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AN ABSTRACT NONLOCAL SECOND ORDER EVOLUTION PROBLEM

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Abstract. The aim of the paper is to prove theorems on the existence and uniqueness of mild and classical solutions of a semilinear evolution second order equation together with nonlocal conditions. The theory of strongly continuous cosine families of linear operators in a Banach space is applied.

Keywords: nonlocal, second order, evolution problem, Banach space.

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1. INTRODUCTION

In this paper, we consider the abstract semilinear nonlocal second order Cauchy problem

$$u''(t) = Au(t) + f(t, u(t), u'(t)), \ t \in (0, T], \tag{1.1}$$

$$u(0) = x_0, \tag{1.2}$$

$$u'(0) + \sum_{i=1}^{p} h_i u(t_i) = x_1, \tag{1.3}$$

where A is a linear operator from a real Banach space X into itself, $u : [0, T] \to X$, $f : [0, T] \times X^2 \to X$, $x_0, x_1 \in X$, $h_i \in \mathbb{R}$ $(i = 1, 2, \ldots, p)$ and

$$0 < t_1 < t_2 < \ldots < t_p \leq T.$$

We prove two theorems on the existence and uniqueness of mild and classical solutions of the problem (1.1)–(1.3). For this purpose we apply the theory of strongly continuous cosine families of linear operators in a Banach space. We also apply the Banach contraction theorem and the Bochenek theorem (see Theorem 1.1 in this paper).

Let A be the same linear operator as in (1.1). We will need the following assumption:

Assumption ($\mathbf{A}_1$). Operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself.

Recall that the infinitesimal generator of a strongly continuous cosine family $C(t)$ is the operator $A : X \supset \mathcal{D}(A) \to X$ defined by

$$Ax := \frac{d^2}{dt^2} C(t)x_{|t=0}, \; x \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) := \{x \in X : C(t)x \text{ is of class } \mathcal{C}^2 \text{ with respect to } t\}.$$

Let

$$E := \{x \in X : C(t)x \text{ is of class } \mathcal{C}^1 \text{ with respect to } t\}.$$

The associated sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x ds, \; x \in X, \; t \in \mathbb{R}.$$

From Assumption (A_1) it follows (see [4]) that there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|} \quad \text{and} \quad \|S(t)\| \leq Me^{\omega|t|} \quad \text{for } t \in \mathbb{R}.$$

We also will use the following assumption:

Assumption ($\mathbf{A}_2$). The adjoint operator A^* is densely defined in X^*, that is, $\overline{\mathcal{D}(A^*)} = X^*$.

The results obtained here are based on those by Bochenek [1], and Travis and Webb [4]. Second order evolution equations with parameters were considered by Bochenek and Winiarska [2]. Theorems on existence and uniqueness of mild and classical solutions for a semilinear functional – differential evolution Cauchy problem of the first order one can find in [3].

For convenience of the reader, a result obtained by J. Bochenek (see [1]) will be presented here.

Let us consider the Cauchy problem

$$u''(t) = Au(t) + h(t), \quad t \in (0, T], \tag{1.4}$$

$$u(0) = x_0, \tag{1.5}$$

$$u'(0) = x_1. \tag{1.6}$$

A function $u : [0, T] \to X$ is said to be a classical solution of the problem (1.4)–(1.6) if

$$u \in \mathcal{C}^1([0, T], X) \cap \mathcal{C}^2((0, T], X), \tag{a}$$

$$u(0) = x_0 \quad \text{and} \quad u'(0) = x_1, \tag{b}$$

$$u''(t) = Au(t) + h(t) \quad \text{for} \quad t \in (0, T]. \tag{c}$$

Theorem 1.1. *Suppose that:*

(i) *Assumptions* (A_1) *and* (A_2) *are satisfied,*
(ii) $h : [0,T] \to X$ *is Lipshitz continuous,*
(iii) $x_0 \in \mathcal{D}(A)$ *and* $x_1 \in E$.

Then u *given by the formula*

$$u(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)ds, \quad t \in [0,T],$$

is the unique classical solution of the problem (1.4)–(1.6).

2. THEOREM ON MILD SOLUTIONS

A function $u \in \mathcal{C}^1([0,T],X)$ and satisfying the integral equation

$$u(t) = C(t)x_0 + S(t)x_1 - S(t)\left(\sum_{i=1}^{p} h_i u(t_i)\right) + \\ + \int_0^t S(t-s)f(s,u(s),u'(s))ds, \quad t \in [0,T],$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1)–(1.3).

Theorem 2.1. *Suppose that:*

(i) *Assumption* (A_1) *is satisfied,*
(ii) $f : [0,T] \times X^2 \to X$ *is continuous with respect to the first variable* $t \in [0,T]$ *and there exists a positive constant* L *such that*

$$\|f(s,z_1,z_2) - f(s,\tilde{z}_1,\tilde{z}_2)\| \le L_1 \sum_{i=1}^{2} \|z_i - \tilde{z}_i\| \quad \text{for } s \in [0,T],\ z_i, \tilde{z}_i \in X\ (i=1,2),$$

(iii) $2C(TL_1 + \sum_{i=1}^{p} |h_i|) < 1$,
where $C := \sup\{\|C(t)\| + \|S(t)\| + \|S'(t)\| : t \in [0,T]\}$,
(iv) $x_0 \in E$ *and* $x_1 \in X$.

Then nonlocal Cauchy problem (1.1)–(1.3) *has a unique mild solution.*

Proof. Let the operator $F : \mathcal{C}^1([0,T],X) \to \mathcal{C}^1([0,T,],X)$ be given by

$$(Fu)(t) = C(t)x_0 + S(t)x_1 - S(t)\left(\sum_{i=1}^{p} h_i u(t_i)\right) + \\ + \int_0^t S(t-s)f(s,u(s),u'(s))ds, \ \ t \in [0,T].$$

Now, we shall show that F is a contraction on the Banach space $\mathcal{C}^1([0,T,],X)$ equipped with the norm

$$\|w\|_1 := \sup\{\|w(t)\| + \|w'(t)\| \,:\, t \in [0,T]\}.$$

To do it observe that

$$\begin{aligned}\|(Fw)(t) - (F\tilde{w})(t)\| &= \left\| S(t)\left(\sum_{i=1}^{p} h_i(\tilde{w}(t_i) - w(t_i))\right) + \right.\\ &\quad \left. + \int_0^t S(t-s)\big(f(s,w(s),w'(s)) - f(s,\tilde{w}(s),\tilde{w}'(s))\big)ds \right\| \le \\ &\le C\left(\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1 + \\ &\quad + \int_0^t \|S(t-s)\|\, L_1\big(\, \|w(s)-\tilde{w}(s)\| + \|w'(s)-\tilde{w}'(s)\|\,\big)ds \le \\ &\le C\left(TL_1 + \sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1\,,\end{aligned}$$

and

$$\begin{aligned}\|(Fw)'(t) - (F\tilde{w})'(t)\| &= \left\| S'(t)\left(\sum_{i=1}^{p} h_i(\tilde{w}(t_i) - w(t_i))\right) + \right.\\ &\quad \left. + \int_0^t C(t-s)\big(f(s,w(s),w'(s)) - f(s,\tilde{w}(s),\tilde{w}'(s))\big)ds \right\| \le \\ &\le C\left(\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1 + \\ &\quad + \int_0^t \|C(t-s)\|\, L_1\big(\, \|w(s)-\tilde{w}(s)\| + \|w'(s)-\tilde{w}'(s)\|\,\big)ds \le \\ &\le C\left(\sum_{i=1}^{p}|h_i| + TL_1\right)\|w-\tilde{w}\|_1,\ t \in [0,T].\end{aligned}$$

Consequently,

$$\|Fw - F\tilde{w}\|_1 \le 2C\left(TL_1 + \sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1 \quad \text{for } w,\tilde{w} \in \mathcal{C}^1([0,T],X).$$

Therefore, in space $\mathcal{C}^1([0,T],X)$ there is the only one fixed point of F and this point is the mild solution of the nonlocal Cauchy problem (1.1)–(1.3). So, the proof of Theorem 2.1 is complete. □

Remark 2.2. The application of a Bielecki norm in the proof of Theorem 2.1 does not give any benefit.

3. THEOREM ABOUT CLASSICAL SOLUTIONS

A function $u : [0,T] \to X$ is said to be a classical solution of the problem (1.1)–(1.3) if

$$u \in \mathcal{C}^1([0,T],X) \cap \mathcal{C}^2((0,T],X), \tag{a}$$

$$u(0) = x_0 \quad \text{and} \quad u'(0) + \sum_{i=1}^{p} h_i u_i(t_i) = x_1, \tag{b}$$

$$u''(t) = Au(t) + f(t,u(t),u'(t)) \quad \text{for } t \in [0,T]. \tag{c}$$

Theorem 3.1. *Suppose that:*

(i) *Assumptions* (A_1) *and* (A_2) *are satisfied,*

(ii) *There exists a positive constant* L_2 *such that*

$$\|f(s,z_1,z_2) - f(\tilde{s},\tilde{z}_1,\tilde{z}_2)\| \le L_2\left(|s-\tilde{s}| + \sum_{i=1}^{2} \|z_i - \tilde{z}_i\| \right)$$

for $s,\tilde{s} \in [0,T]$, $z_i, \tilde{z}_i \in X$ $(i = 1,2)$.

(iii) $2C(TL_2 + \sum_{i=1}^{p} |h_i|) < 1$.

(iv) $x_0 \in E$ *and* $x_1 \in X$.

Then nonlocal Cauchy problem (1.1)–(1.3) *has a unique mild solution* u. *Moreover, if*

$$x_0 \in \mathcal{D}(A),\ x_1 \in E \quad \text{and} \quad u(t_i) \in E\ (i = 1,2,\ldots,p)$$

then u *is the unique classical solution of nonlocal problem* (1.1)–(1.3).

Proof. Since the assumptions of Theorem 2.1 are satisfied, nonlocal Cauchy problem (1.1)–(1.3) possesses a unique mild solution which is denoted by u.

Now, we shall show that u is the classical solution of problem (1.1)–(1.3).

Firstly we shall prove that u and u' satisfy the Lipschitz condition on $[0,T]$. Let t and $t+h$ be any two points belonging to $[0,T]$. Observe that

$$\begin{aligned} u(t+h)-u(t) = {} & C(t+h)x_0 + S(t+h)x_1 - S(t+h)\left(\sum_{i=1}^{p} h_i u(t_i)\right) + \\ & + \int_0^{t+h} S(t+h-s) f(s,u(s),u'(s))ds - \\ & - C(t)x_0 - S(t)x_1 + S(t)\left(\sum_{i=1}^{p} h_i u(t_i)\right) - \\ & - \int_0^{t} S(t-s) f(s,u(s),u'(s))ds. \end{aligned}$$

Since

$$C(t)x_0 + S(t)\left(x_1 - \sum_{i=1}^{p} h_i u(t_i)\right)$$

is of class $\mathcal{C}^2$ in $[0,T]$, there are $C_1>0$ and $C_2>0$ such that

$$\left\| \big(C(t+h)-C(t)\big)x_0 + \big(S(t+h)-S(t)\big)\left(x_1 - \sum_{i=1}^{p} h_i u(t_i)\right)\right\| \le C_1\,|h|$$

and

$$\left\| \Big(\big(C(t+h)-C(t)\big)x_0\Big)' + \Big(\big(S(t+h)-S(t)\big)\big(x_1 - \sum_{i=1}^{p} h_i u(t_i)\big)\Big)'\right\| \le C_2\,|h|\,.$$

Hence

$$\begin{aligned} \|u(t+h)-u(t)\| \le {} & C_1\,|h| + \left\| \int_0^{t} S(s)\big(f(t+h-s,u(t+h-s),u'(t+h-s)) - \right. \\ & \left. - f(t-s,u(t-s),u'(t-s))\big)ds \right\| + \\ & + \left\| \int_t^{t+h} S(s) f(t+h-s,u(t+h-s),u'(t+h-s))ds \right\| \le \\ & \le C_1\,|h| + \int_0^{t} Me^{\omega T} L_2\big(\,|h| + \|u(t+h-s)-u(t-s)\| + \\ & + \|u'(t+h-s)-u'(t-s)\|\,\big)ds + Me^{\omega T} N\,|h|\,, \end{aligned}$$

where

$$N := \sup\{\|f(s,u(s),u'(s))\| \; : \; s \in [0,T]\}.$$

From this we get

$$\|u(t+h)-u(t)\| \le C_3\,|h| + C_4 \int_0^t \big(\,\|u(s+h)-u(s)\| + \|u'(s+h)-u'(s)\|\,\big)ds. \quad (3.1)$$

Moreover, we have

$$u'(t) = \Big(C(t)x_0 + S(t)\Big(x_1 - \sum_{i=1}^{p} h_i u(t_i)\Big)\Big)' + \int_0^t C(t-s)f(s,u(s),u'(s))ds.$$

From the above formula we obtain, analogously,

$$\|u'(t+h)-u'(t)\| \le C_5\,|h| + C_6 \int_0^t \big(\,\|u(s+h)-u(s)\| + \|u'(s+h)-u'(s)\|\,\big)ds. \quad (3.2)$$

By inequalities (3.1) and (3.2), we get

$$\begin{aligned} &\|u(t+h)-u(t)\| + \|u'(t+h)-u'(t)\| \le \\ &\le C_*\,|h| + C_{**} \int_0^t \big(\,\|u(s+h)-u(s)\| + \|u'(s+h)-u'(s)\|\,\big)ds. \end{aligned}$$

From Gronwall's inequality, we have

$$\|u(t+h)-u(t)\| + \|u'(t+h)-u'(t)\| \le \tilde{C}\,|h|\,, \quad (3.3)$$

where $\tilde{C}$ is a positive constant.

By (3.3), it follows that u and u' satisfy the Lipshitz condition on $[0,T]$ with constant $\tilde{C}$. This implies that the mapping

$$[0,T] \ni t \mapsto f(t,u(t),u'(t)) \in X$$

also satisfies the Lipshitz condition.

The above property of f together with the assumptions of Theorem 3.1 imply, by Theorem 1.1 and by Theorem 2.1, that the linear Cauchy problem

$$\begin{aligned} v''(t) &= Av(t) + f(t,u(t),u'(t)), \; t \in [0,T], \\ v(0) &= x_0, \\ v'(0) &= x_1 - \sum_{i=1}^{p} h_i u(t_i) \end{aligned}$$

has a unique classical solution v such that

$$v(t) = C(t)x_0 + S(t)\left(x_1 - \sum_{i=1}^{p} h_i u(t_i)\right) + \\ + \int_0^t S(t-s)f(s, u(s), u'(s))ds = u(t), \ t \in [0, T].$$

Consequently, u is the unique classical solution of semilinear Cauchy problem (1.1)–(1.3) and, therefore, the proof of Theorem 3.1 is complete. □

Remark 3.2. If $h_i = 0$ $(i = 1, 2, \dots, p)$ then Theorem 3.1 is a particular case of the Bochenek theorem (see [1, Theorem 5]).

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