle, bihomogeneously traceable graph.

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1. DOES THE THOMASSEN’S CONJECTURE IMPLY N=NP?

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By a graph we mean a simple loopless finite undirected graph $G = (V(G), E(G))$. A graph $G$ is Hamilton-connected if $G$ has a hamiltonian $(x,y)$-path for any $x, y \in V(G)$, and, for an integer $k \geq 1$, $G$ is $k$-Hamilton-connected if $G - X$ is Hamilton-connected for any $X \subseteq V(G)$ with $|X| = k$. Denote $E^+(G) = \{xy \mid x, y \in V(G)\}$, and for $X \subseteq E^+(G)$ set $G + X = (V(G), E(G) \cup X)$ (i.e., $X$ is a set of “new” edges that are “added” to $G$; if $e_1 = \{x, y\} \in E(G)$ and $e_2 = \{x, y\} \in X$, we consider $e_1$ and $e_2$ as parallel edges of $G + X$). A graph $G$ is said to be $k$-edge-Hamilton-connected if, for any $X \subseteq E^+(G)$ such that $|X| = k$ and the the edges of $X$ determine a path system, the graph $G + X$ has a hamiltonian cycle containing all edges in $X$. The following facts are easy to observe.

1. A graph $G$ is 1-edge-Hamilton-connected if and only if $G$ is Hamilton-connected.
2. A graph \( G \) is 2-edge-Hamilton-connected if and only if:
   (i) \( G \) is 1-Hamilton-connected (i.e., \( G - x \) is Hamilton-connected for any vertex \( x \in V(G) \)), and
   (ii) for any four distinct vertices \( x_1, x_2, x_3, x_4 \in V(G) \), \( G \) has a path factor consisting of 2 paths \( P_1, P_2 \) such that both \( P_1 \) and \( P_2 \) have one endvertex in \( \{x_1, x_2\} \) and one endvertex in \( \{x_3, x_4\} \).
3. If \( G \) is 2-edge-Hamilton-connected, then \( G \) is 4-connected.

Consider the following two decision problems.

\( k \)-E-HC

**Instance:** A graph \( G \).

**Question:** Is \( G \) \( k \)-edge-Hamilton-connected?

\( k \)-E-HCL

**Instance:** A line graph \( G \).

**Question:** Is \( G \) \( k \)-edge-Hamilton-connected?

(i.e., \( k \)-E-HCL is \( k \)-E-HC restricted to line graphs).

**Question 1:** Determine the complexity of 2-E-HCL.

The following facts are known:

- **HAM**
  
  **Instance:** A graph \( G \).
  
  **Question:** Does \( G \) contain a hamiltonian cycle?  
  
  \( \text{HAM} \in \text{NPC} \), even if restricted to line graphs.

- **H-PATH**
  
  **Instance:** A graph \( G \) and distinct vertices \( u, v \in V(G) \).
  
  **Question:** Does \( G \) contain a hamiltonian \((u, v)\)-path?  
  
  \( \text{H-PATH} \in \text{NPC} \), even if restricted to line graphs [1].

- **H-CONN**
  
  **Instance:** A graph \( G \).
  
  **Question:** Is \( G \) Hamilton-connected?  
  
  \( \text{H-CONN} \in \text{NPC} \) [3].

- **1-H-CONN**
  
  **Instance:** A graph \( G \).
  
  **Question:** Is \( G \) 1-Hamilton-connected?  
  
  \( \text{1-H-CONN} \in \text{NPC} \) [6].

Thus, a common guess would be that probably 2-E-HCL \( \in \text{NPC} \).

**Question 2:** Why is Question 1 interesting?

The following conjecture was posed in [5].

**Conjecture** [Thomassen]. Every 4-connected line graph is hamiltonian.

There are many known equivalent versions of the Thomassen’s conjecture; among others, we mention the following.

**Theorem.** The following statements are equivalent:

(i) Every 4-connected line graph is hamiltonian.
(ii) Every 4-connected line graph is 2-edge-Hamilton-connected [4].
(iii) Every snark has a dominating cycle [2].

Thus, if the Thomassen’s conjecture is true, then a line graph \( G \) is 2-edge-Hamilton-connected if and only if \( G \) is 4-connected, implying that 2-E-HCL is polynomial. Consequently, proving the “common guess” 2-E-HCL \( \in \text{NPC} \) would mean

- disproving the Thomassen’s conjecture,
- proving the existence of a snark with no dominating cycle,

unless \( P=NP \).

REFERENCES


2. DOMINATING CYCLES AND HAMILTONIAN PRISMS

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The prism over a graph \( G \), denoted \( G \square K_2 \), is the Cartesian product of \( G \) and \( K_2 \). It consists of two disjoint copies of \( G \) and a perfect matching connecting a vertex in one copy of \( G \) to its “clone” in the other copy.

A graph \( G \) is hamiltonian if it has a hamiltonian cycle and traceable if it has a hamiltonian path. Define a \( k \)-walk in a graph to be a spanning closed walk in which every vertex is visited at most \( k \) times.

The following implications are easy to verify:
\[ G \text{ is hamiltonian} \Rightarrow G \text{ is traceable} \Rightarrow G \square K_2 \text{ is hamiltonian} \Rightarrow G \text{ has a 2-walk.} \]

Thus the question whether \( G \) has a hamiltonian prism (i.e., whether \( G \square K_2 \) is hamiltonian) is “sandwiched” between hamiltonicity and having a 2-walk. Specifically, the property of having a hamiltonian prism can be considered as a “relaxation” of hamiltonicity. More information about prism-hamiltonicity of a graph can be found e.g. in [1] and [2].

A dominating cycle in a graph \( G \) is a cycle \( C \) such that every edge of \( G \) has at least one vertex on \( C \), i.e. such that the graph \( G - C \) is edgeless. Clearly, a hamiltonian cycle is dominating, and hence the property of having a dominating cycle can be considered as another relaxation of hamiltonicity.

There is a natural question whether there is any relation between these two properties.

**Example 1.** Let \( H \) be any 2-connected cubic nonhamiltonian graph, and let \( G \) be obtained from \( H \) by replacing every vertex of \( H \) with a triangle (such a \( G \) is sometimes called the inflation of \( H \)). Then \( G \) is a 2-connected line graph and these are known [2] to be prism-hamiltonian. On the other hand, since \( H \) is nonhamiltonian, any cycle in \( G \) has to miss at least one “new” triangle and hence \( G \) has no dominating cycle. Thus, there are “many” graphs showing that hamiltonian prism does not imply having a dominating cycle.

**Example 2.** The graph in the figure below shows that also the existence of a dominating cycle does not imply having hamiltonian prism.

![Graph showing Example 2](image)

However, all such known examples are of low toughness (recall that \( G \) is 1-tough if, for any \( S \subset V(G) \), the graph \( G - S \) has at most \(|S|\) components). This motivates the following question.

**Conjecture.** Let \( G \) be a 1-tough graph having a dominating cycle. Then \( G \) has hamiltonian prism.

**Comments.** Suppose that \( G \) has a dominating cycle \( C \) of even length. Set \( M = V(G) \setminus V(C) \) and \( N = \{x \in V(C) | x \text{ has a neighbor in } M\} \). Then the graph induced by \( M \cup N \) has a matching containing all vertices from \( M \) (this follows by the toughness assumption and by the Hall’s theorem). Using this matching, it is easy to construct a hamiltonian cycle in \( G \square K_2 \).

The difficult case is when all dominating cycles in \( G \) are of odd length.
3. TWO PROBLEMS ON BIHOMOGENEOUSLY TRACEABLE DIGRAPHS

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We concern ourselves here exclusively with simple finite oriented graphs (i.e. digraphs with no multiple edges, a finite number of vertices, and without cycles of length 2), calling these simply graphs. A graph is called homogeneously traceable, if for every vertex \( v \) there exists a hamiltonian path starting at \( v \). If, additionally, the graph has the property that in every vertex a hamiltonian path ends, we call it bihomogeneously traceable. In this setting, and in a graph on \( n \) vertices, arc-minimality (or 2-diregularity) means that the graph has precisely \( 2n \) edges (i.e. every vertex has in-degree 2 and out-degree 2). We remark that bihomogeneous traceability does not imply hamiltonicity, for instance hypohamiltonian graphs are non-hamiltonian and bihomogeneously traceable.

Z. Skupień [3] presented in 1981 an infinite family of arc-minimal non-hamiltonian bihomogeneously traceable graphs, featuring graphs of all orders greater or equal to 7. Another such infinite family of graphs (but not arc-minimal) was provided independently by S. Hahn and T. Zamfirescu [1] in the same year.

In 1983, L. E. Penn and D. Witte [2] proved that the cartesian product of two oriented cycles of length \( a \) and \( b \) is hypohamiltonian (whence, non-hamiltonian and bihomogeneously traceable) if and only if there exist relatively prime numbers \( m, n \in \mathbb{N} \) such that \( am + nb = ab - 1 \). We note that these graphs are also arc-minimal.

In their 1981 paper, Hahn and Zamfirescu presented two planar non-hamiltonian bihomogeneously traceable graphs, one of which is arc-minimal, and asked the natural question whether infinitely many such graphs do exist. Very recently it was proven that this is indeed the case, see [4].

The following problems, however, are still open.

**Problem 1.** Is there an infinite family of planar arc-minimal non-hamiltonian bihomogeneously traceable oriented graphs?

**Problem 2.** Are there such graphs on all orders greater than some integer? Even if one removes the condition of arc-minimality, this problem is still open.
REFERENCES


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