ON SOME CLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY SUBORDINATION AND SUPERORDINATION

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Abstract. Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $\Sigma_p$ denote the class of meromorphic functions of the form $g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1z + \ldots$, $z \in \hat{U}$, $a_{-p} \neq 0$.

We consider the integral operator $J_{p,\beta,\gamma} : K_{p,\beta,\gamma} \subset \Sigma_p \rightarrow \Sigma_p$ defined by

$$J_{p,\beta,\gamma}(g)(z) = \left[ \frac{\gamma - p\beta}{z^\gamma} \int_0^z \frac{g^\beta(t)(t)^{\gamma-1}}{t^{p/\beta}} dt \right]^{1/\beta}, \quad g \in K_{p,\beta,\gamma}, \ z \in \hat{U}.$$ 

We introduce some new subclasses of the class $\Sigma_p$, associated with subordination and superordination, such that, in some particular cases, these new subclasses are the well-known classes of meromorphic starlike functions and we study the properties of these subclasses with respect to the operator $J_{p,\beta,\gamma}$.

Keywords: meromorphic functions, integral operators, subordination, superordination.

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1. INTRODUCTION AND PRELIMINARIES

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\hat{U} = U \setminus \{0\}$, $H(U) = \{f : U \rightarrow \mathbb{C} : f \text{ is holomorphic in } U\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{N}^*$ let $\Sigma_p$ denote the class of meromorphic functions of the form $g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1z + \ldots$, $z \in \hat{U}$, $a_{-p} \neq 0$.

We will also use the following notations:

$$\Sigma_p^*(\alpha) = \left\{ g \in \Sigma_p : \text{Re} \left[ -\frac{zg'(z)}{g(z)} \right] > \alpha, \ z \in U \right\}, \text{ where } \alpha < p,$$
\[ \Sigma^*_p(\alpha, \delta) = \left\{ g \in \Sigma_p : \alpha < \text{Re} \left[ -\frac{zg'(z)}{g(z)} \right] < \delta, \ z \in U \right\}, \] where \( \alpha < p < \delta, \)

\[ H[a, n] = \{ f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \} \text{ for } a \in \mathbb{C}, \ n \in \mathbb{N}^*, \]

\[ A_n = \{ f \in H(U) : f(z) = z + a_n z^2 + a_{n+2} z^{n+2} + \ldots \}, \ n \in \mathbb{N}^*, \] and for \( n = 1 \) we denote \( A_1 \) by \( A \) and this set is called the class of analytic functions normalized at the origin.

We remark that \( \Sigma^*_p(\alpha) \) is the well-known class of meromorphic starlike functions of order \( \alpha \), when \( 0 \leq \alpha < 1. \)

**Definition 1.1** ([4, p. 4]). Let \( f \) and \( F \) be members of \( H(U) \). The function \( f \) is said to be subordinate to \( F \), written \( f \prec F \) or \( f(z) \prec F(z) \), if there exists a function \( w \) analytic in \( U \), with \( w(0) = 0 \) and \( |w(z)| < 1 \), such that \( f(z) = F(w(z)) \).

**Definition 1.2** ([4, p. 16]). Let \( \psi : \mathbb{C}^3 \times U \to \mathbb{C} \) and let \( h \) be univalent in \( U \). If \( p \) is analytic in \( U \) and satisfies the (second order) differential subordination

\[ \psi(p(z), zp'(z), z^2 p''(z); z) < h(z), \] (1.1)

then \( p \) is called a solution of the differential subordination. The univalent function \( q \) is called a dominant of the solutions of the differential subordination, or more simply, a dominant, if \( p \prec q \) for all \( p \) satisfying (1.1). A dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants \( q \) of (1.1) is said to be the best dominant of (1.1). (Note that the best dominant is unique up to a rotation of \( U \)).

If we require the more restrictive condition \( p \in H[a, n] \), then \( p \) will be called an \((a, n)\)-solution, \( q \) an \((a, n)\)-dominant, and \( \tilde{q} \) the best \((a, n)\)-dominant.

**Definition 1.3** ([?], [1, p. 98]). Let \( \varphi : \mathbb{C}^3 \times U \to \mathbb{C} \) and let \( h \) be analytic in \( U \). If \( p \) and \( \varphi(p(z), zp'(z), z^2 p''(z); z) \) are univalent in \( U \) and satisfy the second order differential superordination

\[ h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z), \quad z \in U, \] (1.2)

then \( p \) is called a solution of the differential superordination. An analytic function \( q \) is called a subordinant of the solutions of the differential superordination, or more simply, a subordinant, if \( q \prec p \) for all \( p \) satisfying (1.2). An univalent subordinant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all subordinants \( q \) of (1.2) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of \( U \).

**Definition 1.4** ([1, p. 99]). We denote by \( Q \) the set of functions \( f \) that are analytic and injective on \( \overline{U} \setminus E(f) \), where

\[ E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}, \]

and they are such that \( f'(\zeta) \neq 0 \) for \( \zeta \in \partial U \setminus E(f) \). The subclass of \( Q \) for which \( f(0) = a \), is denoted by \( Q(a) \).
Definition 1.5 ([4, p. 46]). Let $c$ be a complex number such that $\Re c > 0$, let $n$ be a positive integer, and let

$$C_n = C_n(c) = \frac{n}{\Re c} \left| c \right| \sqrt{1 + \frac{2\Re c}{n} + \Im c}.$$  \hspace{1cm} (1.3)

If $R(z)$ is the univalent function defined in $U$ by $R(z) = \frac{2C_n z}{1 - z^2}$, then the “Open Door” function is defined by

$$R_{c,n}(z) = R \left( \frac{z + b}{1 + bz} \right) = 2C_n \frac{(z + b)(1 + \bar{b} z)}{(1 + b z)^2 - (z + b)^2},$$ \hspace{1cm} (1.4)

where $b = R^{-1}(c)$.

Theorem 1.6 ([4, p. 83]). Let $\beta, \gamma \in \mathbb{C}$ and let $h$ be convex in $U$, with $h(0) = a$. Let $n$ be a positive integer. Suppose that the differential equation

$$q(z) + \frac{n z q'(z)}{\beta q(z) + \gamma} = h(z)$$ \hspace{1cm} (1.5)

has a univalent solution $q$ that satisfies $q(z) \prec h(z)$. If $p \in H[a, n]$ satisfies

$$p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec h(z),$$ \hspace{1cm} (1.6)

then $p(z) \prec q(z)$, and $q$ is the best $(a, n)$-dominant of (1.6).

Corollary 1.7 ([4, p. 84]). Let $h$ be convex in $U$, with $h(0) = a$, and let $m$ and $n$ be positive integers. Let $q_m$ and $q_n$ be univalent solutions of the differential equation (1.5) for $n = m$ and $n$ respectively, with $q_n \prec h$. If $m > n$, then $q_m \prec q_n$.

Theorem 1.8 ([5], [1, p. 114]). Let $\beta, \gamma \in \mathbb{C}$ and let $h$ be convex in $U$ with $h(0) = a$. Suppose that the differential equation

$$q(z) + \frac{z q'(z)}{\beta q(z) + \gamma} = h(z), \quad z \in U,$$

has the univalent solution $q$ with $q(0) = a$, and $q(z) \prec h(z)$. If $p \in H[a, 1] \cap Q$ and $p(z) + \frac{z p'(z)}{\beta p(z) + \gamma}$ is univalent in $U$, then

$$h(z) \prec p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z).$$

The function $q$ is the best subordinant.
Theorem 1.9 ([4, p. 86]). Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let $n$ be a positive integer. Let $R_{\beta a + \gamma, n}$ be as given in (1.4), let $h$ be analytic in $U$ with $h(0) = a$, and let $\text{Re} [\beta a + \gamma] > 0$. If
\[ \beta h(z) + \gamma < R_{\beta a + \gamma, n}(z), \]
then the solution $q$ of
\[ q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z), \tag{1.7} \]
with $q(0) = a$, is analytic in $U$ and satisfies $\text{Re} [\beta q(z) + \gamma] > 0$.
If $a \neq 0$, then the solution for (1.7) is given by
\[ q(z) = z^{\frac{\alpha}{n}} H^{\frac{\alpha}{n}}(z) \left[ \frac{\beta}{n} \int_0^z H^{\frac{\alpha}{n}}(t) t^{\frac{\alpha}{n} - 1} dt \right]^{-1} - \frac{\gamma}{\beta} = \tag{1.8} \]
where
\[ H(z) = z \exp \int_0^z \frac{h(t) - a}{at} dt. \]
If $a = 0$, then the solution is given by
\[ q(z) = H^{\frac{\alpha}{n}}(z) \left[ \frac{\beta}{n} \int_0^z H^{\frac{\alpha}{n}}(t) t^{-1} dt \right]^{-1} - \frac{\gamma}{\beta} = \]
where
\[ H(z) = z \exp \frac{\beta}{\gamma} \int_0^z \frac{h(t)}{t} dt. \]

Theorem 1.10 ([4, p. 97]). Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, and let $n$ be a positive integer. Let $R_{\beta a + \gamma, n}$ be as given in (1.4), let $h$ be analytic in $U$, with $h(0) = a$, $\text{Re} [\beta a + \gamma] > 0$ and
\[ (i) \quad \beta h(z) + \gamma < R_{\beta a + \gamma, n}(z). \]
If $q$ is the analytic solution of the Briot-Bouquet differential equation
\[ q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z) \]
as given in (1.8), and if

(ii) \( h \) is convex or \( Q(z) = \frac{zq'(z)}{\beta q(z) + \gamma} \) is starlike,

then \( q \) and \( h \) are univalent. Furthermore, if \( p \in H[a, n] \) satisfies

\[
p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),
\]

then \( p \prec q \), and \( q \) is the best \((a,n)\)-dominant.

**Theorem 1.11** ([3]). Let \( \beta, \gamma \in \mathbb{C} \) and let \( h \) be a convex function in \( U \), with

\[
\text{Re} [\beta h(z) + \gamma] > 0, \quad z \in U.
\]

Let \( q_m \) and \( q_k \) be the univalent solutions of the Briot-Bouquet differential equation

\[
q(z) + \frac{n z q'(z)}{\beta q(z) + \gamma} = h(z), \quad z \in U, \quad q(0) = h(0),
\]

for \( n = m \) and \( n = k \) respectively. If \( m/k \), then \( q_k(z) \prec q_m(z) \prec h(z) \). So, \( q_k(z) \prec q_1(z) \prec h(z) \).

**Theorem 1.12** ([5], [1, p. 117]). Let \( \beta, \gamma \in \mathbb{C} \) and let the function \( h \in H(U) \) with \( h(0) = a \) and \( \text{Re} c > 0 \), where \( c = \beta a + \gamma \) and suppose that

(i) \( \beta h(z) + \gamma \prec R_{c,1}(z) \).

Let \( q \) be the analytic solution of the Briot-Bouquet differential equation

\[
h(z) = q(z) + \frac{zq'(z)}{\beta q(z) + \gamma}
\]

and suppose that

(ii) \( \frac{zq'(z)}{\beta q(z) + \gamma} \) is starlike in \( U \).

If \( p \in H[a, 1] \cap Q \) and \( p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \) is univalent in \( U \), then

\[
h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z)
\]

and the function \( q \) is the best subordinant.

**Corollary 1.13** ([8]). Let \( p \in \mathbb{N}^*, \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and \( \text{Re}(\gamma - p\beta) > 0 \). If \( g \in \Sigma_p \) and

\[
\beta \frac{z g'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),
\]
then
\[ G(z) = J_{p,\beta,\gamma}(g)(z) = \left[ \frac{\gamma - p\beta}{z^\gamma} \int_0^z g^{\beta}(t)t^{\gamma-1}dt \right]^{\frac{1}{\beta}} \in \Sigma_p, \]
with \( z^p G(z) \neq 0, \ z \in U, \) and
\[ \text{Re} \left[ \beta \frac{zG'(z)}{G(z)} + \gamma \right] > 0, \ \ z \in U. \]
All powers are chosen as principal ones.

We remark that if \( p \in \mathbb{N}^*, \ \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0, \ \text{Re}(\gamma - p\beta) > 0 \) and \( g \in \Sigma_p \) with
\[ \beta \frac{z\gamma g'(z)}{g(z)} + \gamma < R_{\gamma-p\beta}(z), \ \ z \in U, \]
we have from Corollary 1.13 that \( G = J_{p,\beta,\gamma}(g) \in \Sigma_p \) with \( z^p G(z) \neq 0, \ z \in U, \) so
\[ P(z) = -\frac{zG'(z)}{G(z)} \in H[p, p]. \]
Having these conditions, it is easy to see that from
\[ G(z) = \left[ \frac{\gamma - p\beta}{z^\gamma} \int_0^z t^{\gamma-1}g^{\beta}(t)dt \right]^{\frac{1}{\beta}}, \ \ z \in \dot{U}, \]
we obtain
\[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \ \text{where} \ P(z) = -\frac{zG'(z)}{G(z)}. \quad (1.9) \]

2. MAIN RESULTS

In this section we present and prove five theorems and five corollaries concerning the integral operator \( J_{p,\beta,\gamma}. \) We consider some new subclasses of the class \( \Sigma_p, \) associated with superordination and subordination, and we establish the conditions such that when we apply the integral operator \( J_{p,\beta,\gamma} \) to a function which belongs to one of these new subclasses, the result remains in a similar class.

The first result is a simple lemma and we will use it latter to present some examples for the results included in this paper. For this lemma we need the next criteria for convexity:

**Theorem 2.1** ([6]). If \( f \in A_n \) and
\[ |f'''(z)| \leq \frac{n}{n+1}, \ \ z \in U, \]
then
\[ \left| \frac{f''(z)}{f'(z)} \right| \leq 1, \ \ z \in U, \]
hence, \( f \) is convex. The result is sharp for the function

\[
f(z) = z + \frac{z^{n+1}}{(n+1)^2}.
\]

**Lemma 2.2.** Let \( \alpha, \beta, \gamma \in \mathbb{C} \) with \( \gamma \neq 0, \alpha + \gamma \neq 0 \) and \( |\beta| < |\gamma| \). Let \( h \) be the function

\[
h(z) = z + \frac{\alpha z}{\beta z + \gamma}, \quad z \in U.
\]

If we have

\[
4|\alpha/\gamma^2| \leq (|\gamma| - |\beta|)^3|\alpha + \gamma|,
\]

then \( h \) is convex in \( U \).

**Proof.** Since \( |\beta| < |\gamma| \) we have \( \beta z + \gamma \neq 0, z \in U \), so, \( h \in H(U) \). We also have \( h'(0) = \frac{\alpha + \gamma}{\gamma} \neq 0 \), hence \( \frac{\gamma}{\alpha + \gamma} h \in A_1 \).

It is easy to see that

\[
h''(z) = -\frac{2\alpha\beta\gamma}{(\beta z + \gamma)^3}, \quad z \in U,
\]

hence

\[
\left| \frac{\gamma}{\alpha + \gamma} h''(z) \right| = \frac{|\gamma|}{|\alpha + \gamma|} \cdot \frac{2|\alpha/\gamma^2|}{|\beta z + \gamma|^3} \leq \frac{2|\alpha/\gamma^2|}{(|\gamma| - |\beta|)^3|\alpha + \gamma|} \leq \frac{1}{2}, \quad z \in U.
\]

For the last inequality we used the fact that \( 4|\alpha/\gamma^2| \leq (|\gamma| - |\beta|)^3|\alpha + \gamma| \).

Using Theorem 2.1, for \( n = 1 \), we obtain that \( h \) is convex in \( U \).

**Remark 2.3.** 1. It is obvious that if \( h \) is a convex function in \( U \) (with \( h'(0) \neq 0 \)), then \( \delta_1 + \delta_2 h(rz) \) is also a convex function, when \( r \in (0, 1] \), \( \delta_1, \delta_2 \in \mathbb{C} \), \( \delta_2 \neq 0 \).

2. If we consider \( \alpha = |\beta| = 1 \) in the above lemma, then the condition (2.1) becomes

\[
4|\gamma|^2 \leq |\gamma| + 1(|\gamma| - 1)^3.
\]

It is not difficult to verify that the condition (2.2) holds for each real number \( \gamma \geq 3, 2 \).

In other words, the functions

\[
z + \frac{z}{\gamma + z}, \quad z + \frac{z}{\gamma - z}, \quad z \in U,
\]

are convex functions when \( \gamma \geq 3, 2 \).

We mention here that in [7] the authors proved that the function

\[
h(z) = 1 + z + \frac{z}{z + 2}, \quad z \in U,
\]

is convex in \( U \), so the function \( z + \frac{z}{2 + z} \) is also a convex function.
Next, we define some new subclasses of the class $\Sigma_p$, associated with superordination and subordination, such that, in some particular cases, these new subclasses are the well-known classes of meromorphic starlike functions.

**Definition 2.4.** Let $p \in \mathbb{N}^*$ and $h_1, h_2, h \in H(U)$ with $h_1(0) = h_2(0) = h(0) = p$ and $h_1(z) \prec h_2(z)$. We define:

$$\Sigma S_p(h_1, h_2) = \left\{ g \in \Sigma_p : h_1(z) \prec -\frac{zg'(z)}{g(z)} \prec h_2(z) \right\},$$

$$\Sigma S_p(h) = \left\{ g \in \Sigma_p : -\frac{zg'(z)}{g(z)} \prec h(z) \right\}.$$

We remark that if we consider $h(z) = h_{p, \alpha}(z) = \frac{p + (p - 2\alpha)z}{1 - z}$, $z \in U, 0 \leq \alpha < p$, since $h_{p, \alpha}(U) = \{z \in \mathbb{C} : \text{Re } z > \alpha\}$, we have $\Sigma S_p(h_{p, \alpha}) = \Sigma^*_p(\alpha)$.

**Theorem 2.5.** Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\text{Re } (\gamma - p\beta) > 0$. Let $h_1$ and $h_2$ be convex functions in $U$ with $h_1(0) = h_2(0) = p$ and let $g \in \Sigma S_p(h_1, h_2)$ such that

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-\beta p,p}(z), \quad z \in U.$$

Suppose that the Briot-Bouquet differential equations

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z) \quad \text{and} \quad q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U,$$

have the univalent solutions $q^1$ and, respectively, $q^p_2$, with $q^1_2(0) = q^p_2(0) = p$ and $q^1_1 \prec h_1, q^p_2 \prec h_2$.

Let $G = J_{p, \beta, \gamma}(g)$. If $\frac{zg'(z)}{g(z)}$ is univalent in $U$ and $\frac{G'(z)}{G(z)} \in Q$, then

$$G \in \Sigma S_p(q^1_1, q^p_2).$$

The functions $q^1_1$ and $q^p_2$ are the best subordinant and, respectively, the best $(p, p)$-dominant.

**Proof.** From $g \in \Sigma S_p(h_1, h_2)$ we have $\frac{zg'(z)}{g(z)} \in H(U)$ and

$$h_1(z) \prec -\frac{zg'(z)}{g(z)} \prec h_2(z), \quad (2.3)$$

with $h_1 \prec h_2$ and $h_1(0) = h_2(0) = p$.

Let $P(z) = -\frac{zG'(z)}{G(z)}$, $z \in U$. Since $\gamma + \beta \frac{zg'(z)}{g(z)} \prec R_{\gamma-\beta p,p}(z), z \in U$, we have from Corollary 1.13 that $G = J_{p, \beta, \gamma}(g) \in \Sigma_p$ with $z^p G(z) \neq 0, z \in U$. Hence, $P \in H[p, p]$. 

From (1.9) and (2.3), we obtain
\[ h_1(z) \prec P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h_2(z), \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}, \quad z \in U. \]

If we apply Theorem 1.6 (for \( a = n = p \), \( h = h_2 \) and with \( -\beta \) instead of \( \beta \)) to the subordination
\[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h_2(z), \quad z \in U, \]
we get
\[ P(z) < q_1^p(z), \quad z \in U. \tag{2.4} \]

Because \( P \in H[p, p] \cap Q \) and \( P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \) is univalent in \( U \), we may apply Theorem 1.8 (for \( a = p \), \( n = 1 \), \( h = h_1 \) and with \( -\beta \) instead of \( \beta \)) to
\[ h_1(z) \prec P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}, \quad z \in U, \]
and we get
\[ q_1^1(z) \prec P(z), \quad z \in U. \tag{2.5} \]

From (2.4) and (2.5) we have
\[ q_1^1(z) \prec P(z) \prec q_2^p(z), \quad z \in U, \]
which is equivalent to
\[ q_1^1(z) \prec -\frac{zG'(z)}{G(z)} \prec q_2^p(z), \quad z \in U. \tag{2.6} \]

Since \( G \in \Sigma_p \) we have from (2.6) that \( G \in \Sigma_{S_p}(q_1^1, q_2^p) \).

From Theorem 1.6 and Theorem 1.8 we also have that the functions \( q_1^1 \) and \( q_2^p \) are the best subordinant and, respectively, the best \((p, p)\)-dominant. \( \square \)

If we consider in the hypothesis of Theorem 2.5 the condition
\[ \Re \left[ \gamma - \beta h_2(z) \right] > 0, \quad z \in U, \]
instead of
\[ \beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - \beta p, p}(z), \quad z \in U, \]
we get the next result.

**Theorem 2.6.** Let \( p \in \mathbb{N}^* \) and \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and \( \Re (\gamma - p\beta) > 0 \). Let \( h_1 \) and \( h_2 \) be convex functions in \( U \) with \( h_1(0) = h_2(0) = p, \) \( h_1 \prec h_2 \) and
\[ \Re [\gamma - \beta h_2(z)] > 0, \quad z \in U. \]
Let \( g \in \Sigma S_p(h_1, h_2) \) and \( G = J_{p, \beta, \gamma}(g) \). If \( \frac{zq'(z)}{g(z)} \) is univalent in \( U \) and \( \frac{zG'(z)}{G(z)} \in Q \), then

\[
G \in \Sigma S_p(q_1^1, q_2^p),
\]

where \( q_1^1 \) and \( q_2^p \) are the univalent solutions of the Briot-Bouquet differential equations

\[
q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z), \quad z \in U,
\]

(2.7)

and, respectively,

\[
q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U,
\]

(2.8)

with \( q_1^1(0) = q_2^p(0) = p \).

The functions \( q_1^1 \) and \( q_2^p \) are the best subordinant and, respectively, the best \((p, p)\)-dominant.

Proof. From \( g \in \Sigma S_p(h_1, h_2) \) we have

\[
h_1(z) \prec -\frac{zq'(z)}{g(z)} \prec h_2(z), \quad z \in U,
\]

hence

\[
\gamma - \beta h_1(z) \prec \gamma + \beta \frac{zq'(z)}{g(z)} \prec \gamma - \beta h_2(z), \quad z \in U.
\]

(2.9)

Since \( \text{Re} \left[ \gamma - \beta h_2(z) \right] > 0, \ z \in U \), we get from (2.9) that

\[
\text{Re} \left[ \gamma - \beta h_1(z) \right] > 0 \quad \text{and} \quad \text{Re} \left[ \gamma + \beta \frac{zq'(z)}{g(z)} \right] > 0, \quad z \in U.
\]

Now, it is obvious that we have

\[
\gamma + \beta \frac{zq'(z)}{g(z)} \prec R_{\gamma - \beta p, p}(z), \quad z \in U,
\]

\[
\gamma - \beta h_1(z) \prec R_{\gamma - p \beta, 1}(z) \quad \text{and} \quad \gamma - \beta h_2(z) \prec R_{\gamma - p \beta, p}(z), \quad z \in U.
\]

It is easy to see that the conditions from the hypothesis of Theorem 1.9 are fulfilled (for \( h = h_1, a = p \) and \( n = 1 \)) so, the solution \( q_1^1 \) of the equation (2.7) with \( q_1^1(0) = p \) is analytic in \( U \). Analogous we have that the solution \( q_2^p \) of the equation (2.8) with \( q_2^p(0) = p \) is analytic in \( U \).

Since \( h_1 \) and \( h_2 \) are convex functions, we have from Theorem 1.10 that the analytic functions \( q_1^1 \) and \( q_2^p \) are univalent in \( U \), and from Theorem 1.11 (since \( \text{Re} \left[ \gamma - \beta h_1(z) \right] > 0 \) and \( \text{Re} \left[ \gamma - \beta h_2(z) \right] > 0, \ z \in U \)) we have the subordinations \( q_1^1 \prec h_1 \) and \( q_2^p \prec h_2 \).

Therefore, the conditions from the hypothesis of Theorem 2.5 are fulfilled and the result follows using this theorem.
Remark 2.7. Let the conditions from the hypothesis of Theorem 2.6 be fulfilled. If we consider, in addition, that $q_1^p$ and $q_2^p$ are the univalent solutions of the Briot-Bouquet differential equations

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_1(z), \quad z \in U,$$

and, respectively,

$$q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U,$$

with $q_1^p(0) = q_2^p(0) = p$, we have from the above theorem and Corollary 1.7, that

$$q_1^p(z) \prec q_1^1(z) \prec -zG'(z)G(z) \prec q_2^p(z) \prec q_2^1(z), \quad z \in U.$$

Hence $G \in \Sigma S_p(q_1^1, q_2^1)$ is the best choice.

If we consider for Theorem 2.5 only the subordination, we obtain the next result.

Theorem 2.8. Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\Re(\gamma - pb) > 0$. Let $h$ be a convex function in $U$ with $h(0) = p$ and $g \in \Sigma S_p(h)$ such that

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - \beta p, p}(z), \quad z \in U.$$

Suppose that the Briot-Bouquet differential equation

$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U,$$

has the univalent solution $q$ with $q(0) = p$ and $q \prec h$. Then

$$G = J_{p, \beta, \gamma}(g) \in \Sigma S_p(q).$$

The function $q$ is the best $(p, p)$-dominant.

Proof. Let $P(z) = -\frac{zg'(z)}{G(z)}$, $z \in U$. We know from Corollary 1.13 that $G \in \Sigma_p$ with $z^pG(z) \neq 0$, $z \in U$, so $P \in H[p, p]$.

Since $P$ is analytic in $U$, we have from (1.9) that

$$P(z) + \frac{zp^p(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad z \in U.$$

Because $g \in \Sigma S_p(h)$ we have $-\frac{zg'(z)}{g(z)} \prec h(z)$, $z \in U$, hence

$$P(z) + \frac{zp^p(z)}{\gamma - \beta P(z)} \prec h(z), \quad z \in U.$$
Using Theorem 1.6 (for \( a = n = p \) and with \(-\beta\) instead of \( \beta \)) we get\( P \prec q \), so

\[
-\frac{zG'(z)}{G(z)} \prec q(z), \quad z \in U.
\] (2.10)

Since \( G \in \Sigma_p \) we obtain from (2.10) that

\[ G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q). \]

We also have from Theorem 1.6 that the function \( q \) is the best \((p,p)\)-dominant. □

**Theorem 2.9.** Let \( p \in \mathbb{N}^* \) and \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and \( \text{Re}[\gamma - p\beta] > 0 \). Let \( h_1 \) and \( h_2 \) be analytic functions in \( U \) with \( h_1(0) = h_2(0) = p \), \( h_1 \prec h_2 \) and

(i) \( \gamma - \beta h_2(z) \prec R_{\gamma-p\beta,1}(z), \quad z \in U. \)

If \( q_1 \) and \( q_2 \) are the analytic solutions of the Briot-Bouquet differential equations

\[
q(z) + \frac{zq'(z)}{\gamma - \beta q(z)} = h_1(z), \quad z \in U,
\] (2.11)

and, respectively,

\[
q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h_2(z), \quad z \in U,
\] (2.12)

with \( q_1(0) = q_2(0) = p \) and if

(ii) \( \frac{zq_1'(z)}{\gamma - \beta q_1(z)} \) is starlike in \( U \),

(iii) \( h_2 \) is convex or \( \frac{zq_2'(z)}{\gamma - \beta q_2(z)} \) is starlike,

then \( q_1 \) and \( q_2 \) are univalent in \( U \).

Moreover, if \( g \in \Sigma S_p(h_1, h_2) \) such that \( \frac{zq'(z)}{g(z)} \) is univalent in \( U \) and \( \frac{zG'(z)}{G(z)} \in Q \), where \( G = J_{p,\beta,\gamma}(g) \), then

\[ G \in \Sigma S_p(q_1, q_2). \]

The functions \( q_1 \) and \( q_2 \) are the best subordinant and, respectively, the best \((p,p)\)-dominant.

**Proof.** From \( h_1 \prec h_2 \) and (i) we have

\[
\gamma - \beta h_1(z) \prec \gamma - \beta h_2(z) \prec R_{\gamma-p\beta,1}(z), \quad z \in U.
\] (2.13)

From (2.13), using also the fact that \( R_{\gamma-p\beta,1}(z) \prec R_{\gamma-p\beta,p}(z), \quad z \in U \), we have

\[
\gamma - \beta h_1(z) \prec R_{\gamma-p\beta,1}(z), \quad \gamma - \beta h_2(z) \prec R_{\gamma-p\beta,p}(z), \quad z \in U.
\]
Therefore, from Theorem 1.9 (for \( n = 1 \) and \( h = h_1 \), respectively \( n = p \) and \( h = h_2 \)) we have the existence of the analytic solutions \( q_1 \) and \( q_2 \) of the equation (2.11), respectively (2.12).

Since we have conditions (\( ii \)) and (\( iii \)) in the hypothesis, we obtain from Theorem 1.10 the univalence of \( q_1 \) and \( q_2 \).

From \( g \in \Sigma S_p(h_1, h_2) \) and (i) we have

\[
\gamma - \beta h_1(z) < \gamma + \beta \frac{zg'(z)}{g(z)} < \gamma - \beta h_2(z) < R_{\gamma-p\beta,1}(z), \quad z \in U. \tag{2.14}
\]

Since \( R_{\gamma-p\beta,1}(z) < R_{\gamma-p\beta,p}(z) \), \( z \in U \), we have from (2.14)

\[
\gamma + \beta \frac{zg'(z)}{g(z)} < R_{\gamma-p\beta,p}(z), \quad z \in U.
\]

Using Corollary 1.13 we have \( G = J_{p,\beta,\gamma}(g) \in \Sigma_p \) with \( z^p G(z) \neq 0 \), \( z \in U \). Consequently,

\[
P \in H[p,p], \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}, \quad z \in U.
\]

From (1.9) and \( g \in \Sigma S_p(h_1, h_2) \) we obtain

\[
h_1(z) < P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} < h_2(z), \quad z \in U. \tag{2.15}
\]

It is easy to see that we have \( P \in H[p,p] \cap Q \) and \( P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \) univalent in \( U \).

We remark that the conditions from the hypotheses of Theorem 1.10 and Theorem 1.12 are met, so, using these two theorems we get from (2.15) that

\[
q_1(z) < P(z) < q_2(z), \quad z \in U. \tag{2.16}
\]

Since \( P(z) = -\frac{zG'(z)}{G(z)}, z \in U \), and \( G \in \Sigma_p \) we obtain from (2.16) that

\[
G \in \Sigma S_p(q_1, q_2).
\]

Of course, we also have from Theorem 1.10 and Theorem 1.12, that the functions \( q_1 \) and \( q_2 \) are the best subordinant and, respectively, the best \((p,p)\)-dominant. \( \Box \)

From Theorem 1.9, since \( p \neq 0 \), we have that the solutions \( q_1 \) and \( q_2 \) (from the above theorem) are given by:

\[
q_1(z) = z^\gamma H_1^{-p\beta}(z) \left[ -\beta \int_0^z \frac{H_1^{-p\beta}(t)t^{\gamma-1}dt}{H_1(tz)} \right]^{-1} + \frac{\gamma}{\beta} = \left[ -\beta \int_0^1 \frac{H_1(tz)}{H_1(z)} t^{-p\beta} t^{\gamma-1}dt \right]^{-1} + \frac{\gamma}{\beta},
\]
\[ q_2(z) = z^\gamma H_2^{-\beta}(z) \left[ \frac{-\beta}{p} \int_0^z H_2^{-\beta}(t) t^{\gamma-1} dt \right]^{-1} + \frac{\gamma}{\beta} = \]
\[ = \left[ \frac{-\beta}{p} \int_0^1 \left[ \frac{H_2(tz)}{H_2(z)} \right]^{-\beta} t^{\gamma-1} dt \right]^{-1} + \frac{\gamma}{\beta}, \]

where
\[ H_k(z) = z \exp \int_0^z \frac{h_k(t) - p \beta}{pt} dt, \quad k = 1, 2. \]

If we consider only the subordination for Theorem 2.9 we obtain the next result.

**Theorem 2.10.** Let \( p \in \mathbb{N}^* \) and \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and \( \Re(\gamma - p\beta) > 0 \). Also let \( h \in H(U) \) with \( h(0) = p \) such that

1. \( \gamma - \beta h(z) \prec R_{\gamma-p\beta,p}(z), \quad z \in U. \)

If \( q \) is the analytic solution of the Briot-Bouquet differential equation
\[ q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U, \]

with \( q(0) = p \) and if
\[ (ii) \quad h \text{ is convex or } \frac{zq'(z)}{\gamma - \beta q(z)} \text{ is starlike}, \]

then \( q \) is univalent in \( U \).

Moreover, if \( g \in \Sigma S_p(h) \) and \( G = J_{p,\beta,\gamma}(g) \), then \( G \in \Sigma S_p(q) \).

The function \( q \) is the best \( (p,p) \)-dominant.

**Proof.** The fact that the function \( q \) is univalent in \( U \) results from Theorem 1.10. Since \( g \in \Sigma S_p(h) \) we have
\[ -\frac{zg'(z)}{g(z)} \prec h(z), \quad z \in U, \quad (2.17) \]

and using \( (i) \) we obtain
\[ \gamma + \beta \frac{zg'(z)}{g(z)} \prec R_{\gamma-p\beta,p}(z), \quad z \in U. \]

Using now Corollary 1.13 we get that \( G = J_{p,\beta,\gamma}(g) \in \Sigma_p \) with \( z^p G(z) \neq 0, \quad z \in U \).

Hence, \( P \in H[p,p] \), where \( P(z) = -\frac{zG'(z)}{G(z)} \), \( z \in U \). We know that
\[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad z \in U, \]
and using (2.17) we get

\[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \prec h(z), \quad z \in U. \]

Using now Theorem 1.10 for \( a = n = p \) and with \(-\beta\) instead of \(\beta\), we obtain that

\[ P(z) \prec q(z), \quad z \in U. \tag{2.18} \]

Since \( G \in \Sigma_p \) we have from (2.18) that \( G \in \Sigma S_p(q) \).

It is obvious that the function \( q \) is the best \((p, p)\)-dominant. \( \square \)

If we consider, in the above theorem, that the function \( h \) is convex we obtain the corollary:

**Corollary 2.11.** Let \( p \in \mathbb{N}^* \) and \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and \( \text{Re} (\gamma - p\beta) > 0 \). Also let \( g \in \Sigma S_p(h) \) with \( h \) convex in \( U \), \( h(0) = p \). If the function \( h \) satisfies the condition

\[ \gamma - \beta h(z) \prec R_{\gamma - \beta p, p}(z), \quad z \in U, \]

then

\[ G = J_{p, \beta, \gamma}(g) \in \Sigma S_p(q), \]

where \( q \) is the univalent solution of the Briot-Bouquet differential equation

\[ q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U, \]

with \( q(0) = p \).

The function \( q \) is the best \((p, p)\)-dominant.

Next, we present an application for the above corollary, when \( \beta = 1, \gamma \in \mathbb{R} \), for a particular function \( h \). We will use the notation \( J_{p, \gamma} \) instead of \( J_{p,1, \gamma} \).

**Corollary 2.12.** Let \( p \in \mathbb{N}^* \) and \( \gamma \geq p + 3 \) such that \( 4p(\gamma - p)^2 \leq \gamma(\gamma - p - 1)^3 \).

If \( g \in \Sigma S_p(h) \) with \( h(z) = p + z + \frac{pz}{\gamma - p - z} \), then

\[ G = J_{p, \gamma}(g) \in \Sigma S_p(p + z), \]

which is equivalent to

\[ \left| \frac{zG'(z)}{G(z)} + p \right| < 1, \quad z \in U. \]

Therefore,

\[ p - 1 < \text{Re} \left[ -\frac{zG'(z)}{G(z)} \right] < p + 1, \quad z \in U, \]

this meaning that \( G \in \Sigma^*_p(p - 1, p + 1) \).
Proof. Considering \( \alpha = p, \beta = -1, \gamma \to \gamma - p \) in Lemma 2.2, we remark that the conditions from this lemma are met in the hypothesis of this corollary, so, the function 
\[
h(z) = p + z + \frac{pz}{\gamma - p - z}
\]
is convex in \( U \).

It is easy to see that the function \( q(z) = p + z \) is the univalent solution for the differential equation
\[
q(z) + \frac{pzq'(z)}{\gamma - q(z)} = h(z), \quad z \in U, \quad \text{with } q(0) = p.
\]

Next we verify that \( |\text{Im } h(z)| < C_p(\gamma - p), z \in U \), which is equivalent to
\[
|\text{Im } \left(z + \frac{pz}{\gamma - p - z}\right)| < \sqrt{p^2 + 2p(\gamma - p)}, \quad z \in U.
\]

We have
\[
|\text{Im } h(z)| = \left| \frac{pz}{\gamma - p - z} \right| \leq |\text{Im } z| + p(\gamma - p) \left| \frac{1}{z - \gamma + p} \right|.
\]

If we denote \( \gamma - p \) with \( a \) we have from the hypothesis \( a \geq 3 \) and
\[
|\text{Im } \frac{1}{z - a}| = \frac{|\text{Im } z|}{|z - a|^2} < \frac{1}{|z - a|^2} \leq \frac{1}{(a - \text{Re } z)^2} \leq \frac{1}{a}, \quad z \in U, \quad a \geq 3,
\]
so
\[
|\text{Im } \frac{1}{z - \gamma + p}| < \frac{1}{\gamma - p}, \quad z \in U.
\]

Therefore, we get
\[
|\text{Im } \left(z + \frac{pz}{\gamma - p - z}\right)| < p + 1, \quad z \in U, \quad \text{so } |\text{Im } h(z)| < p + 1, \quad z \in U.
\]

Now it is obvious that we have \( |\text{Im } h(z)| < \sqrt{p^2 + 2p(\gamma - p)} = C_p(\gamma - p) \), hence
\[
|\text{Im } (\gamma - h(z))| < C_p(\gamma - p), \quad z \in U, \quad \text{this means that}
\]
\[
\gamma - h(z) \prec R_{\gamma - p, p}(z), \quad z \in U.
\]

Therefore, from Corollary 2.11, we obtain
\[
G = J_{p, \gamma}(g) \in \Sigma S_p(p + z),
\]
which is equivalent to
\[
\left| \frac{zG'(z)}{G(z)} + p \right| < 1, \quad z \in U.
\]

If we consider for Corollary 2.11 the condition \( \text{Re } [\gamma - \beta h(z)] > 0, z \in U \), instead of \( \gamma - \beta h(z) < R_{\gamma - p, p}(z), z \in U \), we get:

**Corollary 2.13.** Let \( p \in \mathbb{N}^* \) and \( \beta, \gamma \in \mathbb{C} \) with \( \beta \neq 0 \) and \( \text{Re } (\gamma - p\beta) > 0 \). Also let \( g \in \Sigma S_p(h) \) with \( h \) convex in \( U \) and \( h(0) = p \). If
\[
\text{Re } [\gamma - \beta h(z)] > 0, \quad z \in U,
\]
then
$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(q),$$
where $q$ is the univalent solution of the Briot-Bouquet differential equation
$$q(z) + \frac{pzq'(z)}{\gamma - \beta q(z)} = h(z), \quad z \in U, \, q(0) = p.$$

The function $q$ is the best $(p,p)$-dominant.

**Proof.** The result follows from Corollary 2.11.

Since for Corollary 2.13 we have $q \prec h$ (see Theorem 1.11), we get the next corollary:

**Corollary 2.14.** Let $p \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\text{Re}(\gamma - p\beta) > 0$. Also let $g \in \Sigma S_p(h)$ with $h$ convex in $U$ and $h(0) = p$. If
$$\text{Re} [\gamma - \beta h(z)] > 0, \quad z \in U,$$
then
$$G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(h).$$

Furthermore, using Corollary 2.14 for a particular function $h$, we present a result which was also obtained in [8] but using a different method.

We consider $h(z) = h_{p,\alpha}(z) = \frac{p + (p - 2\alpha)z}{1 - z}$, $z \in U$, where $p \in \mathbb{N}^*$ and $0 \leq \alpha < p$. It is not difficult to see that $h_{p,\alpha}(U) = \{z \in \mathbb{C}/\text{Re} z > \alpha\}$ and $h_{p,\alpha}(0) = p$.

Using the notations given at the beginning of this paper we have
$$g \in \Sigma S_p(h_{p,\alpha}) \Leftrightarrow g \in \Sigma^*_p(\alpha).$$

We now get the next result:

**Corollary 2.15.** [8] Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\frac{\text{Re} \gamma}{\beta} \leq \alpha < p$. Then we have
$$g \in \Sigma^*_p(\alpha) \Rightarrow G = J_{p,\beta,\gamma}(g) \in \Sigma^*_p(\alpha).$$

**Proof.** From $\frac{\text{Re} \gamma}{\beta} \leq \alpha < p$ and $\beta < 0$ we have $\text{Re} \gamma - \beta \alpha \geq 0$ and $\text{Re} \gamma - p\beta > 0$.

It is easy to see that
$$\text{Re} (\gamma - \beta h_{p,\alpha}(z)) > \text{Re} (\gamma - \alpha \beta) \geq 0, \quad z \in U,$$
hence $\text{Re} [\gamma - \beta h_{p,\alpha}(z)] > 0, \, z \in U$.

We know that $g \in \Sigma^*_p(\alpha) \Leftrightarrow g \in \Sigma S_p(h_{p,\alpha})$.

Since the conditions from Corollary 2.14 holds, we get $G = J_{p,\beta,\gamma}(g) \in \Sigma S_p(h_{p,\alpha})$ which is equivalent to $G \in \Sigma^*_p(\alpha)$. \qed
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