SOME PROPERTIES OF SET-VALUED SINE FAMILIES

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Abstract. Let $\{F_t : t \geq 0\}$ be a family of continuous additive set-valued functions defined on a convex cone K in a normed linear space X with nonempty convex compact values in X. It is shown that (under some assumptions) a regular sine family associated with $\{F_t : t \geq 0\}$ is continuous and $\{F_t : t \geq 0\}$ is a continuous cosine family.

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1. INTRODUCTION

Our primary objective in this paper is to introduce some basic properties of families of set-valued functions satisfying the functional equation

$$G_{t+s}(x) = G_{t-s}(x) + 2F_t(G_s(x)),$$

which are called here sine families and refer to the trigonometric functional equation

$$g(t+s) - g(t-s) = 2f(t)g(s)$$

considered e.g. in [1, p. 138], [2, p. 365].

Sine families are strongly connected with cosine families, which have been considered by various authors. Cosine families of continuous linear operators were investigated e.g. in [4–7] and [16], whereas the set-valued case in [14], [10, 11] and [12].

A set-valued regular sine family appeared (non-explicitly) in the paper [10] as a Hukuhara derivative of a cosine family of continuous additive set-valued functions.

2. PRELIMINARIES

Throughout the paper, we assume that all linear spaces are real. Let X be a normed linear space. $n(X)$ denotes the set of all nonempty subsets of X, whereas $b(X)$ stands for the set of all bounded members of $n(X)$ and $c(X)$ stands for the set of all compact members of $n(X)$. Moreover, by $bcl(X)$ we denote all closed members of $b(X)$, by $bccl(X)$ all convex members of $bcl(X)$ and by $cc(X)$ all convex members of $c(X)$.

By $B(x_0, r)$ we denote the open ball of the radius r centered at a point x_0.

A subset K of the space X is called a *cone* if $tK \subset K$ for all $t \in [0, \infty)$. We say that a cone is *convex* if it is a convex set.

Let K be a convex cone in X. Assume that $\{F_t : t \geq 0\}$ is a family of set-valued functions $F_t \colon K \to n(X)$, $t \geq 0$.

A family $\{G_t : t \geq 0\}$ of set-valued functions $G_t \colon K \to n(K)$, $t \geq 0$, is called a *sine family associated with family* $\{F_t : t \geq 0\}$, if

$$G_{t+s}(x) = G_{t-s}(x) + 2F_t(G_s(x)) \tag{2.1}$$

for $0 \leq s \leq t$ and $x \in K$, where $F_t(G_s(x)) := \bigcup\{F_t(y) : y \in G_s(x)\}$.

Example 2.1. Let $K = [0, \infty)$, $G_t(x) = \{x\sin t\}$ and $F_t(x) = \{x\cos t\}$ for $t \geq 0$. Then $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$.

Example 2.2. Let $K = [0, \infty)$, $G_t(x) = [0, \sinh t]x$ and $F_t(x) = [1, \cosh t]x$ for $t \geq 0$. Then $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$.

A family $\{F_t : t \geq 0\}$ of set-valued functions $F_t \colon K \to n(K)$, $t \geq 0$, is called a *cosine family*, if

$$F_0(x) = \{x\} \tag{2.2}$$

for all $x \in K$ and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)) \tag{2.3}$$

whenever $0 \leq s \leq t$ and $x \in K$.

Take a set-valued function $\phi \colon K \to n(Y)$, where Y is a normed linear space. We say that ϕ is *lower semi-continuous at a point* $t_0 \in K$ if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$\phi(t_0) \subset \phi(t) + V$$

for all $t \in (t_0 + U) \cap K$. We say that ϕ is *upper semi-continuous at a point* $t_0 \in K$ if for every neighbourhood V of zero in Y there exists a neighbourhood U of zero in X such that

$$\phi(t) \subset \phi(t_0) + V$$

for all $t \in (t_0 + U) \cap K$. ϕ is *continuous at* $t_0 \in K$ if it is both lower semi-continuous and upper semi-continuous at x_0. It is *continuous on* K if it is continuous at each point of K. It is easy to prove that a set-valued function $\phi \colon K \to bcl(Y)$ is continuous if and only if a single valued function $K \ni x \mapsto \phi(x) \in bcl(Y)$ is continuous with respect to the Hausdorff metric derived from the norm in Y.

A sine family $\{G_t : t \geq 0\}$ is *continuous* if the function $t \mapsto G_t(x)$ is continuous for every $x \in K$.

A set-valued function $F \colon K \to n(X)$ is said to be *additive* if

$$F(x + y) = F(x) + F(y) \tag{2.4}$$

for all $x, y \in X$. F is *linear* if (2.4) holds true and it is *homogeneous*, i.e.

$$F(\lambda x) = \lambda F(x) \tag{2.5}$$

for all $x \in K$, $\lambda \geq 0$. An additive and continuous set-valued function with values in $cc(X)$ is linear (cf. Theorem 5.3 in [9]). We say F is *midconcave* if

$$F\Big[\frac{1}{2}(x+y)\Big] \subset \frac{1}{2}[F(x) + F(y)]$$

for all $x, y \in K$ (cf. [9]).

Proposition 2.3. *Let X be a normed linear space and let K be a convex cone in X. Assume that $\{F_t : t \geq 0\}$ is a family of set-valued functions $F_t \colon K \to n(X)$, such that F_0 is upper semi-continuous linear with compact values and $x \in F_0(x)$ for $x \in K$. If $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$ and $G_0(x) \in cc(K)$ for $x \in K$, then $G_0(x) = \{0\}$ for $x \in K$.*

Indeed, putting $t = 0$ and $s = 0$ in (2.1), by the cancellation law (cf. [13]) we obtain the equality $\{0\} = F_0(G_0(x))$, $x \in K$. Since $y \in F_0(y)$ for all $y \in K$, this equality yields $G_0(x) = \{0\}$ for $x \in K$.

A family $\{G_t : t \geq 0\}$ is *increasing* if $G_s(x) \subset G_t(x)$ for every $x \in K$ and $0 \leq s \leq t$.

The two following propositions are easy to prove.

Proposition 2.4. *Let X be a normed linear space and let K be a convex cone in X. Assume that $\{F_t : t \geq 0\}$ is a family of set-valued functions $F_t \colon K \to n(X)$, such that $x \in F_t(x)$ for $x \in K$, $t \geq 0$. If $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$, then the inclusion*

$$G_u(x) + 2G_v(x) \subset G_{u+2v}(x) \tag{2.6}$$

holds for every $u, v \geq 0$, $x \in K$.

Proposition 2.5. *Let X be a normed linear space and let K be a convex cone in X. If a family $\{G_t : t \geq 0\}$ of set-valued functions $G_t \colon K \to n(X)$, such that $0 \in G_t(x)$ for $t \geq 0$, $x \in K$, fulfils inclusion (2.6), then it is increasing.*

Let $\{F_t : t \geq 0\}$ be a family of set-valued functions $F_t \colon K \to n(K)$. We write $\lim\limits_{t \to 0^+} F_t(x) = \{x\}$ if

$$\lim_{t \to 0^+} d(F_t(x), \{x\}) = 0,$$

where d is the Hausdorff distance derived from the norm in X.

A cosine family $\{F_t : t \geq 0\}$ is *regular* if the above equality is satisfied for each $x \in K$ (cf. [14]).

A sine family $\{G_t : t \geq 0\}$ is *regular* if $\lim\limits_{t \to 0^+} \frac{G_t(x)}{t} = \{x\}$.

Example 2.6. Let $K = (-\infty, \infty)$ and $F_t(x) = [1, \cosh t]x$ for $t \geq 0$. Then $\{F_t : t \geq 0\}$ is a regular cosine family.

The sine family from Example 2.1 is regular, whereas the sine family given in Example 2.2 is not regular. Indeed, since $\lim_{t\to 0^+} \frac{\sin t}{t} = 1$ and $\lim_{t\to 0^+} \frac{\sinh t}{t} = 1$ we have

$$\lim_{t\to 0^+} \frac{\{x \sin t\}}{t} = \{x\}$$

and

$$\lim_{t\to 0^+} \frac{[0, \sinh t]x}{t} = [0, x].$$

Let A, B, C be sets of $cc(X)$. We say that a set C is the *Hukuhara difference* of A and B, i.e., $C = A - B$ if $B + C = A$. If this difference exists, then it is unique (see Lemma 1 in [13]).

The next lemma follows directly from the definition of Hukuhara difference.

Lemma 2.7. *Let X be a normed linear space and let A, B, C, D be sets of $cc(X)$. Then:*

(a) *$A - A$ exists and $A - A = \{0\}$;*
(b) *$A - \{0\}$ exists and $A - \{0\} = A$;*
(c) *if the differences $A - C$, $C - D$, $D - B$ exist, then the differences $A - B$, $(A - B) - (C - D)$ exist and $(A - B) - (C - D) = (A - C) + (D - B)$.*

From the definition of a sine family we obtain

Lemma 2.8. *Let X be a normed linear space, K be a convex cone in X and let $G_t \colon K \to cc(K)$, $F_t \colon K \to cc(X)$ for $t \geq 0$. If $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$, then for all $u, v \in [0, \infty)$ with $u \leq v$ and all $x \in K$ there exist Hukuhara differences*

$$G_v(x) - G_u(x).$$

In the next section we will make use of the following lemma.

Lemma 2.9 ([15, Lemma 3])**.** *Let X be a normed linear space and K be a convex cone in X. Assume that $F \colon K \to cc(K)$ is a continuous additive set-valued function and $A, B \in cc(K)$. If there exists the difference $A - B$, then there exists $F(A) - F(B)$ and $F(A) - F(B) = F(A - B)$.*

3. MAIN RESULTS

We give some interesting properties of sine families, in particular continuity and a connection with cosine families.

Theorem 3.1. *Let X be a normed linear space and K be a convex cone in X. Assume that $\{F_t : t \geq 0\}$ is a family of upper semi-continuous at zero set-valued functions $F_t \colon K \to n(X)$, $t \geq 0$, such that $x \in F_t(x)$ for $x \in K$, $t \geq 0$, F_0 is upper semi-continuous linear with compact values and $F_t(0) = \{0\}$ for $t \geq 0$. Then a sine family $\{G_t : t \geq 0\}$ of set-valued functions $G_t \colon K \to b(K)$ associated with the family $\{F_t : t \geq 0\}$, such that G_0 has convex compact values and $0 \in G_t(x)$ for $x \in K$, $t \geq 0$ is continuous.*

Proof. Let us fix $x \in K$ arbitrarily and put $\phi(t) := G_t(x)$. From (2.6) we have

$$\phi(u) + 2\phi(v) \subset \phi(u + 2v)$$

for $u \geq 0$, $v \geq 0$. Putting $u = v$ we get

$$3\phi(u) \subset \phi(3u),$$

and therefore

$$\phi\Big(\frac{u}{3}\Big) \subset \frac{1}{3}\phi(u).$$

Thus

$$\phi\Big(\frac{u}{3^n}\Big) \subset \frac{1}{3^n}\phi(u)$$

for $u \geq 0$ and $n \in \mathbb{N}$. Taking $u = 1$ we obtain $\phi(\frac{1}{3^n}) \subset \frac{1}{3^n}\phi(1)$ for $n \in \mathbb{N}$. Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $\frac{1}{3^n}\phi(1) \subset B(0, \varepsilon)$. By the monotonicity of ϕ

$$\phi(w) \subset B(0, \varepsilon) \tag{3.1}$$

for $0 \leq w < \frac{1}{3^n}$. Since $\phi(0) = \{0\}$ (Proposition 2.3), ϕ is upper semi-continuous at 0.

Let us fix $u \in (0, \infty)$ arbitrarily. We shall prove that ϕ is upper semi-continuous at u. It is easily seen, that it suffices to show that ϕ is upper semi-continuous on the right. Suppose that V is a neighbourhood of zero in X. Setting $t = u$ in (2.1) and using the monotonicity of ϕ, we obtain

$$\phi(u + s) = \phi(u - s) + 2F_u(\phi(s)) \subset \phi(u) + 2F_u(\phi(s)) \tag{3.2}$$

for all $s \in (0, u)$. Since F_u is upper semi-continuous at 0 and $F_u(0) = \{0\}$, there exists $\varepsilon > 0$ such that

$$F_u(y) \subset \frac{1}{2}V$$

for $y \in B(0, \varepsilon) \cap K$. By (3.1) there is some positive integer n such that

$$F_u(\phi(s)) \subset \frac{1}{2}V \quad \text{for} \quad s \in \Big[0, \frac{1}{3^n}\Big).$$

Hence, for $w \in \big(u, u + \frac{1}{3^n}\big)$ we have

$$\phi(w) \subset \phi(u) + V,$$

which shows that ϕ is upper semi-continuous at u.

Now it remains to show that ϕ is lower semi-continuous. Let us fix $u \in [0, \infty)$. It is easily seen, that it suffices to show that ϕ is lower semi-continuous on the left at $u \in (0, \infty)$. Let us fix a neighbourhood V of zero in X. Using (3.2) and the monotonicity of ϕ, we get

$$\phi(u) \subset \phi(u + s) = \phi(u - s) + 2F_u(\phi(s))$$

for all $s \in (0, u)$. A similar reasoning as before shows that there is some positive integer n such that $\phi(u) \subset \phi(w) + V$, for all $w \in \big(u - \frac{1}{3^n}, u\big)$, thus ϕ is lower semi-continuous in u. This completes the proof. □

Lemma 3.2. *Let X be a normed linear space, K be a convex cone in X, $G_t\colon K \to cc(K)$, $F_t\colon K \to cc(X)$, $t \geq 0$ and let F_0 be upper semi-continuous linear. If $\{G_t : t \geq 0\}$ is a regular sine family associated with the family $\{F_t : t \geq 0\}$ and $x \in F_t(x)$, $x \in K$, $t \geq 0$, then*

$$x \in \frac{G_s(x)}{s} \tag{3.3}$$

for all $x \in K$ and $s > 0$.

Proof. From (2.1), Proposition 2.3 and by $x \in F_t(x)$ we have

$$G_s(x) = G_0(x) + 2F_{\frac{s}{2}}(G_{\frac{s}{2}}(x)) \supset 2G_{\frac{s}{2}}(x),$$

thus

$$\frac{G_{\frac{s}{2^n}}(x)}{\frac{s}{2^n}} \subset \frac{G_s(x)}{s} \quad \text{for} \quad n \in \mathbb{N}.$$

Regularity of $\{G_t : t \geq 0\}$ implies

$$\frac{G_{\frac{s}{2^n}}(x)}{\frac{s}{2^n}} \to \{x\} \text{ as } n \to \infty,$$

therefore

$$x \in \frac{G_s(x)}{s}$$

for all $x \in K$ and $s > 0$. □

Theorem 3.3. *Let X be a normed linear space and K be a convex cone in X. Assume that $\{F_t : t \geq 0\}$ is a family of upper semi-continuous at zero additive set-valued functions $F_t\colon K \to cc(X)$, $t \geq 0$, such that $x \in F_t(x)$ for $x \in K$, $t \geq 0$ and F_0 is upper semi-continuous linear. If a sine family $\{G_t : t \geq 0\}$ of set-valued functions $G_t\colon K \to cc(K)$ associated with the family $\{F_t : t \geq 0\}$ is regular, then it is continuous.*

Proof. Let us fix $x \in K$ arbitrarily and put $\psi(t) := G_t(x) - tx$, $t \geq 0$. Then $0 \in \psi(x)$, $t \geq 0$. Indeed, by Lemma 3.2 and Proposition 2.3 we have

$$tx \in G_t(x)$$

for $t \geq 0$. Hence

$$0 \in G_t(x) - tx = \psi(t), \quad t \geq 0.$$

From (2.6) we have

$$\begin{aligned}\psi(u) + 2\psi(v) &= G_u(x) - ux + 2G_v(x) - 2vx = \\ &= G_u(x) + 2G_v(x) - (u+2v)x \subset G_{u+2v}(x) - (u+2v)x = \psi(u+2v),\end{aligned}$$

i.e.,

$$\psi(u) + 2\psi(v) \subset \psi(u+2v)$$

for $u \geq 0$, $v \geq 0$. In the same way as in the proof of Theorem 3.1 we obtain that for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\psi(w) \subset B(0, \varepsilon) \tag{3.4}$$

for all $w \in [0, \frac{1}{3^n})$, and that ψ is upper semi-continuous at 0.

Let us fix $u \in (0, \infty)$ arbitrarily. We shall prove that ψ is upper semi-continuous at u. Since ψ is increasing (Proposition 2.5), it suffices to show that ψ is upper semi-continuous on the right at u. Suppose that V is a symmetric convex neighbourhood of zero in X. Setting $t = u$ in (2.1) we obtain

$$\begin{aligned}\psi(u+s) &= G_{u+s}(x) - (u+s)x = [G_{u-s}(x) - (u-s)x] + 2F_u(G_s(x)) - 2sx = \\ &= \psi(u-s) + 2F_u(\psi(s) + sx) - 2sx = \\ &= \psi(u-s) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx\end{aligned}$$

i.e.,

$$\psi(u+s) = \psi(u-s) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx \tag{3.5}$$

for all $s \in (0, u)$. Hence, by the monotonicity of ψ

$$\psi(u+s) \subset \psi(u) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx$$

for $s \in (0, u)$. Since F_u is upper semi-continuous at zero and $F_u(0) = \{0\}$, there exists $\varepsilon > 0$ such that

$$F_u(y) \subset \frac{1}{6}V$$

for $y \in B(0, \varepsilon) \cap K$. By (3.4) there is some positive integer n such that

$$F_u(\psi(s)) \subset \frac{1}{6}V \quad \text{for} \quad s \in \Big[0, \frac{1}{3^n}\Big).$$

Moreover, we can assume that n is large enough in order that

$$F_u(sx) \subset \frac{1}{6}V, \quad sx \in \frac{1}{6}V$$

for $s \in \big[0, \frac{1}{3^n}\big)$. Hence, for $w \in \big(u, u + \frac{1}{3^n}\big)$ we have

$$\psi(w) \subset \psi(u) + V,$$

which shows that ψ is upper semi-continuous at u.

It remains to show that ψ is lower semi-continuous. Let us fix $u \in [0, \infty)$. It is easily seen, that it suffices to show that ψ is lower semi-continuous on the left at $u \in (0, \infty)$. Let us fix a symmetric convex neighbourhood V of zero in X. Using the monotonicity of ψ and (3.5), we get

$$\psi(u) \subset \psi(u+s) = \psi(u-s) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx$$

for all $s \in (0, u)$. A similar reasoning as before shows that there is a positive integer n such that $\psi(u) \subset \psi(w) + V$ for all $w \in \big(u - \frac{1}{3^n}, u\big)$. Therefore ψ is lower semi-continuous in u, which completes the proof. □

Remark 3.4. Let X be a normed linear space, K be a convex cone in X, $G_t\colon K \to cc(K)$, $F_t\colon K \to cc(X)$ for $t \geq 0$. If $\{G_t : t \geq 0\}$ is a regular sine family associated with the family $\{F_t : t \geq 0\}$ and all F_t are continuous and additive, then the family $\{F_t : t \geq 0\}$ is unique.

Assume that $\{F_t : t \geq 0\}$ and $\{H_t : t \geq 0\}$ are two families of continuous and additive set-valued functions such that

$$G_{t+s}(x) = G_{t-s}(x) + 2F_t(G_s(x))$$

and

$$G_{t+s}(x) = G_{t-s}(x) + 2H_t(G_s(x)).$$

Then

$$G_{t-s}(x) + 2F_t(G_s(x)) = G_{t-s}(x) + 2H_t(G_s(x))$$

and by the cancellation law $F_t(G_s(x)) = H_t(G_s(x))$ for all $0 \leq s \leq t$. Using (2.5) we get

$$F_t\Big(\frac{G_s(x)}{s}\Big) = H_t\Big(\frac{G_s(x)}{s}\Big).$$

Letting s tend to 0 from the right, by regularity of $\{G_t : t \geq 0\}$ we obtain

$$F_t(x) = H_t(x).$$

Example 3.5. Let $K = [0, \infty)$, $G_t(x) = t[0, x]$, $F_t(x) = \{x\}$ and $H_t(x) = [0, x]$ for $t \geq 0$, $x \in K$. Then $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$ and with the family $\{H_t : t \geq 0\}$.

Indeed, we have

$$\begin{aligned} G_{t+s}(x) &= (t+s)[0, x] = (t-s)[0, x] + 2s[0, x] = \\ &= G_{t-s}(x) + 2G_s(x) = G_{t-s}(x) + 2F_t(G_s(x)) \end{aligned}$$

and

$$\begin{aligned} G_{t+s}(x) &= (t+s)[0, x] = (t-s)[0, x] + 2s[0, x] = \\ &= G_{t-s}(x) + 2H_t(s[0, x]) = G_{t-s}(x) + 2H_t(G_s(x)). \end{aligned}$$

Observe that all F_t and H_t are continuous and additive, but the sine family $\{G_t : t \geq 0\}$ is not regular, since

$$\lim_{t \to 0^+} \frac{G_t(x)}{t} = [0, x].$$

Theorem 3.6. *Let X be a real normed additive space, K a convex cone in X and let $\{F_t : t \geq 0\}$ be a family of continuous additive set-valued functions $F_t\colon K \to cc(K)$, such that $F_0(x) = \{x\}$, $x \in K$. Assume that $\{G_t : t \geq 0\}$ is a regular sine family of set-valued functions $G_t\colon K \to cc(K)$ associated with the family $\{F_t : t \geq 0\}$. Then:*

(a) *$\{F_t : t \geq 0\}$ is a cosine family,*

(b) *if moreover*

$$x \in F_t(x) \tag{3.6}$$

for $x \in K$ and $t \geq 0$, then $\{F_t : t \geq 0\}$ is a continuous cosine family. In particular it is regular.

Proof. (a) Let us take s, u, v such that $0 \leq s \leq v - u$, $0 \leq s \leq u$ and $0 \leq u \leq v$. From (2.1) we get

$$G_{v+u+s}(x) = G_{v+u-s}(x) + 2F_{v+u}(G_s(x)), \tag{3.7}$$

$$G_{v-u+s}(x) = G_{v-u-s}(x) + 2F_{v-u}(G_s(x)), \tag{3.8}$$

$$G_{v+u+s}(x) = G_{v-u-s}(x) + 2F_v(G_{u+s}(x)), \tag{3.9}$$

$$G_{v+u-s}(x) = G_{v-u+s}(x) + 2F_v(G_{u-s}(x)), \tag{3.10}$$

for all $x \in K$. By Lemma 2.7 and Lemma 2.9, we have therefore

$$\begin{aligned} 2F_v(2F_u(G_s(x))) &= 2F_v[G_{u+s}(x) - G_{u-s}(x)] = 2F_v(G_{u+s}(x)) - 2F_v(G_{u-s}(x)) = \\ &= [G_{v+u+s}(x) - G_{v-u-s}(x)] - [G_{v+u-s}(x) - G_{v-u+s}(x)] = \\ &= [G_{v+u+s}(x) - G_{v+u-s}(x)] + [G_{v-u+s}(x) - G_{v-u-s}(x)] = \\ &= 2F_{v+u}(G_s(x)) + 2F_{v-u}(G_s(x)). \end{aligned}$$

Since F_t are linear, we can write

$$2F_v\Big(F_u\Big(\frac{G_s(x)}{s}\Big)\Big) = F_{v+u}\Big(\frac{G_s(x)}{s}\Big) + F_{v-u}\Big(\frac{G_s(x)}{s}\Big).$$

Letting s tend to 0 we obtain from continuity of F_t

$$2F_v(F_u(x)) = F_{v+u}(x) + F_{v-u}(x).$$

(b) The proof will be divided into three steps.
Step 1. From (2.3) and (3.6) follows the inclusion

$$F_{t+s}(x) + F_{t-s}(x) \supset 2F_t(x)$$

for $0 \leq s \leq t$, which implies that set-valued functions $u \mapsto F_u(x)$ $(x \in K)$ are midconcave in $[0, \infty)$ (cf. [11, the proof of Theorem 3]).

For fixed $s > 0$ and $t > 0$ from (2.1) and Lemma 3.2 we obtain

$$F_t(x) \subset F_t\Big(\frac{G_s(x)}{s}\Big) = \frac{G_{t+s}(x) - G_{t-s}(x)}{2s}$$

for all $x \in K$. Since set-valued functions

$$t \mapsto \frac{G_{t+s}(x) - G_{t-s}(x)}{2s}$$

are continuous in (s, ∞) (cf. Theorem 3.3), from Theorem 4.3 in [9] set-valued functions

$$t \mapsto F_t(x)$$

for $x \in K$ are continuous in (s, ∞), thus also in $(0, \infty)$. Continuity and midconcavity of set-valued functions $t \mapsto F_t(x)$ imply their concavity, i.e.,

$$F_{\lambda t+(1-\lambda)s}(x) \subset \lambda F_t(x) + (1-\lambda)F_s(x), \quad \lambda \in [0,1],\ s,t > 0,\ x \in K$$

(cf. Theorem 4.1 in [9]). We get therefore convexity of functions

$$\psi(t) := \operatorname{diam}(F_t(x))$$

in $(0, \infty)$ for all $x \in K$.

Indeed, let $\lambda \in [0,1]$ and $s, t \in (0, \infty)$. By the concavity of the functions $t \mapsto F_t(x)$ we have

$$\begin{aligned}\psi(\lambda t + (1-\lambda)s) &= \operatorname{diam}[F_{\lambda t+(1-\lambda)s}(x)] \leq \operatorname{diam}[\lambda F_t(x) + (1-\lambda)F_s(x)] \leq \\ &\leq \operatorname{diam}[\lambda F_t(x)] + \operatorname{diam}[(1-\lambda)F_s(x)] = \\ &= \lambda \operatorname{diam}[F_t(x)] + (1-\lambda)\operatorname{diam}[F_s(x)] = \lambda\psi(t) + (1-\lambda)\psi(s).\end{aligned}$$

Step 2. For $t > 0$ and $x \in K$ we have

$$F_t(x) + x = 2F^2_{\frac{t}{2}}(x).$$

From (3.6) we obtain

$$F_t(x) + x = F^2_{\frac{t}{2}}(x) + F^2_{\frac{t}{2}}(x) \supset F_{\frac{t}{2}}(x) + x,$$

and therefore

$$F_{\frac{t}{2}}(x) \subset F_t(x).$$

Hence the sequence $(F_{\frac{t}{2^n}}(x))$ is descending. Put

$$H_t(x) := \bigcap_{n=0}^{\infty} F_{\frac{t}{2^n}}(x).$$

From the inclusion

$$F_{\frac{t}{2^n}}(x) + x = 2F^2_{\frac{t}{2^{n+1}}}(x) \supset F_{\frac{t}{2^{n+1}}}(x) + F_{\frac{t}{2^{n+1}}}(x) \supset 2H_t(x)$$

and Lemma 2 in [8] it follows that

$$H_t(x) + x = \bigcap_{n=0}^{\infty} F_{\frac{t}{2^n}}(x) + x = \bigcap_{n=0}^{\infty} [F_{\frac{t}{2^n}}(x) + x] \supset 2H_t(x).$$

Therefore, by the cancellation law we get

$$H_t(x) = \{x\}$$

for $t > 0$ and $x \in K$. Thus $\lim_{n\to\infty} F_{\frac{t}{2^n}}(x) = \{x\}$ (cf. Lemma 3 in [8]), whence $\lim_{n\to\infty} \psi\left(\frac{t}{2^n}\right) = 0$. Since ψ is convex, we have

$$\lim_{s\to 0^+} \psi(s) = 0.$$

Step 3. Fix $\varepsilon > 0$. There is $\eta > 0$ such that

$$\psi(s) < \varepsilon \quad \text{for} \quad s \in (0, \eta).$$

Let $s \in (0, \eta)$ and $y \in F_s(x)$. We have then

$$\|y - x\| \leq \operatorname{diam}(F_s(x)) = \psi(s) < \varepsilon.$$

Hence

$$F_s(x) \subset B(x, \varepsilon)$$

and

$$\lim_{s\to 0^+} F_s(x) = \{x\}. \qquad \square$$

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