Opuscula Mathematica • Vol. 31 • No. 1 • 2011

EXISTENCE AND UNIQUENESS OF ANTI-PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR n-TH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

Ling Liu, Yongkun Li

Abstract. In this paper, we use the method of coincide degree theory to establish new results on the existence and uniqueness of anti-periodic solutions for a class of nonlinear n-th order functional differential equations of the form

$$x^{(n)}(t) = F(t, x_t, x_t^{(n-1)}, x(t), x^{(n-1)}(t), x(t-\tau(t)), x^{(n-1)}(t-\sigma(t))).$$

Keywords: anti-periodic solution, coincidence degree, nonlinear n-th-order equation, delay.

Mathematics Subject Classification: 34K13.

1. INTRODUCTION

Consider the nonlinear nth-order functional differential equation

$$x^{(n)}(t) = F(t, x_t, x_t^{(n-1)}, x(t), x^{(n-1)}(t), x(t-\tau(t)), x^{(n-1)}(t-\sigma(t))), \tag{1.1}$$

where $F : \mathbb{R}^7 \to \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ are continuous $\frac{T}{2}$-periodic functions, $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous differential $\frac{T}{2}$-periodic function, $\sigma^L = \max_{t\in[0,T]} |\sigma'(t)| < 1, x_t(\theta) = x(t+\theta)$ for $\theta \in \mathbb{R}$, and $T > 0$ is a constant.

Clearly, when $F = p(t) - f(x^{(n-1)}(t)) - g(x(t-\tau(t)))$ Eq. (1.1) reduces to

$$x^{(n)}(t) + f(x^{(n-1)}(t)) + g(x(t-\tau(t))) = p(t),$$

which has been discussed in [1]. And when $n = 2$ and $F = p(t) - f(x'(t)) - g(x(t-\tau(t)))$ or $F = p(t) - f(t, x(t))x'(t) - g(x(t-\tau(t)))$, Eq. (1.1) reduces to

$$x''(t) + f(x'(t)) + g(x(t-\tau(t))) = p(t) \text{ or } x''(t) + f(t, x(t))x'(t) + g(x(t-\tau(t)) = p(t)$$

which has been known as the delayed Rayleigh equation [2–6] or the delayed Liénard equation [7–9], respectively. Therefore, we can consider Eq. (1.1) as a generalized higher-order delayed Rayleigh equation or delayed Liénard equation.

Arising from problems in applied sciences, anti-periodic problems of nonlinear differential equations have been extensively studied by many authors during the past twenty years, see [10–18] and references therein. For example, anti-periodic trigonometric polynomials are important in the study of interpolation problems [19, 20], and anti-periodic wavelets are discussed in [21]. However, to the best of our knowledge, there exists few results for the existence and uniqueness of anti-periodic solutions of Equation (1.1) by applying the method of coincidence degree.

A primary purpose of this paper is to study the existence and uniqueness of anti-periodic solutions of Eq. (1.1) by using the method of coincidence degree theory.

The organization of this paper is as follows. In Section 2, we give some lemmas needed in later sections. In Section 3, by using the method of coincidence degree, we establish some sufficient conditions for the existence and uniqueness of anti-periodic solutions of Eq. (1.1). An illustrative example is given in Section 4.

2. PRELIMINARIES

The following continuation theorem of coincidence degree theory is crucial in the arguments of our main results which are cited from [22].

Let $\mathbb{X}$, $\mathbb{Y}$ be Banach spaces, $L : \operatorname{Dom} L \subset \mathbb{X} \to \dim \mathbb{Y}$ be a linear mapping, and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{co} \dim \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $\mathbb{Y}$. If L is a Fredholm mapping of index zero and there exist continuous projector $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im}(I - Q)$, it follows that mapping $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I-P)\mathbb{X} \to \operatorname{Im} L$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to \mathbb{X}$ is compact.

Lemma 2.1 ([22]). *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \overline{\Omega} \neq \varnothing$ and $N : \overline{\Omega} \to \mathbb{Y}$ is L-compact. Further, we also assume that*

(H) $Lx - Nx \neq \lambda(-Lx - N(-x))$ *for all* $x \in D(L) \cap \partial\Omega, \lambda \in (0, 1]$.

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \overline{\Omega}$.

Let $x : \mathbb{R} \to \mathbb{R}$ be continuous, $x(t)$ is said to be anti-periodic on $\mathbb{R}$ if,

$$x(t + \frac{T}{2}) = -x(t), \quad \text{for all } t \in \mathbb{R}.$$

We will adopt the following notations:

$$C_T^k := \{x \in C^k(\mathbb{R},\mathbb{R}), x \text{ is } T\text{-periodic}\}, \quad k \in \{0,1,2,\cdots\},$$

$$|x|_2 = \Bigg(\int\limits_0^T |x(t)|^2 dt\Bigg)^{1/2}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x^{(k)}|_\infty = \max_{t\in[0,T]} |x^{(k)}(t)|,$$

$$C_T^{k,\frac{1}{2}} := \Big\{x \in C_T^k, x\Big(t + \frac{T}{2}\Big) = -x(t), \quad \text{for} \quad \text{all} \quad t \in \mathbb{R}\Big\}.$$

It is clear that $C_T^{k,\frac{1}{2}}$ is a linear normed space endowed with the norm $\|\cdot\|$ defined by

$$\|x\| = \max\{|x|_\infty, |x'|_\infty, \cdots, |x^{(k)}|_\infty\}, \text{ for all } x \in C_T^{k,\frac{1}{2}}.$$

For the sake of convenience, we introduce the following assumptions:

(H_1) There exist nonnegative constants $\alpha, \beta, \gamma, \delta, \epsilon$ and η such that

$$\begin{aligned}|F(t,x_1,x_2,x_3,x_4,x_5,x_6) - F(t,y_1,y_2,y_3,y_4,y_5,y_6)| \le {} & \alpha|x_1-y_1| + \beta|x_2-y_2| + \\ & + \gamma|x_3-y_3| + \delta|x_4-y_4| + \\ & + \epsilon|x_5-y_5| + \eta|x_6-y_6|\end{aligned}$$

for all $(t,x_1,x_2,x_3,x_4,x_5,x_6),(t,y_1,y_2,y_3,y_4,y_5,y_6) \in \mathbb{R}^7$.

(H_2) There exists a nonnegative constant m such that

$$m|x-y| \le |F(t,u_1,u_2,u_3,x,u_4,u_5) - F(t,u_1,u_2,u_3,y,u_4,u_5)|$$

for all $t,x,y \in \mathbb{R}$ and some constants $u_1,u_2,u_3,u_4,u_5 \in \mathbb{R}$.

(H_3) For all $(t,x,y,z,g,h,j) \in \mathbb{R}^7$,

$$F\Big(t + \frac{T}{2}, -x, -y, -z, -g, -h, -j\Big) = -F(t,x,y,z,g,h,j).$$

Lemma 2.2 ([23]). *If* $x \in C^2(\mathbb{R},\mathbb{R}), x(t+T) = x(t)$, *then*

$$|x'|_2 \le \frac{T}{2\pi}|x''|_2.$$

Lemma 2.3 ([24]). *If* $x \in C_T^1$ *and* $\int_0^T x(t)\,\mathrm{dt} = 0$, *then*

$$|x|_2 \le \frac{T}{2\pi}|x'|_2.$$

Lemma 2.4. *If $x \in C_T^k$, then*

$$\int_0^T |x^{(k)}(t-\sigma(t))|dt \le \frac{1}{1-\sigma^L}\int_0^T |x^{(k)}(t)|dt.$$

Proof. Since $\sigma^L = \max_{t\in[0,T]} |\sigma'(t)| < 1$, then $t-\sigma(t)$ has its inverse function and represents the inverse function of $t-\sigma(t)$ by $\mu(t)$. Let $t-\sigma(t)=s$, then $t=\mu(s)$ and

$$\begin{aligned}\int_0^T |x^{(k)}(t-\sigma(t))|dt &= \int_{-\sigma(0)}^{T-\sigma(T)} \mu'(s)|x^{(k)}(s)|\mathrm{d}s = \\ &= \int_{-\sigma(0)}^{T-\sigma(T)} \frac{|x^{(k)}(s)|}{1-\sigma'(\mu(s))}\mathrm{d}s \le \\ &\le \frac{1}{1-\sigma^L}\int_{-\sigma(0)}^{T-\sigma(T)} |x^{(k)}(s)|\mathrm{d}s \le \\ &\le \frac{1}{1-\sigma^L}\int_0^T |x^{(k)}(s)|\mathrm{d}s.\end{aligned}$$

This completes the proof of this lemma. □

Lemma 2.5. *Assume that one of the following conditions is satisfied:*

(H_4) *Suppose that* (H_1) *holds, and* $\left[\alpha(\frac{T}{2\pi})^n + \left(\beta+\delta+\frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\right)\frac{T}{2\pi} + \frac{\gamma+\epsilon}{\pi^{n-1}}\frac{T^n}{2^n}\right] < 1.$

(H_5) *Suppose that* $(H_1)-(H_2)$ *hold, and* $0 \le \delta < m$.

Then Eq. (1.1) *has at most one anti-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two anti-periodic solutions of Eq. (1.1). Then we have

$$(x_1(t)-x_2(t))^{(n)} = F_1(t) - F_2(t), \tag{2.1}$$

where $F_i(t) = F(t, x_{it}, x_{it}^{(n-1)}, x_i(t), x_i^{(n-1)}(t), x_i(t-\tau(t)), x_i^{(n-1)}(t-\sigma(t))), i = 1, 2.$ Set $z(t) = x_1(t) - x_2(t)$. Hence we get from (2.1) that

$$z^{(n)} = F_1(t) - F_2(t). \tag{2.2}$$

Since $z(t) = x_1(t) - x_2(t)$ is an anti-periodic function on $\mathbb{R}$, then

$$\int_0^T z(t)dt = 0.$$

It follows that there exists a constant $\widetilde{\gamma} \in [0, T]$ such that

$$z(\widetilde{\gamma}) = 0. \tag{2.3}$$

Then, we have

$$\begin{aligned} |z(t)| &= \left| z(\widetilde{\gamma}) + \int_{\widetilde{\gamma}}^{t} z'(s)\mathrm{d}s \right| \le \\ &\le \int_{\widetilde{\gamma}}^{t} |z'(s)|\mathrm{d}s, \quad t \in [\widetilde{\gamma}, \widetilde{\gamma} + T] \end{aligned}$$

and

$$\begin{aligned} |z(t)| &= |z(t-T)| = \\ &= \left| z(\widetilde{\gamma}) - \int_{t-T}^{\widetilde{\gamma}} z'(s)\mathrm{d}s \right| \le \\ &\le \int_{t-T}^{\widetilde{\gamma}} |z'(s)|\mathrm{d}s, \quad t \in [\widetilde{\gamma}, \widetilde{\gamma} + T]. \end{aligned}$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |z|_\infty &= \max_{t \in [0,T]} |z(t)| = \\ &= \max_{t \in [\widetilde{\gamma}, \widetilde{\gamma}+T]} |z(t)| \le \\ &\le \max_{t \in [\widetilde{\gamma}, \widetilde{\gamma}+T]} \left\{ \frac{1}{2} \left(\int_{\widetilde{\gamma}}^{t} |z'(s)|\mathrm{d}s + \int_{t-T}^{\widetilde{\gamma}} |z'(s)|\mathrm{d}s \right) \right\} \le \\ &\le \frac{1}{2} \int_0^T |z'(s)|\mathrm{d} \le \\ &\le \frac{1}{2} \sqrt{T} |z'|_2. \end{aligned} \tag{2.4}$$

Now suppose that (H_4) (or (H_5)) holds. We shall consider two cases as follows.

Case 1. If (H_4) holds, multiplying both sides of (2.2) by $z^{(n)}(t)$ and then integrating them from 0 to T, we have from (H_1) and (2.4) that

$$\begin{aligned}
|z^{(n)}|_2^2 &= \int_0^T |z^{(n)}(t)|^2 dt = \int_0^T |F_2(t)-F_1(t)||z^{(n)}(t)|dt \le \alpha \int_0^T |x_2(t+\theta)-x_1(t+\theta)||z^{(n)}(t)|dt+\\
&\quad + \beta \int_0^T |x_2^{(n-1)}(t+\theta)-x_1^{(n-1)}(t+\theta)||z^{(n)}(t)|dt + \gamma \int_0^T |x_2(t)-x_1(t)||z^{(n)}(t)|dt+\\
&\quad + \delta \int_0^T |x_2^{(n-1)}(t)-x_1^{(n-1)}(t)||z^{(n)}(t)|dt + \epsilon \int_0^T |x_2(t-\tau(t))-x_1(t-\tau(t))||z^{(n)}(t)|dt+\\
&\quad + \eta \int_0^T |x_2^{(n-1)}(t-\sigma(t))-x_1^{(n-1)}(t-\sigma(t))||z^{(n)}(t)|dt \le\\
&\le \alpha\Big(\int_0^T |x_2(t+\theta)-x_1(t+\theta)|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T |z^{(n)}(t)|^2 dt\Big)^{\frac{1}{2}} +\\
&\quad + \beta\Big(\int_0^T |x_2^{(n-1)}(t+\theta)-x_1^{(n-1)}(t+\theta)|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T |z^{(n)}(t)|^2 dt\Big)^{\frac{1}{2}} +\\
&\quad + \gamma|z|_\infty \int_0^T |z^{(n)}(t)|dt + \epsilon|z|_\infty \int_0^T |z^{(n)}(t)|dt + \delta\Big(\int_0^T |z^{(n-1)}(t)|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T |z^{(n)}(t)|^2 dt\Big)^{\frac{1}{2}} +\\
&\quad + \eta\Big(\int_0^T |x_2^{(n-1)}(t-\sigma(t))-x_1^{(n-1)}(t-\sigma(t))|^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^T |z^{(n)}(t)|^2 dt\Big)^{\frac{1}{2}} \le\\
&\le \alpha\Big(\int_\theta^{T+\theta} |x_2(s)-x_1(s)|^2 \mathrm{d}s\Big)^{\frac{1}{2}} |z^{(n)}|_2 + \beta\Big(\int_\theta^{T+\theta} |x_2^{(n-1)}(s)-x_1^{(n-1)}(s)|^2 \mathrm{d}s\Big)^{\frac{1}{2}} |z^{(n)}|_2+\\
&\quad + \gamma\sqrt{T}|z|_\infty|z^{(n)}|_2 + \epsilon\sqrt{T}|z|_\infty|z^{(n)}|_2 + \delta|z^{(n-1)}|_2|z^{(n)}|_2+\\
&\quad + \frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\Big(\int_0^T |x_2^{(n-1)}(s)-x_1^{(n-1)}(s)|^2 \mathrm{d}s |z^{(n)}|_2\Big)^{\frac{1}{2}} \le\\
&\le \alpha|z|_2|z^{(n)}|_2 + \beta|z^{(n-1)}|_2|z^{(n)}|_2 + \gamma\sqrt{T}\cdot\frac{1}{2}\sqrt{T}|z'|_2|z^{(n)}|_2 + \epsilon\sqrt{T}\cdot\frac{1}{2}\sqrt{T}|z'|_2|z^{(n)}|_2+\\
&\quad + \delta|z^{(n-1)}|_2|z^{(n)}|_2 + \frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}|z^{(n-1)}|_2|z^{(n)}|_2 =\\
&= \alpha|z|_2|z^{(n)}|_2 + \Big(\beta+\delta+\frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\Big)|z^{(n-1)}|_2|z^{(n)}|_2 + (\gamma+\epsilon)\frac{T}{2}|z'|_2|z^{(n)}|_2 \le\\
&\le \alpha(\frac{T}{2\pi})^n|z^{(n)}|_2^2 + \Big(\beta+\delta+\frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\Big)\frac{T}{2\pi}|z^{(n)}|_2^2 + (\gamma+\epsilon)\frac{T}{2}(\frac{T}{2\pi})^{n-1}|z^{(n)}|_2^2 =\\
&= \Big[\alpha(\frac{T}{2\pi})^n + \Big(\beta+\delta+\frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\Big)\frac{T}{2\pi} + \frac{\gamma+\epsilon}{\pi^{n-1}}\frac{T^n}{2^n}\Big]|z^{(n)}|_2^2.
\end{aligned}$$

It follows from (H_4) that

$$z^{(n)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}. \tag{2.5}$$

Since $z^{(n-2)}(0) = z^{(n-2)}(T)$, there exists a constant $\xi_{n-1} \in [0, T]$ such that $z^{(n-1)}(\xi_{n-1}) = 0$, then, in view of (2.5), we get

$$z^{(n-1)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}. \tag{2.6}$$

By using a similar argument as that in the proof of (2.6), in view of (2.3), we can show

$$z(t) \equiv z'(t) \equiv \ldots \equiv z^{(n-2)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}.$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Therefore, Eq. (1.1) has at most one anti-periodic solution.

Case 2. If (H_5) holds, multiplying both sides of (2.2) by $z^{(n-1)}(t)$ and then integrating them from 0 to T, together with (2.4), we can obtain from (H_1) and (H_2) that

$$\begin{aligned} m|z^{(n-1)}|_2^2 &= \int_0^T m|x_1^{(n-1)}(t) - x_2^{(n-1)}(t)|^2 dt \le \\ &\le \int_0^T |F(t, u_1, u_2, u_3, x_1^{(n-1)}, u_4, u_5) - F(t, u_1, u_2, u_3, x_2^{(n-1)}, u_4, u_5)| \times \\ &\times |x_1^{(n-1)} - x_2^{(n-1)}| dt \le \\ &\le \int_0^T \delta |x_1^{(n-1)}(t) - x_2^{(n-1)}(t)||z^{(n-1)}(t)| dt = \\ &= \delta |z^{(n-1)}|_2^2. \end{aligned} \tag{2.7}$$

By using a similar argument as that in the proof of Case 1, in view of (2.3), (H_5) and (2.7), we obtain

$$z(t) \equiv z'(t) \equiv \cdots \equiv z^{(n-1)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}.$$

Hence, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Therefore, Eq. (1.1) has at most one anti-periodic solution. The proof of Lemma 2.5 is now complete. □

3. MAIN RESULTS

Theorem 3.1. *Let (H_3) hold. Assume that either condition (H_4) or condition (H_5) is satisfied. Then Eq. (1.1) has a unique anti-periodic solution.*

Proof. Let

$$\mathbb{X} = \left\{ x \in C_T^{n-1, \frac{1}{2}} : x\Big(t + \frac{T}{2}\Big) = -x(t), \text{ for all } t \in \mathbb{R} \right\}$$

and

$$\mathbb{Y} = \left\{ x \in C_T^{n-2, \frac{1}{2}} : x\Big(t + \frac{T}{2}\Big) = -x(t), \text{ for all } t \in \mathbb{R} \right\}.$$

Then $\mathbb{X}$ and $\mathbb{Y}$ are two Banach spaces with the norms

$$\|x\|_{\mathbb{X}} = \max\{|x|_\infty, |x'|_\infty, \cdots, |x^{(n-1)}|_\infty\}$$

and

$$\|x\|_{\mathbb{Y}} = \max\{|x|_\infty, |x'|_\infty, \cdots, |x^{(n-2)}|_\infty\},$$

respectively.

Define a linear operator $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ by setting

$$Lx = x^{(n)} \text{ for all } x \in D(L),$$

where $D(L) = \{x \in \mathbb{X} : x^{(n)} \in L^2[0,T]\}$ and $N : \mathbb{X} \to \mathbb{Y}$ by setting

$$Nx = F(t, x_t, x_t^{(n-1)}, x(t), x^{(n-1)}(t), x(t-\tau(t)), x^{(n-1)}(t-\sigma(t))).$$

It is easy to see that

$$\text{Ker } L = 0 \text{ and } \text{Im } L = \left\{x \in \mathbb{Y} : \int_0^T x(s)\mathrm{d}s = 0\right\} = \mathbb{Y}.$$

Thus dim Ker $L = 0 =$ codim Im L, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P : \mathbb{X} \to \text{Ker } L$ and the averaging projector $Q : \mathbb{Y} \to \mathbb{Y}$ by

$$Px = \frac{1}{T}\int_0^T x(s)\mathrm{d}s$$

and

$$Qy = \frac{1}{T}\int_0^T y(s)\mathrm{d}s.$$

Hence Im $P =$ Ker L and Ker $Q =$ ImL. Denoting by $L_P^{-1} : \text{Im } L \to D(L) \cap \text{Ker } P$ the inverse of $L\mid_{D(L)\cap \text{Ker } P}$, we have

$$L_P^{-1}x(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!}h_i + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}x(s)\mathrm{d}s,$$

in which h_i $(i = 0, 1, \cdots, n-1)$ are decided by $EZ = B$, where

$$E = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 2 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & 2 & 0 \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & 2 \end{pmatrix}_{n\times n,} \qquad Z = \begin{pmatrix} h_{n-1} \\ h_{n-2} \\ h_{n-3} \\ \vdots \\ h_1 \\ h_0 \end{pmatrix}_{n\times 1,}$$

$B = (b_1, b_2, \cdots, b_n)^T$, $b_i = -\frac{1}{i!}\int_0^{\frac{T}{2}} (\frac{T}{2} - s)^i x(s)\mathrm{d}s$ $(i = 0, 1, \cdots, n-1)$, $c_j = \frac{(\frac{T}{2})^j}{j!}$ $(j = 1, 2, \cdots, n-1)$.

Clearly, QN and $L_p^{-1}(I - Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that $QN(\overline{\Omega}), L_P^{-1}(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 2.1, we need to find appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x))$, $\lambda \in (0, 1]$, we have

$$x^{(n)} = \frac{1}{1+\lambda}G(t, x) - \frac{\lambda}{1+\lambda}G(t, -x), \tag{3.1}$$

where

$$G(t, x) = F(t, x_t, x_t^{(n-1)}, x(t), x^{(n-1)}(t), x(t - \tau(t)), x^{(n-1)}(t - \sigma(t)))$$

and

$$G(t, -x) = F(t, -x_t, -x_t^{(n-1)}, -x(t), -x^{(n-1)}(t), -x(t - \tau(t)), -x^{(n-1)}(t - \sigma(t))).$$

Suppose that $x \in \mathbb{X}$ is an arbitrary anti-periodic solution of Eq. (3.1). Then, by using a similar argument as that in the proof of (2.4), we have

$$|x|_\infty \le \frac{1}{2}\sqrt{T}|x'|_2. \tag{3.2}$$

In view of (H_4) and (H_5), we consider two cases as follows.

Case 1. If (H_4) holds, multiplying both sides of Eq. (3.1) by $x^{(n)}$ and then integrating it from 0 to T, in view of Lemma 2.2 - Lemma 2.4, we obtain

$$
\begin{aligned}
|x^{(n)}|_2^2 &= \int_0^T |x^{(n)}(t)|^2\,dt = \int_0^T \Big|\frac{1}{1+\lambda}G(t,x) - \frac{\lambda}{1+\lambda}G(t,-x)\Big||x^{(n)}(t)|\,dt \le \\
&\le \Big(\frac{1}{1+\lambda} + \frac{\lambda}{1+\lambda}\Big)\int_0^T \max\Big\{|G(t,x)|, |G(t,-x)|\Big\}|x^{(n)}(t)|dt = \\
&= \int_0^T \max\Big\{|G(t,x)|, |G(t,-x)|\Big\}|x^{(n)}(t)|dt \le \\
&\le \int_0^T \max\Big\{|G(t,x)-G(t,0)|, |G(t,-x)-G(t,0)|\Big\}|x^{(n)}(t)|dt + \int_0^T |G(t,0)||x^{(n)}(t)|dt \le \\
&\le \alpha\int_0^T |x(t+\theta)||x^{(n)}(t)|dt + \beta\int_0^T |x^{(n-1)}(t+\theta)||x^{(n)}(t)|dt + \gamma\int_0^T |x(t)||x^{(n)}(t)|dt + \\
&\quad + \delta\int_0^T |x^{(n-1)}(t)||x^{(n)}(t)|dt + \epsilon\int_0^T |x(t-\tau(t))||x^{(n)}(t)|dt + \\
&\quad + \eta\int_0^T |x^{(n-1)}(t-\sigma(t))||x^{(n)}(t)|dt + \int_0^T |G(t,0)||x^{(n)}(t)|dt \le \\
&\le \alpha|x|_2|x^{(n)}|_2 + \beta|x^{(n-1)}|_2|x^{(n)}|_2 + \gamma\sqrt{T}\cdot\frac{1}{2}\sqrt{T}|x'|_2|x^{(n)}|_2 + \epsilon\sqrt{T}\cdot\frac{1}{2}\sqrt{T}|x'|_2|x^{(n)}|_2 + \\
&\quad + \delta|x^{(n-1)}|_2|x^{(n)}|_2 + \frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}|x^{(n-1)}|_2|x^{(n)}|_2 + \max_{t\in[0,T]}|G(t,0)|\int_0^T |x^{(n)}(t)|dt = \\
&= \alpha|x|_2|x^{(n)}|_2 + \bigg(\beta+\delta+\frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\bigg)|x^{(n-1)}|_2|x^{(n)}|_2 + (\gamma+\epsilon)\frac{T}{2}|x'|_2|x^{(n)}|_2 + \\
&\quad + \max_{t\in[0,T]}|G(t,0)|\sqrt{T}|x^{(n)}|_2 \le \\
&\le \alpha\Big(\frac{T}{2\pi}\Big)^n|x^{(n)}|_2^2 + \bigg(\beta+\delta+\frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\bigg)\frac{T}{2\pi}|x^{(n)}|_2^2 + (\gamma+\epsilon)\frac{T}{2}\Big(\frac{T}{2\pi}\Big)^{n-1}|x^{(n)}|_2^2 + \\
&\quad + \max_{t\in[0,T]}|G(t,0)|\sqrt{T}|x^{(n)}|_2 = \\
&= \bigg[\alpha\Big(\frac{T}{2\pi}\Big)^n + \bigg(\beta+\delta+\frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\bigg)\frac{T}{2\pi} + \frac{\gamma+\epsilon}{\pi^{n-1}}\frac{T^n}{2^n}\bigg]|x^{(n)}|_2^2 + \max_{t\in[0,T]}|G(t,0)|\sqrt{T}|x^{(n)}|_2,
\end{aligned}
\tag{3.3}
$$

which, together with (H_4), implies that there exists a positive constant D_1 such that

$$
|x^{(j)}|_2 \le \Big(\frac{T}{2\pi}\Big)^{n-j}|x^{(n)}|_2 < D_1, \quad j = 1, 2, \cdots, n. \tag{3.4}
$$

Since $x^{(j)}(0) = x^{(j)}(T)$ $(j = 0, 1, 2, \cdots, n-1)$, it follows that there exists a constant $\zeta_j \in [0, T]$ such that

$$
x^{(j+1)}(\zeta_j) = 0
$$

and

$$|x^{(j+1)}(t)| = |x^{(j+1)}(\zeta_j) + \int_{\zeta_j}^{t} x^{(j+2)}(s)\mathrm{d}s| \le \int_0^t |x^{(j+2)}(t)|dt \le \sqrt{T}|x^{(j+2)}|_2, \quad (3.5)$$

where $j = 0, 1, 2, \cdots, n-2, t \in [0, T]$.

Together with (3.2) and (3.4), (3.5) implies that there exists a positive constant D_2 such that

$$|x^{(j)}|_\infty \le \sqrt{T}|x^{(j+1)}|_2 \le D_2, \quad j = 0, 1, 2, \cdots, n-1,$$

which implies that, for all possible anti-periodic solutions $x(t)$ of (3.1), there exists a constant M_1 such that

$$\max_{1\le j\le n-1} |x^{(j)}|_\infty < M_1. \quad (3.6)$$

Case 2. If (H_5) holds, multiplying both sides of Eq. (3.1) by $x^{(n-1)}(t)$ and then integrating them from 0 to T, by (H_5) and (3.2), we have

$$\begin{aligned} m|x^{(n-1)}|_2^2 &= \int_0^T m|x^{(n-1)}(t)||x^{(n-1)}(t)|dt \le \\ &\le \int_0^T |F(t, u_1, u_2, u_3, x^{(n-1)}(t), u_4, u_5) - F(t, u_1, u_2, u_3, 0, u_4, u_5)||x^{(n-1)}(t)|dt \le \\ &\le \int_0^T \delta|x^{(n-1)}(t)||x^{(n-1)}(t)|dt = \\ &= \delta|x^{(n-1)}|_2^2, \end{aligned}$$

which implies from (H_5) that there exists a positive constant $\overline{D_2} > 0$ such that

$$|x^{(j)}|_\infty \le \sqrt{T}|x^{(x^{(j+1)})}|_2 \le \overline{D_2}, \quad j = 0, 1, 2, \cdots, n-2. \quad (3.7)$$

Multiplying both sides of Eq. (3.1) by $x^{(n)}(t)$ and then integrating it from 0 to T, by (H_5), (3.2), (3.3) and (3.7), we obtain

$$\begin{aligned} |x^{(n)}|_2^2 &= \int_0^T |x^{(n)}(t)|^2 dt \le \\ &\le \alpha|x|_2|x^{(n)}|_2 + \left(\beta + \delta + \frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\right)|x^{(n-1)}|_2|x^{(n)}|_2 + (\gamma + \epsilon)\frac{T}{2}|x'|_2|x^{(n)}|_2 + \\ &\quad + \max_{t\in[0,T]} |G(t, 0)|\sqrt{T}|x^{(n)}|_2 \le \\ &\le \alpha\overline{D_2}|x^{(n)}|_2 + \left(\beta + \delta + \frac{\eta}{(1-\sigma^L)^{\frac{1}{2}}}\right)\overline{D_2}|x^{(n)}|_2 + (\gamma + \epsilon)\frac{T}{2}\overline{D_2}|x^{(n)}|_2 + \\ &\quad + \max_{t\in[0,T]} |G(t, 0)|\sqrt{T}|x^{(n)}|_2, \end{aligned}$$

it follows from (3.5) that there exists a positive constant $\overline{D_1}$

$$|x^{(n-1)}(t)| \leq \sqrt{T}|x^{(n)}|_2 \leq \overline{D_1}. \tag{3.8}$$

Therefore, in view of (3.7) and (3.8), for all possible anti-periodic solutions $x(t)$ of (3.1), there exists a constant $\widetilde{M_1}$ such that

$$\max_{1\leq j\leq n-1} |x^{(j)}|_\infty < \widetilde{M_1},$$

which, together with (3.6), implies that

$$\max_{1\leq j\leq n-1} |x^{(j)}|_\infty < M_1 + \widetilde{M_1} + 1 := M.$$

Take

$$\Omega = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < M\}.$$

It is clear that Ω satisfies all the requirement in Lemma 2.1 and that condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 2.1 and Lemma 2.5 that Eq. (3.1) has a unique anti-periodic solution. This completes the proof. □

4. AN EXAMPLE

Example 4.1. Let $F(t, x, y, z, g, h, j) = -\frac{1}{4}y\cos t - \frac{1}{8}g - \frac{1}{6\pi}h - \frac{3}{8}j\cos^4 t$, for all $t, y, g, h, j \in \mathbb{R}$. Then the following equation

$$x'' + \frac{1}{4}(\cos t)x'(t+2) + \frac{1}{8}x'(t) + \frac{1}{6\pi}x(t - \cos^2 t) + \frac{3}{8}x'(t - \sin^2 t) = \frac{1}{40}\sin t \tag{4.1}$$

has a unique anti-periodic solution with period 2π.

Proof. By (4.1), we have $\alpha = \gamma = 0, \delta = \frac{1}{8}, \epsilon = \frac{1}{6\pi}, \eta = \frac{3}{8}$. It is obvious that assumptions (H_3) and (H_4) hold. Hence, by Theorem 3.1, Eq. (4.1) has a unique anti-periodic solution with period 2π. □

Acknowledgments

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 10971183.

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Yongkun Li
yklie@ynu.edu.cn

Yunnan University
Department of Mathematics
Kunming, Yunnan 650091, P.R. China

Ling Liu

Yunnan University
Department of Mathematics
Kunming, Yunnan 650091, P.R. China

Received: April 18, 2010.
Revised: June 3, 2010.
Accepted: June 7, 2010.