EXISTENCE AND UNIQUENESS THEOREM FOR A HAMMERSTEIN NONLINEAR INTEGRAL EQUATION

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Abstract. The existence of a solution, as well as some properties of the obtained solution for a Hammerstein type nonlinear integral equation have been investigated. For a certain class of functions the uniqueness theorem has also been proved.

Keywords: iteration, Wiener-Hopf operator, pointwise convergence, Hammerstein type equation.

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1. INTRODUCTION

Let us consider the following class of Hammerstein type nonlinear integral equations

$$\varphi(x) = \int_0^{+\infty} K(x-t)\varphi^\alpha(t)dt, \quad x \in (0,+\infty), \quad \alpha \in (0,1), \tag{1.1}$$

with respect to an unknown function $\varphi(x) \geq 0$. The kernel $K(x) \geq 0$ is an integrable function on $(-\infty,+\infty)$ such that

$$\int_{-\infty}^{+\infty} K(t)dt = 1, \quad \nu = \nu_+ - \nu_- < 0, \tag{1.2}$$

where $\nu_+ = \int_0^\infty tK(t)dt < +\infty$ and $\nu_- = \int_{-\infty}^0 tK(-t)dt < +\infty$.

In the present paper we prove the existence of a positive, monotonic increasing and bounded solution $\varphi(x) \leq 1$. Moreover, we show that $\lim_{x\to+\infty} \varphi(x) = 1$. We also prove that, by putting an additional condition on the kernel, the obtained solution is continuous on $[0,+\infty)$ and unique in a certain class of functions.

2. PRELIMINARIES

Let E be one of the following Banach spaces: $L_p(0,+\infty)$ for $p \geq 1$, $M[0,+\infty)$, $C_M[0,+\infty)$, $C_0[0,+\infty)$, where $M[0,+\infty)$ is the space of bounded functions on $[0,+\infty]$, $C_M[0,+\infty)$ is the space of continuous and bounded functions on $[0,+\infty)$, and finally $C_0[0,+\infty)$ is the the space of continuous functions, possessing zero limit at infinity.

We denote by $\mathcal{K}$ the Wiener-Hopf type integral operator with the kernel $K(x)$

$$(\mathcal{K}f)(x) = \int_0^{+\infty} K(x-t)f(t)dt, \quad x > 0, \quad f \in E, \quad \mathcal{K} : E \to E. \tag{2.1}$$

It is known (see [1, §1, Theorem 1.1]) that given condition (1.2) the operator $I - \mathcal{K}$ permits the following volteryan factorization

$$I - \mathcal{K} = (I - V_-)(I - V_+) \tag{2.2}$$

as an equality of operators acting in space E. Here

$$(V_- f)(x) = \int_x^{+\infty} v_-(t-x)f(t)dt, \quad (V_+ f)(x) = \int_0^x v_+(x-t)f(t)dt, \tag{2.3}$$

where $0 \leq v_\pm \in L_1(0,+\infty)$, and

$$\gamma_- = \int_0^{+\infty} v_-(x)dx = 1, \quad \gamma_+ = \int_0^{+\infty} v_+(x)dx < 1. \tag{2.4}$$

The existence of the solution of the corresponding linear equation

$$S(x) = \int_0^{+\infty} K(x-t)S(t)dt, \quad x > 0 \tag{2.5}$$

was proved in [3]. Using factorization (2.2), it was proved that the problem (2.5), such that (1.2) holds, has a positive solution, possessing the following properties (see [1, §3, p. 188]):

(a) $1 \leq S(x) \leq (1-\gamma_+)^{-1}$, $x > 0$,
(b) $S(x) \uparrow$ by x on $[0,+\infty)$, i.e. $S(x)$ is increasing on $[0,+\infty)$,
(c) $\lim\limits_{x\to+\infty} S(x) = (1-\gamma_+)^{-1}$.

3. BASIC RESULT

We introduce the following iteration for equation (1.1):

$$\varphi_{n+1}(x) = \int_0^{+\infty} K(x-t)\varphi_n^{\alpha}(t)dt, \quad x>0, \quad \alpha\in(0,1), \quad n=0,1,2,\dots, \qquad (3.1)$$
$$\varphi_0(x)\equiv 1, \quad x>0.$$

By induction, it is easy to check that the following statements are true:

$j_1)$ $\varphi_n(x)\downarrow$ by n,
$j_2)$ $\varphi_n(x)\geq(1-\gamma_+)S(x)$, $n=0,1,2,\dots$
$j_3)$ $\varphi_n(x)\uparrow$ by x on $[0,+\infty)$, $n=0,1,2,\dots$

For example, we prove $j_2)$ and $j_3)$. When $n=0$, inequality $j_2)$ immediately follows from the double inequality $1\leq S(x)\leq(1-\gamma_+)^{-1}$. Assuming that $\varphi_n(x)\geq$ $\geq(1-\gamma_+)S(x)$ we have

$$\varphi_{n+1}(x)\geq(1-\gamma_+)^{\alpha}\int_0^{+\infty}K(x-t)S^{\alpha}(t)dt\geq(1-\gamma_+)\int_0^{+\infty}K(x-t)S(t)dt=(1-\gamma_+)S(x),$$

because $\alpha\in(0,1)$ and $0<(1-\gamma_+)S(x)\leq 1$.

Now we prove statement $j_3)$. Let $x_1,x_2\in[0,+\infty)$ be arbitrary numbers such that $x_1>x_2$. We may rewrite iteration (3.1) in the following form:

$$\varphi_{n+1}(x)=\int_{-\infty}^{x}K(\tau)\varphi_n^{\alpha}(x-\tau)d\tau, \quad n=0,1,2,\dots, \quad \varphi_0(x)\equiv 1,$$

It is obvious that $\varphi_0(x)$ is increasing by x. Assuming that $\varphi_n(x)$ is an increasing function by x we have

$$\varphi_{n+1}(x_1)-\varphi_{n+1}(x_2)=\int_{-\infty}^{x_1}K(t)\varphi_n^{\alpha}(x_1-t)dt-\int_{-\infty}^{x_2}K(t)\varphi_n^{\alpha}(x_2-t)dt\geq$$
$$\geq\int_{-\infty}^{x_1}K(t)\varphi_n^{\alpha}(x_2-t)dt-\int_{-\infty}^{x_2}K(t)\varphi_n^{\alpha}(x_2-t)dt=$$
$$=\int_{x_2}^{x_1}K(t)\varphi_n^{\alpha}(x_2-t)dt\geq 0.$$

We proved that $j_3)$ holds.

It follows from $j_1)$ and $j_2)$ that the sequence of functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ has the pointwise limit

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x)\leq 1. \qquad (3.2)$$

From B. Levi's theorem (see [2]) we deduce that the limit function satisfies equation (1.1). It follows from j_3) that

$$\varphi(x) \uparrow \text{ by } x \text{ on } (0,+\infty). \tag{3.3}$$

Taking into account j_2) and (3.2) we obtain the following double inequalities:

$$1-\gamma_+ \leq (1-\gamma_+)S(x) \leq \varphi(x) \leq 1, \tag{3.4}$$

$$\lim_{x\to\infty} \varphi(x) = 1. \tag{3.5}$$

Now we prove that if

$$0 < \gamma_+ < 1 - \frac{1}{e}, \tag{3.6}$$

then $\varphi \in C[0,+\infty)$ and a solution of equation (1.1) in the following class of functions

$$\mathfrak{M} = \{f \in M[0,+\infty) : f(x) \geq 1-\gamma_+ \text{ for all } x \in [0,+\infty)\} \tag{3.7}$$

is unique.

First we show the continuity of the obtained solution assuming that condition (3.6) is fulfilled. By induction, we show that the following inequality holds

$$|\varphi_{n+1}(x) - \varphi_n(x)| \leq (\alpha e^{1-\alpha})^n, \quad n = 0,1,2,\ldots. \tag{3.8}$$

In the case of $n=0$ the inequality is obvious, because

$$|\varphi_1(x) - \varphi_0(x)| = 1 - \int\limits_{-\infty}^{x} K(\tau)d\tau \leq 1.$$

Assume that (3.8) is true for any $n = p \in \mathbb{N}$. Taking into account the inequality

$$|x_1^\alpha - x_2^\alpha| \leq \alpha \left(\frac{1}{1-\gamma_+}\right)^{1-\alpha} |x_1 - x_2| \quad \text{for all} \quad x_1, x_2 \in [1-\gamma_+, +\infty) \tag{3.9}$$

we obtain from (3.1) that

$$\begin{aligned}|\varphi_{p+2}(x) - \varphi_{p+1}(x)| &\leq \int\limits_0^{+\infty} K(x-t)|\varphi_{p+1}^\alpha(t) - \varphi_p^\alpha(t)|dt \leq \\ &\leq \alpha \left(\frac{1}{1-\gamma_+}\right)^{1-\alpha} \int\limits_0^{+\infty} K(x-t)|\varphi_{p+1}(t) - \varphi_p(t)|dt \leq \\ &\leq \alpha \left(\frac{1}{1-\gamma_+}\right)^{1-\alpha} \alpha^p e^{p-\alpha p} \int\limits_{-\infty}^{x} K(\tau)d\tau \leq \alpha^{(p+1)} e^{(1-\alpha)(p+1)}.\end{aligned}$$

As $e^{\alpha-1} > \alpha$, $\alpha \in (0,1)$, then $q = \alpha e^{1-\alpha} \in (0,1)$. Therefore, in accordance with the Weierstrass theorem, from (3.8) it follows that the convergence of sequences of functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ is uniform. By induction, the reader may easily convince himself that $\varphi_n(x) \in C[0,+\infty)$. Thus, from the Dini inverse theorem it follows that the limit function φ is continuous.

Now we prove uniqueness of a solution of equation (1.1) in the class $\mathfrak{M}$. We assume that equation (1.1) has two different solutions φ and φ^*, which belong to $\mathfrak{M}$. Then from (1.1), (3.6) and (3.9) we have

$$|\varphi(x) - \varphi^*(x)| \leq \alpha e^{1-\alpha} \int_0^{+\infty} K(x-t)|\varphi(t) - \varphi^*(t)|dt. \tag{3.10}$$

We set

$$\delta = \sup_{x \in \mathbb{R}^+} |\varphi(x) - \varphi^*(x)|$$

Then from (3.10) we infer that $\delta \leq q\delta$ or $\delta = 0$. Therefore, $\varphi(x) = \varphi^*(x)$. In this way we prove that the following theorem holds.

Theorem 3.1. *Assume that condition* (1.2) *is fulfilled. Then equation* (1.1) *has a positive, monotonic increasing and bounded solution* $\varphi(x)$ *such that* $\lim_{x\to+\infty} \varphi(x) = 1$. *Moreover, if condition* (3.6) *holds then the obtained solution is continuous and unique in the class* $\mathfrak{M}$.

Example 3.2. Assume that $K(x)$ has the following form:

$$K(x) = \begin{cases} \beta e^{-x}; & x > 0 \\ (1-\beta)e^{x}; & x < 0 \end{cases} \quad \beta \in \left(0, \frac{1}{2}\right). \tag{3.11}$$

Opening brackets in (2.2), from operator equality we come to Yengibaryan's nonlinear factorization equation (see [1]).

$$v_{\pm}(x) = K(\pm x) + \int_0^{+\infty} v_{\mp}(t) v_{\pm}(x+t)dt, \quad x > 0. \tag{3.12}$$

From (3.11) and (3.12) it follows that $v_+ = 2\beta e^{-x}$ $(x > 0)$, $v_- = e^{x}$ $(x < 0)$, i.e. $\gamma_+ = 2\beta$, $\gamma_- = 1$. If $\beta \in \left(0, \frac{1}{2}\left(1 - \frac{1}{e}\right)\right)$, then both conditions (1.2) and (3.6) are fulfilled. Equation (1.1) with kernel (3.11) is reduced to the following ordinary differential equation

$$\varphi''(x) + (1-2\beta)\alpha\varphi^{\alpha-1}(x)\varphi'(x) - \varphi(x) = 0. \tag{3.13}$$

From the proof it follows that equation (3.13) possesses positive, bounded and monotonic increasing solution, which tends to 1 when $x \to +\infty$.

Remark 3.3. It should be noted that if we assume a weaker condition $0 < \gamma_+ < \left(1 - \frac{1}{\alpha}\right)^{\frac{1}{1-\alpha}}$ instead of (3.6) then the assertion of the theorem remains true.

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