THE FEJER-RIESZ TYPE RESULT
FOR SOME WEIGHTED HILBERT SPACES
OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this note we prove Fejer-Riesz inequality type results for some weighted Hilbert spaces of analytic functions in the unit disc. We describe also a class of such spaces for which Fejer-Riesz inequality type results do not hold.

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1. INTRODUCTION

Denote by $U$ the unit disc in the complex plane $\mathbb{C}$. Given $s > -1$ set

$$A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : \int_U |f(z)|^2 (1 - |z|^2)^s dm(z) < +\infty \right\}. \quad (1.1)$$

The spaces $A^{2,s}(U)$ were considered by many authors; see e.g. [1,2,6–8]. Given a function $f$ holomorphic in $U$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in U$, one can prove by integrating in polar coordinates and using the formula

$$(p + 1)^{1+s} \int_0^1 u^p (1 - u)^s du \to \Gamma(s + 1) \quad (1.2)$$

as $p \to \infty$ (where $\Gamma$ is Euler’s gamma function) that

$$f \in A^{2,s}(U) \text{ iff } \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty. \quad (1.3)$$
In particular for $s = 0$ we have $A^{2,0}(U) = L^2 H(U)$, the space of all holomorphic functions in $U$ which are $L^2$-integrable with respect to the Lebesgue measure in $U$; this space is called the Bergman space. For $s = 0$ the condition (1.3) has the form

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)} < +\infty.$$ 

Moreover, for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic in $U$ we have

$$\int_U |f(z)|^2 (1 - |z|^2)^s dm(z) = \sum_{n=0}^{\infty} |a_n|^2 \int_U |z|^{2n} (1 - |z|^2)^s dm(z) = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_0^1 t^{2n+1} (1 - t^2)^s dt.$$ 

If $s \leq -1$, for every $n = 0, 1, \ldots$, it holds

$$\int_0^1 t^{2n+1} (1 - t^2)^s dt = +\infty; \quad (1.4)$$

therefore, for $s \leq -1$ the spaces $A^{2,s}(U)$ defined by the integral condition (1.1) consist only of the zero function; on the other hand, the series condition

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty \quad (1.5)$$

in the right-hand side of (1.3) gives non-zero Hilbert spaces of holomorphic functions in $U$. In this note we will consider such spaces for $s \leq -1$. We recall that for $s = -1$ we obtain the condition $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$, the well-known condition characterizing functions from the Hardy space $H^2(U)$.

For functions from the space $H^2(U)$ it is the Fejer-Riesz inequality is well-known; see e.g. [p. 46][3]; it follows from this that if $f \in H^2(U)$ then for every $z \in \partial U$ it holds

$$\int_0^1 |f(tz)|^2 dt < +\infty.$$ 

(This inequality is valid for all $H^p$-spaces with $1 \leq p < +\infty$).

In [5] we have proved a result of similar type for the spaces $A^{2,s-1}(U)$ with $s > 0$:

**Proposition 1.1** ([5, Theorem 1]). *Let $s$ be a positive number. Suppose that $f \in A^{2,s-1}(U)$. Then for every $z \in \partial U$

$$\int_0^1 |f(tz)|^2 (1 - t^2)^s dt < +\infty. \quad (1.6)$$

In this note we consider the spaces $A^{2,s}(U)$ with $s \leq -1$ and for some range of the exponent $s$ we prove a result similar to that in Proposition 1.1.
2. THE RESULT

First of all, because of (1.4), for \( s \leq -1 \) we define the spaces \( A^{2,s}(U) \) by the series condition (1.5): For \( s \leq -1 \) set

\[
A^{2,s}(U) = \left\{ f \text{ holomorphic in } U : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ in } U \text{ and } \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty \right\}.
\]  

(2.1)

Moreover, note that condition (1.6) makes sense for \( s > -1 \). We want therefore to prove the following result for exponent \( s \) with \( -2 < s \leq -1 \): If \( s \) is a given number with \( -2 < s \leq -1 \), \( f \) is holomorphic in \( U \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), and

\[
\sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} < +\infty,
\]  

then for every \( z \in \partial U \):

\[
\int_{0}^{1} |f(tz)|^2 (1 - t^2)^{s+1} dt < +\infty.
\]  

(2.3)

Because of the technical difficulties we are able to prove only a weaker result, which we now describe:

Let \( \{b_k\}_{k=0}^{\infty} \) be a new sequence, obtained from \( \{a_n\}_{n=0}^{\infty} \) in such a way that we delete all numbers \( \{a_n\} \) with \( a_n = 0 \), and then reorder the remaining numbers \( a_n \) to get the new sequence \( \{a'_k\}_{k=0}^{\infty} \) with \( |a'_0| \geq |a'_1| \geq \ldots \), we define then \( b_k = |a'_k|, k = 0, 1, \ldots \).

It is not difficult to prove that also

\[
\sum_{n=0}^{\infty} \frac{b_n^2}{(n+1)^{1+s}} < +\infty.
\]  

(2.4)

The additional condition which we assume in order to be able to provide the proof is the following:

The sequence \( \left\{ \frac{b_n^2}{(n+1)^{1+s}} \right\}_{k=0}^{\infty} \) is decreasing.  

(2.5)

This condition seems to be superfluous for the result described in (2.2) and (2.3) to hold, but as already mentioned we cannot prove (2.3) without assuming it.

**Proposition 2.1.** Suppose that \( -2 < s \leq -1 \); let the function \( f \), holomorphic in \( U \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) satisfy (2.2), and suppose that condition (2.5) holds. Then for every \( z \in \partial D \)

\[
\int_{0}^{1} |f(tz)|^2 (1 - t^2)^{s+1} dt < +\infty.
\]  

(2.6)
Proof. It is sufficient to prove that for a given $z \in \partial U$

\[
\int_0^1 |f(tz)|^2 (1 - t)^{s+1} dt < +\infty.
\]  

(2.7)

We have

\[
\int_0^1 |f(tz)|^2 (1 - t)^{s+1} dt = \int_0^1 \left| \sum_{n=0}^{\infty} a_n t^n \right|^2 (1 - t)^{s+1} dt \leq \sum_{k,l=0}^{\infty} |a_k||a_l| \int_0^1 (1 - t)^{s+1} t^{k+l} dt.
\]

(2.8)

By (1.2),

\[
(p + 1)^{s+2} \int_0^1 (1 - t)^{s+1} t^p dt \to \Gamma(s + 2)
\]

as $p \to \infty$; therefore to prove that the series on the right-hand side of (2.8) is finite it is sufficient to prove that

\[
\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1 + k + l)^{s+2}}
\]

(2.9)

is convergent. We refer now to the process described in [4, pp. 278–280]. It follows from this that

\[
\sum_{k,l=0}^{\infty} \frac{1}{(1 + k + l)^{s+2}} |a_k||a_l| \leq \sum_{k,l=0}^{\infty} \frac{1}{(1 + k + l)^{s+2}} b_k b_l,
\]

(2.10)

where $\{b_k\}_{k=0}^{\infty}$ is the sequence defined above; the series in [4], formula (32), differs from our series (2.9), but the reasoning in [4] described on pages 278–280 can be applied to (2.9); the only required property is that the sequence

\[
\left\{ \frac{1}{(1 + p)^{s+2}} \right\}_{p=0}^{\infty}
\]

is decreasing, similarly as the sequence $\left\{ \frac{1}{1+p} \right\}_{p=0}^{\infty}$ considered in [4].

Therefore, to prove that (2.9) is convergent it is sufficient to prove that the sequence

\[
\sum_{k,l=0}^{\infty} \frac{1}{(1 + k + l)^{s+2}} b_k b_l
\]

on the right-hand side of (2.10) is convergent. Since the sequence $\{b_k\}_{p=0}^{\infty}$ is decreasing we may use the two-sided version of the well-known Cauchy’s concentration principle
for testing the convergence of the series \( \sum_{k=0}^{\infty} c_k \) with \( c_0 \geq c_1 \geq c_2 \geq \cdots \geq 0 \); this principle says that such a series is convergent iff the series \( \sum_{l=0}^{\infty} 2^l c_{2^l} \) is convergent. By applying this two-sided Cauchy’s principle to our double series

\[
\sum_{k,l=0}^{\infty} \frac{1}{(1 + k + l)^{s/2}} b_k b_l
\]

we obtain that the series \( \sum_{k,l=0}^{\infty} \frac{1}{(1 + k + l)^{s/2} + 2} b_k b_l \) is convergent iff the series \( \sum_{r,l=0}^{\infty} \frac{2^{r+l}}{(2^{r} + 2^{l})^{s/2} + 2} b_{2^r} b_{2^l} \) is convergent. Moreover since \( (1 + 2^r + 2^l)^{s/2} \) is estimated from below and from above by a constant time of \( (2^r)^{s/2} + (2^l)^{s/2} \) it suffices to show that the series

\[
\sum_{r,l=0}^{\infty} \frac{2^{r+l}}{(2^{r})^{s/2} + (2^l)^{s/2}} b_{2^r} b_{2^l}
\]

is convergent. We have

\[
\sum_{r,l=0}^{\infty} \frac{2^{r+l}}{(2^{r})^{s/2} + (2^l)^{s/2}} b_{2^r} b_{2^l} =
\]

\[
= \sum_{r,p=0}^{\infty} \frac{2^{r+p}}{(2^{r})^{s/2} + (2^{p})^{s/2}} b_{2^r} b_{2^p} =
\]

\[
= \sum_{r,p=0}^{\infty} \frac{2^{r+p}}{(2^{r})^{s/2} + (2^{p})^{s/2}} b_{2^r} b_{2^p} \leq
\]

\[
\leq \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{2^{r+s}} b_{2^r} \frac{2^{p}}{(2^{p})^{s/2}} b_{2^p} \leq
\]

\[
\leq \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{2^{r+s}} \frac{1}{2^{p}(1+s/2)} b_{2^r} b_{2^p} =
\]

\[
= \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \left( \frac{1}{(2^{r})^{s/2} b_{2^r}} \right) \left( \frac{1}{(2^{p})^{s/2} b_{2^p}} \right) \frac{1}{2^{p}(1+s/2)} \leq
\]

\[
\leq \sum_{p=0}^{\infty} \left( \sum_{r=0}^{\infty} \frac{1}{(2^{r})^{s/2} b_{2^r}} \right)^{1/2} \left( \sum_{r=0}^{\infty} \frac{1}{(2^{p})^{s/2} b_{2^p}} \right)^{1/2} \frac{1}{2^{p}(1+s/2)} \leq
\]

\[
\leq \sum_{p=0}^{\infty} \left( \sum_{r=0}^{\infty} \frac{1}{2^{r} b_{2^r}} \right)^{1/2} \frac{1}{2^{p}(1+s/2)}.
\]

By conditions (2.9) and (2.5) it follows from the aforementioned Cauchy’s concentration principle that the series

\[
\sum_{r=0}^{\infty} \frac{2^{r}}{(1 + 2^{r})^{1+s} b_{2^r}}
\]
is convergent; hence also the series
\[ \sum_{r=0}^{\infty} \frac{1}{2^{rs}} b_{2r}^2 \]
is convergent. Therefore the right-hand side of (2.11) is estimated by a constant time of
\[ \sum_{p=0}^{\infty} \frac{1}{2^{p(1+s/2)}}. \]
Since \(-2 < s \leq -1\), this last sequence as well as both sequences in (2.10) are convergent; this proves that the integral in (2.7) is finite, and so Proposition 2.1 is proved. \( \square \)

Note that we can prove relatively easily that for \( f \) and \( s \) like in Proposition 2.1 the integral
\[ \int_0^1 |f(tz)|^2 (1 - t^2)^{\sigma+1} dt \]
for \( \sigma > s \) is convergent, if we assume that \( f \) satisfies (2.2); we do not need any further condition like e.g. (2.5).

In fact, using a similar argument to that used in the proof of Proposition 2.1 with \( s \) replaced by \( \sigma \) we see that it is sufficient to show that the series
\[ \sum_{k,l=0}^{\infty} |a_k| |a_l| \frac{1}{(1 + k + l)^{\sigma+2}} \]
is convergent. We have by Hölder’s inequality
\[
\sum_{k,l=0}^{\infty} |a_k| |a_l| \frac{1}{(1 + k + l)^{\sigma+2}} = \sum_{k,l=0}^{\infty} |a_k| |a_l| \frac{(k + 1)^{(s+1)/2}(l + 1)^{(s+1)/2}}{(1 + k + l)^{\sigma+2}} \leq \sum_{k,l=0}^{\infty} \frac{|a_k|^2 |a_l|^2}{(k + 1)^{s+1}(l + 1)^{s+1}} \left( \sum_{k,l=0}^{\infty} \frac{(k + 1)^{s+1}(l + 1)^{s+1}}{(1 + k + l)^{2(\sigma+2)}} \right)^{1/2}.
\]
The first series in the right-hand side of the above inequality is convergent by (2.2). As to the second series it is sufficient to show that the series
\[ \sum_{k,l=1}^{\infty} \frac{k^{s+1} l^{s+1}}{(k + l)^{2(\sigma+2)}} \]
is convergent. By using the above mentioned Cauchy’s concentration principle we see that it remains to show that the series
\[ \sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r + 2^t)^{2(\sigma+2)}} 2^{r(s+1)} 2^{t(s+1)} \]

is convergent. But
\[
\sum_{r,t=1}^{\infty} \frac{2^r 2^t}{(2^r + 2^t)^{2(\sigma+2)}} 2^r 2^t 2^{(s+1)} 2^{(s+1)} = \sum_{r,p=1}^{\infty} \frac{2^r 2^r + p}{(2^r + 2^r + p)^{2(\sigma+2)}} 2^r 2^{(s+1)} 2^{(r+p)(s+1)}
\]
and this last series is a constant time of
\[
\sum_{r=1}^{\infty} \left(2^{(s-\sigma)}\right)^r \sum_{p=1}^{\infty} \left(\frac{1}{2^{2\sigma-s}}\right)^p.
\]
(2.12)

Since \(-2 < s < \sigma\), both series in (2.12) converge.

Note that this method does not work for \(\sigma = s\), since in that case the first series in (2.12) is divergent.

Now we shall prove that the result of Proposition 2.1 is the best possible in the following sense:

**Proposition 2.2.** Let \(s\) be a number such that \(-2 < s \leq -1\). Then there exists a function \(f \in A^{2,s}(U)\) such that for all \(\sigma\) with \(-2 < \sigma < s\),
\[
\int_0^1 |f(t)|^2 (1 - t^2)^{\sigma+1} dt = +\infty.
\]
(2.13)

**Proof.** Like in [5, Theorem 2], set
\[
f(z) = \sum_{n=2}^{\infty} \frac{n^{s/2}}{\log n} z^n.
\]
We have
\[
\sum_{n=2}^{\infty} \frac{|a_n|^2}{(n+1)^{1+s}} = \sum_{n=2}^{\infty} \frac{n^{s}}{(n+1)^{1+s} \log^2 n}
\]
and this last series is convergent, so in virtue of (2.1), \(f \in A^{2,s}(U)\).

Moreover, with \(\sigma\) like in the assumption of Proposition 2.2 we have
\[
\int_0^1 |f(t)|^2 (1 - t^2)^{\sigma+1} dt \approx \int_0^1 |f(t)|^2 (1 - t)^{\sigma+1} dt = \sum_{k,l=2}^{\infty} a_k a_l \int_0^1 t^{k+l}(1 - t)^{\sigma+1} dt =
\]
\[
= \sum_{k,l=2}^{\infty} \frac{k^{s/2} l^{s/2}}{\log k \log l} \int_0^1 t^{k+l}(1 - t)^{\sigma+1} dt.
\]
By virtue of (1.2) instead of the last series we may consider the series
\[
\sum_{k,l=2}^{\infty} \frac{k^{s/2} l^{s/2}}{\log k \log l (1 + k + l)^{2+\sigma}}.
\]
(2.14)
If \( \tau \) is any number with \( \sigma < \tau < s \), this last series is bounded from below by a constant time of the series
\[
\sum_{k,l=2}^{\infty} \frac{k^{s/2}l^{s/2}}{(k+l)^{2+\tau}}.
\] (2.15)

To this series we can apply the already mentioned two-sided Cauchy’s concentration principle. It follows that our series behaves like the series
\[
\sum_{r,t=1}^{\infty} \frac{2^{r}2^{s}r^{s/2}2^{t}s/2}{(2^{r} + 2^{t})^{2+\tau}}.
\] (2.16)

But taking the subseries of the last series, consisting only of terms with \( r = t \) we obtain
\[
\sum_{r=1}^{\infty} \frac{2^{rs}2^{r}2^{r}}{(2^{r+1})^{2+\tau}} = \frac{1}{2^{2+\tau}} \sum_{r=1}^{\infty} \left( \frac{2^{s}+s}{2^{2+\tau}} \right)^{r} = \frac{1}{2^{2+\tau}} \sum_{r=1}^{\infty} (2^{s-\tau})^{r}.
\]

Since this last series is divergent, so are the series (2.16), (2.15) and (2.14), and therefore the integral (2.13) is infinite. This ends the proof.

Finally we want to consider the spaces \( A^{2,s}(U) \) with \( s \leq -2 \). In this case for \( f \in A^{2,s}(U) \), \( f(z) = \sum_{n=0}^{\infty} a_{n}z^{n} \), neither the condition
\[
\int_{U} |f(z)|^2 (1 - |z|^2)^{s} dm(z) < +\infty
\] (2.17)

nor
\[
\int_{0}^{1} |f(tz)|^2 (1 - t^2)^{s+1} dt < +\infty,
\] (2.18)

\( z \in \partial D \), hold. As explained before, the integral condition (2.17) was already replaced by condition
\[
\sum_{n=0}^{\infty} \frac{|a_{n}|^2}{(n+1)^{1+s}} < +\infty.
\] (2.19)

Moreover, if the coefficients \( a_{n} \) are non-negative, then it follows from (1.2), (2.8), and (2.9) that for \( s > -2 \) the expression
\[
\int_{0}^{1} |f(tz)|^2 (1 - t^2)^{s+1} dt
\]
is estimated from below and from above by a constant time the sum of the series
\[
\sum_{k,l=0}^{\infty} |a_{k}||a_{l}| \frac{1}{(1+k+l)^{s+2}}.
\]
Therefore we can hope that the right analogue of the Fejer-Riesz type result for \( s \leq -2 \), described already for \(-1 \geq s > -2\), would be the following: If \( f \) is holomorphic in \( U \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), and (2.19) holds, then

\[
\sum_{k,l=0}^{\infty} |a_k||a_l| \frac{1}{(1 + k + l)^{s+2}} < +\infty. \tag{2.20}
\]

We show that such a result does not hold for \( s \leq -2 \); we have namely the following proposition:

**Proposition 2.3.** Suppose that \( s \) is a real number with \( s \leq -2 \). Then there exists the function \( f \) holomorphic in \( U \), \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), such that (2.19) holds, but the series in (2.20) does not converge.

**Proof.** The choice of the function \( f \) is very similar to that in Proposition 2.2. Let

\[
a_n = \frac{1}{(n+1)^{-s/2}\log n}, \quad n = 2, 3, \ldots.
\]

Then

\[
\sum_{n=2}^{\infty} \frac{1}{(n+1)^{1+s}} |a_n|^2 = \sum_{n=2}^{\infty} \frac{1}{(n+1)\log^2 n} < +\infty,
\]

hence condition (2.19) is satisfied. On the other hand

\[
\sum_{k,l=2}^{\infty} a_k a_l \frac{1}{(1 + k + l)^{s+2}} = \sum_{k,l=2}^{\infty} \frac{1}{(1 + k + l)^{s+2}} \frac{1}{(k+1)^{-s/2}\log k} \frac{1}{(l+1)^{-s/2}\log l}. \tag{2.21}
\]

Consider the subseries of the series in the right-hand side of (2.21), consisting of terms with arbitrary \( k \) and with \( l = 2 \). Then we obtain the series

\[
\sum_{k=2}^{\infty} \frac{1}{(k+3)^{s+2}(k+1)^{-s/2}\log k} \frac{1}{3^{-s/2}\log 2}. \tag{2.22}
\]

Note that the terms in this last series are estimated from below by a constant time of \( \frac{1}{k^{(s/2)+2}\log k} \). Since \( s \leq -2 \) it follows that the series

\[
\sum_{k=2}^{\infty} \frac{1}{k^{(s/2)+2}\log k}
\]

is divergent, and so also is the series in (2.22), as well as the original series in (2.21). This ends the proof of Proposition 2.3.

**REFERENCES**


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