ON SELF-ADJOINT OPERATORS
IN KREIN SPACES
CONSTRUCTED BY CLIFFORD ALGEBRA \( Cl_2 \)

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Abstract. Let \( J \) and \( R \) be anti-commuting fundamental symmetries in a Hilbert space \( \mathcal{H} \). The operators \( J \) and \( R \) can be interpreted as basis (generating) elements of the complex Clifford algebra \( Cl_2(J, R) := \text{span}\{I, J, R, iJR\} \). An arbitrary non-trivial fundamental symmetry from \( Cl_2(J, R) \) is determined by the formula \( J_\alpha = \alpha_1 J + \alpha_2 R + \alpha_3 iJR \), where \( \alpha \in \mathbb{S}^2 \).

Let \( S \) be a symmetric operator that commutes with \( Cl_2(J, R) \). The purpose of this paper is to study the sets \( \Sigma_{J_\alpha} (\forall \alpha \in \mathbb{S}^2) \) of self-adjoint extensions of \( S \) in Krein spaces generated by fundamental symmetries \( J_\alpha \) (\( J_\alpha \)-self-adjoint extensions). We show that the sets \( \Sigma_{J_\alpha} \) and \( \Sigma_{J_\beta} \) are unitarily equivalent for different \( \alpha, \beta \in \mathbb{S}^2 \) and describe in detail the structure of operators \( A \in \Sigma_{J_\alpha} \) with empty resolvent set.

Keywords: Krein spaces, extension theory of symmetric operators, operators with empty resolvent set, \( J \)-self-adjoint operators, Clifford algebra \( Cl_2 \).

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1. INTRODUCTION

Let \( \mathcal{H} \) be a Hilbert space with inner product \((\cdot, \cdot)\) and with non-trivial fundamental symmetry \( J \) (i.e., \( J = J^*, J^2 = I \), and \( J \neq \pm I \)).

The space \( \mathcal{H} \) endowed with the indefinite inner product (indefinite metric) \([\cdot, \cdot]_J := (J\cdot, \cdot)\) is called a Krein space \((\mathcal{H}, [\cdot, \cdot]_J)\).

An operator \( A \) acting in \( \mathcal{H} \) is called \( J \)-self-adjoint if \( A \) is self-adjoint with respect to the indefinite metric \([\cdot, \cdot]_J \), i.e., if \( A^* J = J A \).

In contrast to self-adjoint operators in Hilbert spaces (which necessarily have a purely real spectrum), a \( J \)-self-adjoint operator \( A \), in general, has spectrum which is only symmetric with respect to the real axis. In particular, the situation where \( \sigma(A) = \mathbb{C} \) (i.e., \( A \) has empty resolvent set \( \rho(A) = \emptyset \)) is also possible and it may
indicate on a special structure of $A$. To illustrate this point we consider a simple symmetric\(^1\) operator $S$ with deficiency indices $\langle 2, 2 \rangle$ which commutes with $J$:

$$SJ = JS.$$ 

It was recently shown [15, Theorem 4.3] that the existence at least one $J$-self-adjoint extension $A$ of $S$ with empty resolvent set is equivalent to the existence of an additional fundamental symmetry $R$ in $\mathfrak{H}$ such that

$$SR = RS, \quad JR = -RJ.$$  \hspace{1cm} (1.1)

The fundamental symmetries $J$ and $R$ can be interpreted as basis (generating) elements of the complex Clifford algebra $\mathbb{C}l_2(J, R) := \text{span}\{I, J, R, iJR\}$ [11]. Hence, the existence of $J$-self-adjoint extensions of $S$ with empty resolvent set is equivalent to the commutation of $S$ with an arbitrary element of the Clifford algebra $\mathbb{C}l_2(J, R)$.

In the present paper we investigate nonself-adjoint extensions of a densely defined symmetric operator $S$ assuming that $S$ commutes with elements of $\mathbb{C}l_2(J, R)$. Precisely, we show that an arbitrary non-trivial fundamental symmetry $J_{\vec{\alpha}}$ constructed in terms of $\mathbb{C}l_2(J, R)$ is uniquely determined by the choice of vector $\vec{\alpha}$ from the unit sphere $S^2$ (Lemma 2.1) and we study various collections $\Sigma_{J_{\vec{\alpha}}}$ of $J_{\vec{\alpha}}$-self-adjoint extensions of $S$. Such a ‘flexibility’ of fundamental symmetries is inspired by the application to $\mathcal{PT}$-symmetric quantum mechanics [5], where $\mathcal{PT}$-symmetric Hamiltonians are not necessarily can be realized as $\mathcal{P}$-self-adjoint operators [1, 17]. Moreover, for certain models [11], the corresponding $\mathcal{PT}$-symmetric operator realizations can be interpreted as $J_{\vec{\alpha}}$-self-adjoint operators when $\vec{\alpha}$ runs $S^2$.

We show that the sets $\Sigma_{J_{\vec{\alpha}}}$ and $\Sigma_{J_{\vec{\beta}}}$ are unitarily equivalent for different $\vec{\alpha}, \vec{\beta} \in S^2$ (Theorem 2.9) and describe properties of $A \in \Sigma_{J_{\vec{\alpha}}}$ in terms of boundary triplets (subsections 2.4, 2.5).

Denote by $\Xi_{\vec{\alpha}}$ the collection of all operators $A \in \Sigma_{J_{\vec{\alpha}}}$ with empty resolvent set. It follows from our results that, as a rule, an operator $A \in \Xi_{\vec{\alpha}}$ is $J_{\vec{\beta}}$-self-adjoint (i.e., $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$) for a special choice of $\vec{\beta} \in S^2$ which depends on $A$. In this way, for the case of symmetric operators $S$ with deficiency indices $\langle 2, 2 \rangle$, the complete description of $\Xi_{\vec{\alpha}}$ is obtained as the union of operators $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$, $\rho(A) = \emptyset$, $\forall \vec{\beta} \in S^2$ (Theorem 3.3). In the exceptional case when the Weyl function of $S$ is a constant, the set $\Xi_{\vec{\alpha}}$ increases considerably (Corollary 3.5).

The one-dimensional Schrödinger differential expression with non-integrable singularity at zero (the limit-circle case at $x = 0$) is considered as an example of application (Proposition 3.6).

Throughout the paper, $\mathcal{D}(A)$ denotes the domain of a linear operator $A$. $A |_\mathcal{D}$ means the restriction of $A$ onto a set $\mathcal{D}$. The notation $\sigma(A)$ and $\rho(A)$ are used for the spectrum and the resolvent set of $A$.

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\(^1\) with respect to the initial inner product $(\cdot, \cdot)$
2. SETS $\Sigma_{\vec{\alpha}}$ AND THEIR PROPERTIES

2.1. PRELIMINARIES

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$ and let $J$ and $R$ be fundamental symmetries in $\mathcal{H}$ satisfying (1.1).

Denote by $\text{Cl}_2(J,R) := \text{span}\{I, J, R, iJR\}$ a complex Clifford algebra with generating elements $J$ and $R$. Since the operators $I, J, R, iJR$ are linearly independent (due to (1.1)), an arbitrary operator $K \in \text{Cl}_2(J,R)$ can be presented as:

$$K = \alpha_0 I + \alpha_1 J + \alpha_2 R + \alpha_3 iJR, \quad \alpha_j \in \mathbb{C}.$$  \hfill (2.1)

**Lemma 2.1** ([15]). An operator $K$ defined by (2.1) is a non-trivial fundamental symmetry in $\mathcal{H}$ (i.e., $K^2 = I$, $K = K^*$, and $K \neq I$) if and only if

$$K = \alpha_1 J + \alpha_2 R + \alpha_3 iJR,$$  \hfill (2.2)

where $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $\alpha_j \in \mathbb{R}$.

**Proof.** The reality of $\alpha_j$ in (2.2) follows from the self-adjointness of $I, J, R,$ and $iJR$. The condition $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ is equivalent to the relation $K^2 = I$. \hfill $\Box$

**Remark 2.2.** The formula (2.2) establishes a one-to-one correspondence between the set of non-trivial fundamental symmetries $K$ in $\text{Cl}_2(J,R)$ and vectors $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ of the unit sphere $S^2$ in $\mathbb{R}^3$. To underline this relationship we will use the notation $J_{\vec{\alpha}}$ for the fundamental symmetry $K$ determined by (2.2), i.e.,

$$J_{\vec{\alpha}} = \alpha_1 J + \alpha_2 R + \alpha_3 iJR.$$  \hfill (2.3)

In particular, this means that $J_{\vec{\alpha}} = J$ with $\vec{\alpha} = (1, 0, 0)$ and $J_{\vec{\alpha}} = R$ when $\vec{\alpha} = (0, 1, 0)$.

**Lemma 2.3.** Let $\vec{\alpha}, \vec{\beta} \in S^2$. Then

$$J_{\vec{\alpha}}J_{\vec{\beta}} = -J_{-\vec{\beta}}J_{\vec{\alpha}} \quad \text{if and only if} \quad \vec{\alpha} \cdot \vec{\beta} = 0.$$  \hfill (2.4)

and

$$J_{\vec{\alpha}} + J_{\vec{\beta}} = |\vec{\alpha} + \vec{\beta}| J_{\frac{\vec{\alpha} + \vec{\beta}}{|\vec{\alpha} + \vec{\beta}|}} \quad \text{if} \quad \vec{\alpha} \neq -\vec{\beta}.$$  \hfill (2.5)

**Proof.** It immediately follows from Lemma 2.1 and identities (1.1), (2.3). \hfill $\Box$

2.2. DEFINITION AND PROPERTIES OF $\Sigma_{J_{\vec{\alpha}}}$

1. Let $S$ be a closed densely defined symmetric operator with equal deficiency indices in the Hilbert space $\mathcal{H}$. In what follows we suppose that $S$ commutes with all elements of $\text{Cl}_2(J,R)$ or, that is equivalent, $S$ commutes with $J$ and $R$:

$$SJ = JS, \quad SR = RS$$  \hfill (2.6)
Denote by $\Upsilon$ the set of all self-adjoint extensions $A$ of $S$ which commute with $J$ and $R$:

$$\Upsilon = \{ A \supset S : A^* = A, \ AJ = JA, \ AR = RA \}. \quad (2.7)$$

It follows from (2.3) and (2.7) that $\Upsilon$ contains self-adjoint extensions of $S$ which commute with all fundamental symmetries $J \alpha \in \text{Cl}_2(J, R)$.

Let us fix one of them $J \alpha$ and denote by $(H, [\cdot, \cdot]_{J \alpha})$ the corresponding Krein space with the indefinite inner product $[\cdot, \cdot]_{J \alpha} := (J \alpha \cdot, \cdot)$.

Denote by $\Sigma_{J \alpha}$ the collection of all $J \alpha$-self-adjoint extensions of $S$:

$$\Sigma_{J \alpha} = \{ A \supset S : J \alpha A^* = AJ \alpha \}. \quad (2.8)$$

An operator $A \in \Sigma_{J \alpha}$ is a self-adjoint extension of $S$ with respect to the indefinite metric $[\cdot, \cdot]_{J \alpha}$.

**Proposition 2.4.** The following relation holds

$$\bigcap_{\forall \alpha \in S^2} \Sigma_{J \alpha} = \Upsilon.$$

**Proof.** It follows from the definitions above that $\Sigma_{J \alpha} \supset \Upsilon$. Therefore,

$$\bigcap_{\forall \alpha \in S^2} \Sigma_{J \alpha} \supset \Upsilon.$$

Let $A \in \bigcap_{\forall \alpha \in S^2} \Sigma_{J \alpha}$. In particular, this means that $A \in \Sigma_J$, $A \in \Sigma_R$, and $A \in \Sigma_{iJR}$. It follows from the first two relations that $JA^* = AJ$ and $RA^* = AR$. Therefore, $iJRA^* = iJAR = A^*iJR$. Simultaneously, $iJRA^* = AiJR$ since $A \in \Sigma_{iJR}$. Comparing the obtained relations we deduce that $A^*iJR = AiJR$ and hence, $A^* = A$. Thus $A$ is a self-adjoint operator and it commutes with an arbitrary fundamental symmetry $J \alpha \in \text{Cl}_2(J, R)$. Therefore, $A \in \Upsilon$. Proposition 2.4 is proved. \qed

Simple analysis of the proof of Proposition 2.4 leads to the conclusion that

$$\Sigma_{J \alpha} \cap \Sigma_{J \beta} \cap \Sigma_{J \gamma} = \Upsilon$$

for any three linearly independent vectors $\alpha, \beta, \gamma \in S^2$. However,

$$\Sigma_{J \alpha} \cap \Sigma_{J \beta} \supset \Upsilon, \quad \forall \alpha, \beta \in S^2 \quad (2.9)$$

and the intersection $\Sigma_{J \alpha} \cap \Sigma_{J \beta}$ contains operators $A$ with empty resolvent set (i.e., $\rho(A) = \emptyset$ or, that is equivalent, $\sigma(A) = \mathbb{C}$). Let us discuss this phenomena in detail.

Consider two linearly independent vectors $\alpha, \beta \in S^2$. If $\alpha \cdot \beta \neq 0$, we define new vector $\beta' \in S^2$:

$$\beta' = \frac{\alpha + c\beta}{|\alpha + c\beta|}, \quad c = -\frac{1}{\alpha \cdot \beta} \quad (2.10)$$

We refer to [4, 10] for the terminology of the Krein spaces theory.
such that $\vec{\alpha} \cdot \vec{\beta}' = 0$. Then the fundamental symmetry
\[
J_{\vec{\beta}'} = \frac{1}{|\vec{\alpha} + c\vec{\beta}|} J_{\vec{\alpha}} + \frac{c}{|\vec{\alpha} + c\vec{\beta}|} J_{\vec{\beta}}
\] (2.10)
anti-commutes with $J_{\vec{\alpha}}$ (due to Lemma 2.3).

The operator
\[
J_{\vec{\gamma}} = iJ_{\vec{\alpha}}J_{\vec{\beta}'} = \begin{vmatrix} J & R & iJR \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1' & \beta_2' & \beta_3' \end{vmatrix}
\] (2.11)
is a fundamental symmetry in $\mathcal{F}$ which commutes with $S$. Therefore, the orthogonal decomposition of $\mathcal{F}$ constructed by $J_{\vec{\gamma}}$:
\[
\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-, \quad \mathcal{F}_+^\gamma = \frac{1}{2} (I + J_\vec{\gamma}) \mathcal{F}, \quad \mathcal{F}_-^\gamma = \frac{1}{2} (I - J_\vec{\gamma}) \mathcal{F}
\] (2.12)
reduces $S$:
\[
S = \begin{pmatrix} S_{\gamma^+} & 0 \\ 0 & S_{\gamma^-} \end{pmatrix}, \quad S_{\gamma^+} = S \mid \mathcal{F}_+^\gamma, \quad S_{\gamma^-} = S \mid \mathcal{F}_-^\gamma.
\] (2.13)

Since $J_{\vec{\gamma}}$ anti-commutes with $J_{\vec{\alpha}}$ (see (2.11)), the operator $J_{\vec{\alpha}}$ maps $\mathcal{F}_+^\gamma$ onto $\mathcal{F}_+^\gamma$ and operators $S_{\gamma^+}$ and $S_{\gamma^-}$ are unitarily equivalent. Precisely, $S_{\gamma^-} x = J_{\vec{\alpha}} S_{\gamma^+} J_{\vec{\alpha}} x$ for all elements $x \in \mathcal{D}(S_{\gamma^-})$. This means that $S_{\gamma^+}$ and $S_{\gamma^-}$ have equal deficiency indices.\(^3\)

Denote
\[
A_\gamma = \begin{pmatrix} S_{\gamma^+} & 0 \\ 0 & S_{\gamma^-} \end{pmatrix}, \quad A_\gamma^* = \begin{pmatrix} S_{\gamma^+}^* & 0 \\ 0 & S_{\gamma^-}^* \end{pmatrix}.
\] (2.14)
The operators $A_\gamma$ and $A_\gamma^*$ are extensions of $S$ and $\sigma(A_\gamma) = \sigma(A_\gamma^*) = \mathbb{C}$ (since $S_{\gamma^\pm}$ are symmetric operators), i.e., these operators have empty resolvent set.

**Theorem 2.5.** Let $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ be linearly independent vectors. Then the operators $A_\gamma$ and $A_\gamma^*$ defined by (2.14) belong to $\Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$.

**Proof.** Assume that $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$, where $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ are linearly independent vectors. Then
\[
J_{\vec{\alpha}} A = A_\gamma^* J_{\vec{\alpha}}, \quad J_{\vec{\beta}'} A = A_\gamma^* J_{\vec{\beta}'}
\] (2.15)
and hence, $J_{\vec{\beta}'} A = A_\gamma^* J_{\vec{\beta}'}$ due to (2.10). In that case
\[
J_{\vec{\gamma}} A = iJ_{\vec{\alpha}} J_{\vec{\beta}'} A = iJ_{\vec{\alpha}} A_\gamma^* J_{\vec{\beta}'} = iA J_{\vec{\alpha}} J_{\vec{\beta}'} = AJ_{\vec{\gamma}},
\]
where $J_{\vec{\gamma}} = iJ_{\vec{\alpha}} J_{\vec{\beta}'}$ (see (2.11)) is the fundamental symmetry in $\mathcal{F}$.\(^3\)

\(^3\) This also implies that the symmetric operator $S$ commuting with $Cl_2(J, R)$ may have only even deficiency indices.
Since \( A \) commutes with \( J_\vec{\gamma} \), the decomposition (2.12) reduces \( A \) and

\[
A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \quad A_+ = A \upharpoonright_{\mathcal{H}^+_\gamma}, \quad A_- = A \upharpoonright_{\mathcal{H}^-_\gamma},
\]

(2.16)

where \( S^+_\gamma \subseteq A_+ \subseteq S^{*+}_\gamma \) and \( S^-_\gamma \subseteq A_- \subseteq S^{*-}_\gamma \). This means that

\[
A^* = \begin{pmatrix} A^*_+ & 0 \\ 0 & A^*_- \end{pmatrix}, \quad A^*_+ = A^* \upharpoonright_{\mathcal{H}^+_\gamma}, \quad A^*_- = A^* \upharpoonright_{\mathcal{H}^-_\gamma}.
\]

(2.17)

Since \( J_\vec{\alpha} \) anti-commutes with \( J_\vec{\gamma} \), the operator \( J_\vec{\alpha} \) maps \( \mathcal{H}^+_\gamma \) onto \( \mathcal{H}^-_\gamma \). Therefore, the first relation in (2.15) can be rewritten with the use of formulas (2.16) and (2.17) as follows:

\[
J_\vec{\alpha} Ax = J_\vec{\alpha}(A_+x_+ + A_-x_-) = A^*_+J_\vec{\alpha}x_+ + A^*_+J_\vec{\alpha}x_- = A^*J_\vec{\alpha}x,
\]

(2.18)

where \( x = x_+ + x_- \in \mathcal{D}(A), \ x_\pm \in \mathcal{D}(A_\pm) \).

The identity (2.18) holds for all \( x_\pm \in \mathcal{D}(A_\pm) \). This means that

\[
J_\vec{\alpha} A_+ = A^*_+ J_\vec{\alpha}, \quad J_\vec{\alpha} A_- = A^*_+ J_\vec{\alpha}.
\]

(2.19)

It follows from (2.10) and (2.11) that the fundamental symmetry \( J_\vec{\beta} \) anti-commutes with \( J_\vec{\gamma} \). Repeating the arguments above for the second relation in (2.15) we obtain

\[
J_\vec{\beta} A_+ = A^*_+ J_\vec{\beta}, \quad J_\vec{\beta} A_- = A^*_+ J_\vec{\beta}.
\]

(2.20)

Thus an operator \( A \) belongs to \( \Sigma_{J_\vec{\alpha}} \cap \Sigma_{J_\vec{\beta}} \) if and only if its counterparts \( A_+ \) and \( A_- \) in (2.16) satisfy relations (2.19) and (2.20). In particular, these relations are satisfied for the cases when \( A_+ = S^+_\gamma, \ A_- = S^{*-}_\gamma \) and \( A_+ = S^{*+}_\gamma, \ A_- = S^-_\gamma \). Hence, the operators \( A_\gamma, \ A^*_\gamma \) defined by (2.14) belong to \( \Sigma_{J_\vec{\alpha}} \cap \Sigma_{J_\vec{\beta}} \). Theorem 2.5 is proved.

Remark 2.6. The operators \( A_\gamma \) and \( A^*_\gamma \) constructed above depend on the choice of \( \vec{\beta} \in \mathbb{S}^2 \). Considering various vectors \( \vec{\beta} \in \mathbb{S}^2 \) in (2.10), (2.11), we obtain a collection of fundamental symmetries \( J_{\vec{\gamma}(\vec{\beta})} \). This gives rise to a one-parameter set of different operators \( A_{\gamma(\vec{\beta})} \) and \( A^*_{\gamma(\vec{\beta})} \) with empty resolvent set which belong to \( \Sigma_{J_\vec{\alpha}} \).

Corollary 2.7. Let \( \vec{\alpha}, \vec{\beta} \in \mathbb{S}^2 \) be linearly independent vectors and let (2.12) be the decomposition of \( \mathcal{H} \) constructed by these vectors. Then, with respect to (2.12), all operators \( A \in \Sigma_{J_\vec{\alpha}} \cap \Sigma_{J_\vec{\beta}} \) are described by the formula

\[
A = \begin{pmatrix} A_+ & 0 \\ 0 & J_\vec{\alpha} A^*_+ J_\vec{\alpha} \end{pmatrix},
\]

(2.21)

where \( A_+ \) is an arbitrary intermediate extension of \( S_{\gamma+} = S \upharpoonright_{\mathcal{H}^+_\gamma} \) (i.e., \( S_{\gamma+} \subseteq A_+ \subseteq \subseteq S^{*+}_{\gamma+} \)).
Proof. If \( A \in \Sigma_{J_\alpha} \cap \Sigma_{J_\beta} \), then the presentation (2.21) follows from (2.16) and the second identity in (2.19).

Conversely, assume that an operator \( A \) is defined by (2.21). Since \( J_\alpha \) and \( J_\beta \) anti-commute with \( J_\gamma \), they admit the presentations \( J_\alpha = \begin{pmatrix} 0 & J_\alpha \\ J_\alpha & 0 \end{pmatrix} \) and \( J_\beta = \begin{pmatrix} 0 & J_\beta \\ J_\beta & 0 \end{pmatrix} \) with respect to (2.12). Then, the operator equality \( J_\alpha A = A^* J_\alpha \) is established by the direct multiplication of the corresponding operator entries. The same procedure for \( J_\beta A = A^* J_\beta \) leads to the verification of relations

\[
J_\alpha J_\beta A_+ = A_+ J_\alpha J_\beta, \quad J_\beta J_\alpha A^*_+ = A^*_+ J_\beta J_\alpha.
\]

(2.22)

To this end we recall that \( J_\gamma \) commutes with \( A_+ \) and

\[
J_\alpha J_\beta = -\frac{1}{c} I - i \frac{[\bar{\alpha} + c\bar{\beta}]}{c} J_\gamma
\]
due to (2.10) and (2.11). Therefore, \( A_+ \) commutes with \( J_\alpha J_\beta \) and the first relation in (2.22) holds. The second relation is established in the same manner, if we take into account that \( J_\gamma \) commutes with \( A_+ \) and \( J_\beta J_\alpha = -\frac{1}{c} I + i \frac{[\bar{\alpha} + c\bar{\beta}]}{c} J_\gamma \). Corollary 2.7 is proved.

Remark 2.8. It follows from the proof that the choice of \( J_\gamma \) in (2.21) is not essential and the similar description of \( \Sigma_{J_\alpha} \cap \Sigma_{J_\beta} \) can be obtained with the help of \( J_\beta \).

2. Denote

\[
W_{\bar{\alpha}, \bar{\beta}} = \begin{cases} J_{\frac{\bar{\alpha} + \bar{\beta}}{|\bar{\alpha} + \bar{\beta}|}} & \text{if } \bar{\alpha} \neq -\bar{\beta}, \\ I & \text{if } \bar{\alpha} = -\bar{\beta}. \end{cases}
\]

(2.23)

It is clear that \( W_{\bar{\alpha}, \bar{\beta}} \) is a fundamental symmetry in \( \mathcal{S} \) and \( W_{\bar{\alpha}, \bar{\beta}} = W_{\bar{\beta}, \bar{\alpha}} \) for any \( \bar{\alpha}, \bar{\beta} \in \mathbb{S}^2 \).

Theorem 2.9. For any \( \bar{\alpha}, \bar{\beta} \in \mathbb{S}^2 \) the sets \( \Sigma_{J_{\bar{\alpha}}} \) and \( \Sigma_{J_{\bar{\beta}}} \) are unitarily equivalent and \( A \in \Sigma_{J_{\bar{\alpha}}} \) if and only if \( W_{\bar{\alpha}, \bar{\beta}} A W_{\bar{\alpha}, \bar{\beta}}^* \in \Sigma_{J_{\bar{\beta}}} \).

Proof. Since \( J_{-\bar{\alpha}} = -J_\bar{\alpha} \) (see (2.3)), the sets \( \Sigma_{J_{\bar{\alpha}}} \) and \( \Sigma_{J_{-\bar{\alpha}}} \) coincide and therefore, the case \( \bar{\alpha} = -\bar{\beta} \) is trivial.

Assume that \( A \in \Sigma_{J_{\bar{\alpha}}} \), \( \bar{\alpha} \neq -\bar{\beta} \) and consider the operator \( W_{\bar{\alpha}, \bar{\beta}} A W_{\bar{\alpha}, \bar{\beta}}^* \), which we denote \( B \) for brevity. Taking into account that \( S \) commutes with \( J_\alpha \) for any choice of \( \bar{\alpha} \in \mathbb{S}^2 \), we deduce from (2.23) that \( W_{\bar{\alpha}, \bar{\beta}} S = SW_{\bar{\alpha}, \bar{\beta}} \) and \( W_{\bar{\alpha}, \bar{\beta}} S^* = S^* W_{\bar{\alpha}, \bar{\beta}} \).

This means that \( Bx = W_{\bar{\alpha}, \bar{\beta}} A W_{\bar{\alpha}, \bar{\beta}}^* x = W_{\bar{\alpha}, \bar{\beta}} S W_{\bar{\alpha}, \bar{\beta}}^* x = Sx \) for all \( x \in \mathcal{D}(S) \) and \( By = W_{\bar{\alpha}, \bar{\beta}} A W_{\bar{\alpha}, \bar{\beta}}^* y = W_{\bar{\alpha}, \bar{\beta}} S^* W_{\bar{\alpha}, \bar{\beta}}^* y = S^* y \) for all \( y \in \mathcal{D}(B) = W_{\bar{\alpha}, \bar{\beta}} \mathcal{D}(A) \). Therefore, \( B \) is an intermediate extension of \( S \) (i.e., \( S \subseteq B \subseteq S^* \)).

It follows from (2.5) and (2.23) that

\[
J_\beta W_{\bar{\alpha}, \bar{\beta}} = \frac{J_\alpha + J_\beta}{|\bar{\alpha} + \bar{\beta}|} = \frac{J_\beta J_\alpha + I}{|\bar{\alpha} + \bar{\beta}|} = \frac{J_\beta J_\alpha}{|\bar{\alpha} + \bar{\beta}|} J_\alpha = W_{\bar{\alpha}, \bar{\beta}} J_\alpha.
\]

(2.24)
Using (2.15) and (2.24), we arrive at the conclusion that

$$J^\beta B^\ast = J^\beta W_{\bar{\alpha},\bar{\beta}} A^\ast W_{\bar{\alpha},\bar{\beta}} = W_{\bar{\alpha},\bar{\beta}} A J_{\bar{\alpha}} W_{\bar{\alpha},\bar{\beta}} = W_{\bar{\alpha},\bar{\beta}} A W_{\bar{\alpha},\bar{\beta}} J^\beta = B J^\beta.$$ 

Therefore, condition $A \in \Sigma_{J^\alpha}$ implies that $B = W_{\bar{\alpha},\bar{\beta}} A W_{\bar{\alpha},\bar{\beta}} \in \Sigma_{J^\beta}$. The inverse implication $B \in \Sigma_{J^\beta} \Rightarrow A = W_{\bar{\alpha},\bar{\beta}} B W_{\bar{\alpha},\bar{\beta}} \in \Sigma_{J^\alpha}$ is established in the same manner. Theorem 2.9 is proved.

**Remark 2.10.** Due to Theorem 2.9, for any $J^\alpha$-self-adjoint extension $A \in \Sigma_{J^\alpha}$ there exists a unitarily equivalent $J_{\beta}$-self-adjoint extension $B \in \Sigma_{J_{\beta}}$. This means that, the spectral analysis of operators from $\bigcup_{\alpha \in \Sigma} \Sigma_{J^\alpha}$ can be reduced to the spectral analysis of $J^\alpha$-self-adjoint extensions from $\Sigma_{J^\alpha}$, where $\bar{\alpha}$ is a fixed vector from $S^2$.

### 2.3. Boundary Triplets and Weyl Function

1. Let $S$ be a closed symmetric operator with equal deficiency indices in the Hilbert space $\mathfrak{H}$. A triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1$ are linear mappings of $D(S^\ast)$ into $\mathcal{H}$, is called a boundary triplet of $S^\ast$ if the abstract Green identity

$$\langle S^\ast x, y \rangle - \langle x, S^\ast y \rangle = \langle \Gamma_1 x, \Gamma_0 y \rangle_\mathcal{H} - \langle \Gamma_0 x, \Gamma_1 y \rangle_\mathcal{H}, \quad x, y \in D(S^\ast)$$

is satisfied and the map $(\Gamma_0, \Gamma_1) : D(S^\ast) \to \mathcal{H} \oplus \mathcal{H}$ is surjective [7,9].

**Lemma 2.11.** Assume that $S$ satisfies the commutation relations (2.6) and $J_\varphi, J_\bar{\varphi} \in \text{Cl}_2(J, R)$ are fixed anti-commuting fundamental symmetries. Then there exists a boundary triplet $(\mathcal{H}, \Gamma_0, \Gamma_1)$ of $S^\ast$ such that the formulas

$$J_\varphi \Gamma_j := \Gamma_j J_\varphi, \quad J_\bar{\varphi} \Gamma_j := \Gamma_j J_\bar{\varphi}, \quad j = 0, 1$$

correctly define anti-commuting fundamental symmetries $J_\varphi$ and $J_\bar{\varphi}$ in the Hilbert space $\mathcal{H}$.

**Proof.** If $S$ satisfies (2.6), then $S$ commutes with an arbitrary fundamental symmetry $J_{\bar{\varphi}} \in \text{Cl}_2(J, R)$ and hence, $S$ admits the representation (2.13) for any vector $\bar{\gamma} \in S^2$.

Let $S_{\gamma_+}$ be a symmetric operator in $\mathfrak{H}_+^\gamma$ from (2.13) and let $(N, \Gamma_0^+, \Gamma_1^+)$ be an arbitrary boundary triplet of $S_{\gamma_+}^\ast$.

Since $J_\varphi$ anti-commutes with $J_{\bar{\varphi}}$, the symmetric operator $S_{\gamma_-}$ in (2.13) can be described as $S_{\gamma_-} = J_\varphi S_{\gamma_+} J_{\bar{\varphi}}$. This means that $(N, \Gamma_0^+, J_{\bar{\varphi}}, \Gamma_1^+ J_\varphi)$ is a boundary triplet of $S_{\gamma_-}$.

It is easy to see that the operators

$$\Gamma_j f = \Gamma_j (f_+ + f_-) = \begin{pmatrix} \Gamma_j^+ f_+ \\ \Gamma_j^+ J_{\bar{\varphi}} f_- \end{pmatrix}$$

(\(f = f_+ + f_- \in D(S^\ast), \ f_\pm \in D(S_{\gamma_\pm}^\ast))\) map $D(S^\ast)$ onto the Hilbert space

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad \mathcal{H}_+ = \begin{pmatrix} N \\ 0 \end{pmatrix}, \quad \mathcal{H}_- = \begin{pmatrix} 0 \\ N \end{pmatrix}$$
and they form a boundary triplet \((H, \Gamma_0, \Gamma_1)\) of \(S^*\) which satisfies (2.26) with
\[
\mathcal{J}_{\vec{\alpha}} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \mathcal{J}_{\vec{\gamma}} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.
\]
(2.28)

It is clear that \(\mathcal{J}_{\vec{\alpha}}\) and \(\mathcal{J}_{\vec{\gamma}}\) are anti-commuting fundamental symmetries in the Hilbert space \(H\).

**Remark 2.12.** The fundamental symmetries \(\mathcal{J}_{\vec{\alpha}}\) and \(\mathcal{J}_{\vec{\gamma}}\) are defined by (2.26) in a similar way that in [13], where symmetric operators commuting with involution has been studied.

**Remark 2.13.** Since \(J\) and \(R\) can be expressed as linear combinations of \(J_{\vec{\alpha}}, J_{\vec{\gamma}}\), and \(iJ_{\vec{\alpha}}J_{\vec{\gamma}}\), formulas (2.26) imply that
\[
\mathcal{J} \Gamma_j := \Gamma_j J, \quad \mathcal{R} \Gamma_j := \Gamma_j R, \quad j = 0, 1,
\]
where \(\mathcal{J}\) and \(\mathcal{R}\) are anti-commuting fundamental symmetries in \(H\). Therefore, an arbitrary boundary triplet \((H, \Gamma_0, \Gamma_1)\) of \(S^*\) with property (2.26) allows one to establish a bijective correspondence between elements of the initial Clifford algebra \(Cl_2(J, R)\) and its “image” \(Cl_2(J, \mathcal{R})\) in the auxiliary space \(H\). In particular, for every \(\mathcal{J}_{\vec{\alpha}} \in Cl_2(J, R)\) defined by (2.3),
\[
\mathcal{J}_{\vec{\alpha}} \Gamma_j = \Gamma_j \mathcal{J}_{\vec{\alpha}}, \quad j = 0, 1,
\]
(2.29)

where \(\mathcal{J}_{\vec{\alpha}} = \alpha_1 \mathcal{J} + \alpha_2 \mathcal{R} + \alpha_3 i \mathcal{J} \mathcal{R}\) belongs to \(Cl_2(\mathcal{J}, \mathcal{R})\).

2. Let \((H, \Gamma_0, \Gamma_1)\) be a boundary triplet of \(S^*\). The Weyl function of \(S\) associated with \((H, \Gamma_0, \Gamma_1)\) is defined as follows:
\[
M(\mu) \Gamma_0 f_\mu = \Gamma_1 f_\mu, \quad \forall f_\mu \in \ker(S^* - \mu I), \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}.
\]
(2.30)

**Lemma 2.14.** Let \((H, \Gamma_0, \Gamma_1)\) be a boundary triplet of \(S^*\) with properties (2.26). Then the corresponding Weyl function \(M(\cdot)\) commutes with every fundamental symmetry \(\mathcal{J}_{\vec{\alpha}} \in Cl_2(\mathcal{J}, \mathcal{R})\):
\[
M(\mu) \mathcal{J}_{\vec{\alpha}} = \mathcal{J}_{\vec{\alpha}} M(\mu), \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}.
\]

**Proof.** It follows from (2.3) and (2.6) that \(S^* \mathcal{J}_{\vec{\alpha}} = \mathcal{J}_{\vec{\alpha}} S^*\) for all \(\vec{\alpha} \in \mathbb{S}^2\). Therefore, \(\mathcal{J}_{\vec{\alpha}} : \ker(S^* - \mu I) \rightarrow \ker(S^* - \mu I)\). In that case, relations (2.29) and (2.30) lead to \(M(\mu) \mathcal{J}_{\vec{\alpha}} \Gamma_0 f_\mu = \mathcal{J}_{\vec{\alpha}} \Gamma_1 f_\mu\). Thus, \(\mathcal{J}_{\vec{\alpha}} M(\mu) \mathcal{J}_{\vec{\alpha}} = M(\mu)\) or \(M(\mu) \mathcal{J}_{\vec{\alpha}} = \mathcal{J}_{\vec{\alpha}} M(\mu)\). \(\square\)

2.4. DESCRIPTION OF \(\Sigma_{\mathcal{J}_{\vec{\alpha}}}\) IN TERMS OF BOUNDARY TRIPLETS

**Theorem 2.15.** Let \((H, \Gamma_0, \Gamma_1)\) be a boundary triplet of \(S^*\) with properties (2.26) for a fixed anti-commuting fundamental symmetries \(J_{\vec{\alpha}}, J_{\vec{\gamma}} \in Cl_2(J, R)\) and let \(\mathcal{J}_{\vec{\alpha}}\) be an arbitrary fundamental symmetry from \(Cl_2(J, R)\). Then operators \(A \in \Sigma_{\mathcal{J}_{\vec{\alpha}}}\) coincide with the restriction of \(S^*\) onto the domains
\[
\mathcal{D}(A) = \{ f \in \mathcal{D}(S^*) : U(\mathcal{J}_{\vec{\alpha}} \Gamma_1 + i \Gamma_0) f = (\mathcal{J}_{\vec{\alpha}} \Gamma_1 - i \Gamma_0) f \},
\]
(2.31)

where \(U\) runs the set of unitary operators in \(H\). The correspondence \(A \leftrightarrow U\) determined by (2.31) is a bijection between the set \(\Sigma_{\mathcal{J}_{\vec{\alpha}}}\) of all \(\mathcal{J}_{\vec{\alpha}}\)-self-adjoint extensions of \(S\) and the set of unitary operators in \(H\).
Proof. An operator $A$ is a $J_{\bar{\alpha}}$-self-adjoint extension of $S$ if and only if $J_{\bar{\alpha}}A$ is a self-adjoint extension of the symmetric operator $J_{\bar{\alpha}}S$. Since
\[
(J_{\bar{\alpha}}S)^* = S^* J_{\bar{\alpha}} = J_{\bar{\alpha}}S^*,
\] (2.32)
the Green identity (2.25) can be rewritten with the use of (2.29) as follows:
\[
(S^* J_{\bar{\alpha}} x, y) - (x, S^* J_{\bar{\alpha}} y) = (J_{\bar{\alpha}} \Gamma_1 x, \Gamma_0 y)_{\mathcal{H}} - (\Gamma_0 x, J_{\bar{\alpha}} \Gamma_1 y)_{\mathcal{H}}.
\]
Recalling the definition of boundary triplet we conclude that (2.25) can be rewritten with the use of (2.29) as follows:
\[
(S^* J_{\bar{\alpha}} x, y) - (x, S^* J_{\bar{\alpha}} y) = (J_{\bar{\alpha}} \Gamma_1 x, \Gamma_0 y)_{\mathcal{H}} - (\Gamma_0 x, J_{\bar{\alpha}} \Gamma_1 y)_{\mathcal{H}}.
\]
Corollary 2.16. If $A \in \Sigma J_{\bar{\alpha}}$ and $A \leftrightarrow U$ in (2.31), then the $J_{\bar{\beta}}$-self-adjoint operator $B = W_{\bar{\alpha},\bar{\beta}} AW_{\bar{\alpha},\bar{\beta}} \in \Sigma J_{\bar{\beta}}$ ($\bar{\alpha} \neq -\bar{\beta}$) is determined by the formula
\[
B = S^* \upharpoonright \{ g \in \mathcal{D}(S^*) \ : \ W_{\bar{\alpha},\bar{\beta}} U W_{\bar{\alpha},\bar{\beta}} (J_{\bar{\beta}} \Gamma_1 + i \Gamma_0) g = (J_{\bar{\beta}} \Gamma_1 - i \Gamma_0) g \},
\]
where $W_{\bar{\alpha},\bar{\beta}} = J_{\bar{\alpha} + \bar{\beta}}$ is a fundamental symmetry in $\mathcal{H}$. Proof. Let $A \in \Sigma J_{\bar{\alpha}}$. Then $B = W_{\bar{\alpha},\bar{\beta}} AW_{\bar{\alpha},\bar{\beta}} \in \Sigma J_{\bar{\beta}}$ by Theorem 2.9 and, in view of Theorem 2.15,
\[
B = S^* \upharpoonright \{ g \in \mathcal{D}(S^*) \ : \ U'(J_{\bar{\beta}} \Gamma_1 + i \Gamma_0) g = (J_{\bar{\beta}} \Gamma_1 - i \Gamma_0) g \},
\] (2.33)
where $U'$ is a unitary operator in $\mathcal{H}$.

It follows from the definition of $B$ that $f \in \mathcal{D}(A)$ if and only if $g = W_{\bar{\alpha},\bar{\beta}} f \in \mathcal{D}(B)$. Hence, we can rewrite (2.33) with the use of (2.24):
\[
U'(J_{\bar{\beta}} \Gamma_1 + i \Gamma_0) g = U' W_{\bar{\alpha},\bar{\beta}}(J_{\bar{\alpha}} \Gamma_1 + i \Gamma_0) f =
\]
\[
= (J_{\bar{\beta}} \Gamma_1 - i \Gamma_0) g =
\]
\[
= W_{\bar{\alpha},\bar{\beta}}(J_{\bar{\alpha}} \Gamma_1 - i \Gamma_0) f,
\] (2.34)
where $W_{\bar{\alpha},\bar{\beta}} \Gamma_j = \Gamma_j W_{\bar{\alpha},\bar{\beta}}$, $j = 0, 1$ (cf. (2.29)).

It follows from (2.23) that $W_{\bar{\alpha},\bar{\beta}} = J_{\bar{\alpha} + \bar{\beta}}$ and hence, $W_{\bar{\alpha},\bar{\beta}}$ is a fundamental symmetry in $\mathcal{H}$. Comparing (2.34) with (2.31), we arrive at the conclusion that $U' = W_{\bar{\alpha},\bar{\beta}} U W_{\bar{\alpha},\bar{\beta}}$. Corollary 2.16 is proved.

Corollary 2.17. A $J_{\bar{\alpha}}$-self-adjoint operator $A \in \Sigma J_{\bar{\alpha}}$ commutes with $J_{\bar{\beta}}$, where $\bar{\alpha} \cdot \bar{\beta} = 0$ if and only if the corresponding unitary operator $U$ in (2.31) satisfies the relation
\[
J_{\bar{\beta}} U = U^{-1} J_{\bar{\beta}}.
\] (2.35)
Proof. Assume that $\vec{\beta} \in S^2$ and $\vec{\alpha} \cdot \vec{\beta} = 0$. Then $J_{\vec{\beta}}J_{\vec{\alpha}} = -J_{\vec{\alpha}}J_{\vec{\beta}}$ due to Lemma 2.3. Since $J_{\vec{\beta}}S^* = S^*J_{\vec{\beta}}$, the commutation relation $AJ_{\vec{\beta}} = J_{\vec{\beta}}A$ is equivalent to the condition
$$\forall f \in \mathcal{D}(A) \Rightarrow J_{\vec{\beta}}f \in \mathcal{D}(A).$$ (2.36)

Let $f \in \mathcal{D}(S^*)$. Recalling that $\mathcal{J}_\vec{\beta}\Gamma_j = \Gamma_jJ_{\vec{\beta}}$, we obtain
$$(\mathcal{J}_\vec{\alpha}\Gamma_1 + i\Gamma_0)J_{\vec{\beta}}f = -\mathcal{J}_{\vec{\beta}}(\mathcal{J}_\vec{\alpha}\Gamma_1 - i\Gamma_0)f,$$
$$(\mathcal{J}_\vec{\alpha}\Gamma_1 - i\Gamma_0)J_{\vec{\beta}}f = -\mathcal{J}_{\vec{\beta}}(\mathcal{J}_\vec{\alpha}\Gamma_1 + i\Gamma_0)f.$$

Combining the last two relations with (2.31), we conclude that (2.36) is equivalent to the identity $\mathcal{J}_\vec{\beta}U^{-1}\mathcal{J}_{\vec{\beta}} = U$. Corollary 2.17 is proved.

Corollary 2.18. A $J_{\vec{\alpha}}$-self-adjoint operator $A \in \Sigma_{J_{\vec{\alpha}}}$ belongs to the subset $\Upsilon$ (see (2.7)) if and only if the corresponding unitary operator $U$ in (2.31) satisfies the equality (2.35) for all $\vec{\beta} \in S^2$ such that $\vec{\alpha} \cdot \vec{\beta} = 0$.

Proof. Since $U$ satisfies (2.35) for all $\vec{\beta} \in S^2$ such that $\vec{\alpha} \cdot \vec{\beta} = 0$, the operator $A \in \Sigma_{J_{\vec{\alpha}}}$ commutes with an arbitrary $J_{\vec{\beta}}$ such that $J_{\vec{\alpha}}J_{\vec{\beta}} = -J_{\vec{\beta}}J_{\vec{\alpha}}$ (due to Lemma 2.3 and Corollary 2.17). In particular, the fundamental symmetry $J_{\vec{\gamma}} = iJ_{\vec{\alpha}}J_{\vec{\beta}}$ anti-commutes with $J_{\vec{\alpha}}$ and hence, $J_{\vec{\gamma}}A = AJ_{\vec{\gamma}}$. On the other hand, since $A \in \Sigma_{J_{\vec{\alpha}}}$, we have $J_{\vec{\alpha}}A = A^*J_{\vec{\alpha}}$ and
$$J_{\vec{\gamma}}A = iJ_{\vec{\alpha}}J_{\vec{\beta}}A = iJ_{\vec{\alpha}}AJ_{\vec{\beta}} = A^*iJ_{\vec{\alpha}}J_{\vec{\beta}} = A^*J_{\vec{\gamma}}.$$ Thus $AJ_{\vec{\gamma}} = A^*J_{\vec{\gamma}}$ and hence, $A = A^*$. This means that the self-adjoint extension $A \supset S$ commutes with all fundamental symmetries from the Clifford algebra $Cl_2(J,R)$. Therefore, $A \in \Upsilon$.

Corollary 2.19. Let $A \in \Sigma_{J_{\vec{\alpha}}}$ be defined by (2.31) with $U = \mathcal{J}_{\vec{\gamma}}$, where $\vec{\gamma} \in S^2$ is an arbitrary vector such that $\vec{\alpha} \cdot \vec{\gamma} = 0$. Then $\sigma(A) = \mathbb{C}$, i.e., $A$ has empty resolvent set.

Proof. Taking into account that $\mathcal{J}_{\vec{\alpha}}\mathcal{J}_{\vec{\gamma}} = -\mathcal{J}_{\vec{\gamma}}\mathcal{J}_{\vec{\alpha}}$ (since $\vec{\alpha} \cdot \vec{\gamma} = 0$), we rewrite the definition (2.31) of $A$:
$$A = S^* \upharpoonright \{ f \in \mathcal{D}(S^*) : \mathcal{J}_{\vec{\alpha}}(\mathcal{J}_{\vec{\gamma}} + I)\Gamma_1f = i(\mathcal{J}_{\vec{\gamma}} + I)\Gamma_0f \}. \quad (2.37)$$

Since relation (2.35) holds when $U = \mathcal{J}_{\vec{\gamma}}$ and $\vec{\beta} = \vec{\gamma}$, the operator $A \in \Sigma_{J_{\vec{\alpha}}}$ commutes with $J_{\vec{\gamma}}$ (Corollary 2.17). Therefore (cf. (2.16)),
$$A = \left( \begin{array}{cc} A_+ & 0 \\ 0 & A_- \end{array} \right), \quad A_+ = A \upharpoonright \mathcal{S}^+\gamma, \quad A_- = A \upharpoonright \mathcal{S}^-\gamma \quad (2.38)$$
with respect to the decomposition (2.12). Here $\mathcal{S}^+\gamma \subseteq A_+ \subseteq S^*_\gamma$ and $\mathcal{S}^-\gamma \subseteq A_- \subseteq S^-\gamma$, where $S^\pm \gamma = \mathcal{S} \upharpoonright \mathcal{S}^\pm\gamma$.

Denote $\mathcal{H}^+ = \frac{1}{2}(I + \mathcal{J}_{\vec{\gamma}})\mathcal{H}$ and $\mathcal{H}^- = \frac{1}{2}(I - \mathcal{J}_{\vec{\gamma}})\mathcal{H}$. Then
$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \quad (2.39)$$
and \((\mathcal{H}^\gamma_+ \oplus \Gamma_0, \Gamma_1)\) are boundary triplets of operators \(S^*_\gamma\) (due to (2.29) and Lemma 2.11).

Let \(f \in \mathcal{D}(S^*_{\gamma^+})\). Then \(\Gamma_j f \in \mathcal{H}^\gamma_j\), \(j = 0, 1\) and the identity in (2.37) takes the form

\[
J_{\tilde{\alpha}} \Gamma_1 f = i \Gamma_0 f.
\]

(2.40)

Since \(J_{\tilde{\alpha}} J_{\tilde{\alpha}} = -J_{\tilde{\alpha}} J_{\tilde{\alpha}}\), the operator \(J_{\tilde{\alpha}}\) maps \(\mathcal{H}^\gamma_+\) onto \(\mathcal{H}^\gamma_+\). Thus, (2.40) may only hold in the case where \(\Gamma_0 f = \Gamma_1 f = 0\). Therefore, the operator \(A_+\) in (2.38) coincides with \(S^*_\gamma\).

Assume now \(f \in \mathcal{D}(S^*_{\gamma^-})\). Then \(\Gamma_j f \in \mathcal{H}^\gamma_j\), \(j = 0, 1\) and the identity in (2.37) vanishes (i.e., \(0 = 0\)). This means that \(A_- = S^*_{\gamma^-}\). Therefore, \(A = A_\gamma\), where \(A_\gamma\) is defined by (2.14) and \(\sigma(A_\gamma) = \mathbb{C}\).

2.5. THE RESOLVENT FORMULA

Let \(\gamma(\mu) = (\Gamma_0 [\ker(S^* - \mu I)]^{-1})^{-1}\) be the \(\gamma\)-field corresponding to the boundary triplet \((\mathcal{H}, \Gamma_0, \Gamma_1)\) of \(S^*\) with properties (2.26). Since \(J_{\tilde{\alpha}}\) maps \(\ker(S^* - \mu I)\) onto \(\ker(S^* - \mu I)\), formula (2.29) implies

\[
\gamma(\mu) J_{\tilde{\alpha}} = J_{\tilde{\alpha}} \gamma(\mu), \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}
\]

for an arbitrary fundamental symmetry \(J_{\tilde{\alpha}} \in C\mathcal{L}_2(J, R)\).

Let \(A_0 = S^* \upharpoonright \ker \Gamma_0\). Then \(A_0\) is a self-adjoint extension of \(S\) (due to the general properties of boundary triplets [9]). Moreover, it follows from (2.7) and Remark 2.13 that \(A_0 \in \Upsilon\).

**Proposition 2.20.** Let \((\mathcal{H}, \Gamma_0, \Gamma_1)\) be a boundary triplet of \(S^*\) with properties (2.26) and let \(A \in \Sigma_{J_{\tilde{\alpha}}}\) be defined by (2.31). Assume that \(A\) is disjoint with \(A_0\) (i.e., \(\mathcal{D}(A) \cap \mathcal{D}(A_0) = \mathcal{D}(S)\)) and \(\mu \in \rho(A) \cap \rho(A_0)\), then

\[
(A - \mu I)^{-1} = (A_0 - \mu I)^{-1} - \gamma(\mu)[M(\mu) - T]^{-1}\gamma^*(\overline{\mu}),
\]

(2.41)

where \(T = i J_{\tilde{\alpha}}(I + U)(I - U)^{-1}\) is a \(J_{\tilde{\alpha}}\)-self-adjoint operator in the Krein space \((\mathcal{H}, \lbrack \cdot, \cdot \rbrack_{J_{\tilde{\alpha}}} )\).

**Proof.** Since \(A\) and \(A_0\) are disjoint, the unitary operator \(U\) which corresponds to the operator \(A \in \Sigma_{J_{\tilde{\alpha}}}\) in (2.31) satisfies the relation \(\ker(I - U) = \{0\}\). This relation and (2.29) allow one to rewrite (2.31) as follows:

\[
A = S^* \upharpoonright \{f \in \mathcal{D}(S^*) \mid TT_0 f = \Gamma_1 f\},
\]

(2.42)

where \(T = i J_{\tilde{\alpha}}(I + U)(I - U)^{-1}\) is a \(J_{\tilde{\alpha}}\)-self-adjoint operator in the Krein space \((\mathcal{H}, \lbrack \cdot, \cdot \rbrack_{J_{\tilde{\alpha}}} )\) (due to self-adjointness of \(i(I + U)(I - U)^{-1}\)). Repeating the standard arguments (see, e.g., [8, p.14]), we deduce (2.41) from (2.42).

**Remark 2.21.** The condition of disjointness of \(A\) and \(A_0\) in Proposition 2.20 is not essential and it is assumed for simplifying the exposition. In particular, this allows one to avoid operators \(A\) with empty resolvent set (see Corollary 2.19 and relation
(2.37)) for which the formula (2.41) has no sense. In the case of an arbitrary $A \in \Sigma_{\mathcal{J}_{\vec{\alpha}}}$ with non-empty resolvent set, the formula (2.41) also remains true if we interpret $T$ as a $\mathcal{J}_{\vec{\alpha}}$-self-adjoint relation in $\mathcal{H}$ (see [12, Theorem 3.22] for a similar result and [6] for the basic definitions of linear relations theory).

3. THE CASE OF DEFICIENCY INDICES $\langle 2,2 \rangle$

In what follows, the symmetric operator $S$ has deficiency indices $\langle 2,2 \rangle$.

1. Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of $S^*$ with properties (2.26) or, that is equivalent, with properties (2.29). Let us fix an arbitrary fundamental symmetry $\mathcal{J}_{\vec{\gamma}} \in \mathcal{C}l_2(\mathcal{J}, \mathcal{R})$ and consider the decomposition $\mathcal{H} = \mathcal{H}_{\gamma}^+ \oplus \mathcal{H}_{\gamma}^-$ constructed by $\mathcal{J}_{\vec{\gamma}}$ (see (2.39)). Then the Weyl function $M(\cdot)$ associated with $(\mathcal{H}, \Gamma_0, \Gamma_1)$ can be rewritten as

$$M(\cdot) = \begin{pmatrix} m_{++}(\cdot) & m_{+-}(\cdot) \\ m_{-+}(\cdot) & m_{--}(\cdot) \end{pmatrix}, \quad m_{xy}(\cdot) : \mathcal{H}_{\gamma}^y \rightarrow \mathcal{H}_{\gamma}^x, \quad x,y \in \{+, -, \},$$

where $m_{xy}(\cdot)$ are scalar functions (since $\dim \mathcal{H} = 2$ and $\dim \mathcal{H}_{\gamma}^\pm = 1$).

According to Lemma 2.14, $M(\cdot)$ commutes with every fundamental symmetry from $\mathcal{C}l_2(\mathcal{J}, \mathcal{R})$. In particular, $\sigma_j M(\cdot) = M(\cdot) \sigma_j$ ($j = 1,3$), where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices. This is possible only in the case

$$m_{+-}(\cdot) = m_{--}(\cdot) = 0, \quad m_{++}(\cdot) = m_{-+}(\cdot),$$

i.e.,

$$M(\cdot) = m(\cdot) E,$$  \hfill (3.1)

where $m(\cdot) = m_{++}(\cdot) = m_{-+}(\cdot)$ is a scalar function defined on $\mathbb{C} \setminus \mathbb{R}$ and $E$ is the identity $2 \times 2$-matrix.

Recalling that $(\mathcal{H}_{\gamma}^+, \Gamma_0, \Gamma_1)$ is a boundary triplet of $S_{\gamma+}^*$ (see the proof of Corollary 2.19) and taking into account the definition (2.30) of Weyl functions, we arrive at the conclusion that $m(\cdot)$ is the Weyl function of $S_{\gamma+} = S \upharpoonright \mathcal{H}_{\gamma+}^\pm$ associated with boundary triplet $(\mathcal{H}_{\gamma}^+, \Gamma_0, \Gamma_1)$.

The following statement is proved.

**Proposition 3.1.** Let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of $S$ defined above. Then the Weyl function $M(\cdot)$ is defined by (3.1), where $m(\cdot)$ is the Weyl function of $S_{\gamma+}$ associated with boundary triplet $(\mathcal{H}_{\gamma}^+, \Gamma_0, \Gamma_1)$. The function $m(\cdot)$ does not depend on the choice of $\vec{\gamma} \in \mathbb{S}^2$.

2. Let $\vec{\alpha}, \vec{\beta} \in \mathbb{S}^2$ be linearly independent vectors. According to Corollary 2.7 all operators $A \in \Sigma_{\mathcal{J}_{\vec{\alpha}}} \cap \Sigma_{\mathcal{J}_{\vec{\beta}}}$ are described by the formula (2.21). This means that spectra of these operators are completely characterized by the spectra of their counterparts $A_+$ in (2.21).
The operator $A_+$ is supposed to be an intermediate extension of $S_{\gamma+}$. Two different situations may occur: 1. $A_+ = S_{\gamma+}$ or $A_+ = S_{\gamma+}^*$; 2. $A_+$ is a quasi-self-adjoint extension\(^4\) of $S$, i.e., $S_{\gamma+} \subset A_+ \subset S_{\gamma+}^*$. In the first case, the operators $A \in \Sigma_{J_{\alpha}} \cap \Sigma_{J_{\beta}}$ have empty resolvent set (Theorem 2.5); in the second case, the spectral properties of $A_+$ (and hence, $A$) are well known (see, e.g., [3, Theorem 1, Appendix I]). Summing up, we arrive at the following conclusion.

**Proposition 3.2.** Let $S$ be a simple symmetric operator with deficiency indices $(2, 2)$ and $A \in \Sigma_{J_{\alpha}} \cap \Sigma_{J_{\beta}}$. Then or $\sigma(A) = \mathbb{C}$ or the spectrum of $A$ consists of the spectral kernel of $S$ and the set of eigenvalues which can have only real accommodation points.

3. Denote by $\Xi_\alpha$ the collection of all operators $A \in \Sigma_{J_{\alpha}}$ with empty resolvent set:

$$\Xi_\alpha = \{A \in \Sigma_{J_{\alpha}} : \rho(A) = \emptyset \}$$

and by $\Xi_{\alpha, \beta}$ the pair of two operators $A_{\gamma(\beta)}$ and $A_{\gamma(\beta)}^*$ with empty resolvent set which are defined by (2.14) for a fixed $\alpha$ and $\beta$.

**Theorem 3.3.** Assume that $S$ is a symmetric operator with deficiency indices $(2, 2)$ and its Weyl function (associated with an arbitrary boundary triplet) differs from constant on $\mathbb{C} \setminus \mathbb{R}$. Then

$$\Xi_\alpha = \bigcup_{\forall \beta \in \mathbb{S}^2, \alpha \cdot \beta = 0} \Xi_{\alpha, \beta}.$$  \hspace{1cm} (3.2)

**Proof.** By Theorem 2.5, $\Xi_\alpha \supset \Xi_{\alpha, \beta}$ for all $\beta \in \mathbb{S}^2$ such that $\alpha \cdot \beta = 0$. Therefore, $\Xi_\alpha \supset \bigcup \Xi_{\alpha, \beta}$.

In the case of deficiency indices $(2, 2)$ of $S$, the set $\Xi_\alpha$ of all $J_{\alpha}$-self-adjoint extensions with empty resolvent set is described in [15]. We briefly outline the principal results.

Denote by $\mathfrak{N}_\mu = \ker(S^* - \mu I)$, $\mu \in \mathbb{C} \setminus \mathbb{R}$, the defect subspaces of $S$ and consider the Hilbert space $\mathfrak{M} = \mathfrak{N}_+ + \mathfrak{N}_-$ with the inner product

$$(x, y)_{\mathfrak{M}} = 2[(x_i, y_i) + (x_{-i}, y_{-i})],$$

where $x = x_i + x_{-i}$ and $y = y_i + y_{-i}$ with $x_i, y_i \in \mathfrak{N}_i$, $x_{-i}, y_{-i} \in \mathfrak{N}_{-i}$.

The operator $Z$ that acts as identity operator $I$ on $\mathfrak{N}_i$ and minus identity operator $-I$ on $\mathfrak{N}_{-i}$ is an example of fundamental symmetry in $\mathfrak{M}$. Other examples can be constructed due to the fact that $S$ commutes with $J_{\beta}$ for all $\beta \in \mathbb{S}^2$. This means that the subspaces $\mathfrak{N}_{\pm i}$ reduce $J_{\beta}$ and the restriction $J_{\beta} | \mathfrak{M}$ gives rise to a fundamental symmetry in the Hilbert space $\mathfrak{M}$. Moreover, according to the properties of $Z$ mentioned above, $J_{\beta}^* Z = Z J_{\beta}$ and $J_{\beta} Z$ is a fundamental symmetry in $\mathfrak{M}$. Therefore, the sesquilinear form

$$[x, y]_{J_{\beta} Z} = (J_{\beta} Z x, y)_{\mathfrak{M}} = 2[(J_{\beta} x_i, y_i) - (J_{\beta} x_{-i}, y_{-i})]$$

defines an indefinite metric on $\mathfrak{M}$.

\(^4\) This class includes self-adjoint extensions also.
According to the von-Neumann formulas, any closed intermediate extension $A$ of $S$ (i.e., $S \subseteq A \subseteq S^*$) is uniquely determined by the choice of a subspace $M \subseteq \mathfrak{M}$:

$$A = S^* \upharpoonright \mathcal{D}(A), \quad \mathcal{D}(A) = \mathcal{D}(S) + M. \quad (3.3)$$

In particular, $J_{\vec{\beta}}$-self-adjoint extensions $A$ of $S$ correspond to hypermaximal neutral subspaces $M$ with respect to $[\cdot, \cdot]_{J_{\vec{\beta}}Z}$. This means that $A \in \Sigma_{J_{\vec{\alpha}}} \cap \Sigma_{J_{\vec{\beta}}}$ if and only if the corresponding subspace $M$ in (3.3) is simultaneously hypermaximal neutral with respect to two different indefinite metrics $[\cdot, \cdot]_{J_{\vec{\alpha}}Z}$ and $[\cdot, \cdot]_{J_{\vec{\beta}}Z}$.

Without loss of generality we assume that $J_{\vec{\alpha}}$ coincides with $J$ in (2.3), i.e., $\vec{\alpha} = (1, 0, 0)$. Then fundamental symmetries $J_{\vec{\beta}}$ which anti-commute with $J$ have the form

$$J_{\vec{\beta}} = \beta_2 R + \beta_3 iJR, \quad \beta_2^2 + \beta_3^2 = 1. \quad (3.4)$$

To specify $M$ we consider an orthonormal basis $\{e_{++}, e_{+-}, e_{-+}, e_{--}\}$ of $\mathfrak{M}$ which satisfies the relations

$$Ze_{++} = e_{++}, \quad Ze_{+-} = e_{+-}, \quad Ze_{-+} = -e_{-+}, \quad Ze_{--} = -e_{--},$$

$$J_{\vec{\alpha}}e_{++} = e_{++}, \quad J_{\vec{\alpha}}e_{+-} = -e_{--}, \quad J_{\vec{\alpha}}e_{-+} = e_{-+}, \quad J_{\vec{\alpha}}e_{--} = -e_{--}, \quad (3.5)$$

$$Re_{++} = e_{++}, \quad Re_{+-} = e_{++}, \quad Re_{-+} = e_{-+}, \quad Re_{--} = e_{--}.$$

The existence of this basis was established in [2] and it was used in [15, Corollary 3.2] to describe the collection of all $M$ in (3.3) which correspond to $J$-self-adjoint extensions of $S$ with empty resolvent set. Such a description depends on properties of Weyl function of $S$. In particular, if the Weyl function differs from the constant for a fixed boundary triplet, then this property remains true for Weyl functions associated with an arbitrary boundary triplet of $S$. Then, using relations (2.7)–(2.9) in [15], we deduce that the Straus characteristic function of $S$ (see [18]) differs from the zero-function on $\mathbb{C} \setminus \mathbb{R}$. In this case, Corollary 3.2 in [15] says that a $J$-self-adjoint extension $A$ has empty resolvent set if and only if the corresponding subspace $M$ coincides with linear span $M = \text{span}\{d_1, d_2\}$, where $d_1 = e_{++} + e^\gamma e_{-+}$, $d_2 = e_{--} + e^{-i\gamma} e_{-+}$, and $\gamma \in [0, 2\pi)$ is an arbitrary parameter.

The operator $A$ will belong to $\Sigma_{J_{\vec{\alpha}}}$ if and only if the subspace $M = \text{span}\{d_1, d_2\}$ turns out to be hypermaximal neutral with respect to $[\cdot, \cdot]_{J_{\vec{\alpha}}Z}$. Since $\dim \mathfrak{M} = 2$ and $\dim \mathfrak{M} = 4$, it suffices to check the neutrality of $M$. The last condition is equivalent to the relations

$$[d_1, d_2]_{J_{\vec{\alpha}}Z} = 0, \quad [d_1, d_1]_{J_{\vec{\alpha}}Z} = 0, \quad [d_2, d_2]_{J_{\vec{\alpha}}Z} = 0.$$

Using (3.4), (3.5), and remembering the orthogonality of $e_{\pm, \pm}$ in $\mathfrak{M}$, we establish that $[d_1, d_2]_{J_{\vec{\beta}}Z} = 0$ for all $\gamma \in [0, 2\pi)$. The next two conditions are transformed to the linear equation

$$(\cos \gamma) \beta_2 - (\sin \gamma) \beta_3 = 0, \quad (3.6)$$

which has the nontrivial solution $\beta_2 = \sin \gamma$, $\beta_3 = \cos \gamma$ for any $\gamma \in [0, 2\pi)$. This means that an arbitrary $J$-self-adjoint extension $A$ with empty resolvent set is also a $J_{\vec{\beta}}$-self-adjoint operator under choosing $\beta_2$ and $\beta_3$ in (3.4) as solutions of (3.6). Theorem 3.3 is proved. □
Corollary 3.4. Let $S$ be a symmetric operator with deficiency indices $(2, 2)$ and let $(\mathcal{H}, \Gamma_0, \Gamma_1)$ be a boundary triplet of $S^*$ with properties (2.26). If the Weyl function of $S$ differs from constant on $\mathbb{C} \setminus \mathbb{R}$, then the set $\Xi_{\vec{\alpha}}$ is described by (2.31) where $U$ runs the set of all fundamental symmetries $\mathcal{F}_\beta \subset \mathcal{Cl}_2(\mathcal{J}, \mathcal{R})$ such that $\vec{\alpha} \cdot \vec{\beta} = 0$.

Proof. It follows from Corollary 2.19 and Theorem 3.3. 

Theorem 3.3 and Corollary 3.4 are not true when the Weyl function of $S$ is a constant. In that case, the set $\Xi_{\vec{\alpha}}$ of $J_{\vec{\alpha}}$-self-adjoint extensions increases considerably and $\Xi_{\vec{\alpha}} \supset \bigcup \Xi_{\vec{\alpha}, \vec{\beta}}$.

Corollary 3.5. Let $S$ be a simple symmetric operator with deficiency indices $(2, 2)$. Then the following statements are equivalent:

(i) the strict inclusion

\[
\Xi_{\vec{\alpha}} \supset \bigcup_{\vec{\beta} \in \mathbb{S}^2, \vec{\alpha} \cdot \vec{\beta} = 0} \Xi_{\vec{\alpha}, \vec{\beta}}
\]

holds,

(ii) the Weyl function $M(\cdot)$ of $S$ is a constant on $\mathbb{C} \setminus \mathbb{R}$,

(iii) $S$ is unitarily equivalent to the symmetric operator in $L_2(\mathbb{R}, \mathbb{C}^2)$:

\[
S' = i \frac{d}{dx}, \quad \mathcal{D}(S') = \{u \in W_2^1(\mathbb{R}, \mathbb{C}^2) : u(0) = 0\}. \tag{3.7}
\]

Proof. Assume that the Weyl function $M(\cdot)$ of $S$ is a constant. By (3.1), the Weyl function $m(\cdot)$ of $S_{\gamma^+} = S \mid_{\mathcal{D}_\gamma}$ is also constant. This means that the Straus characteristic function of the simple symmetric operator $S_{\gamma^+}$ with deficiency indices $(1, 1)$ is zero on $\mathbb{C} \setminus \mathbb{R}$ (see the proof of Theorem 3.3). Therefore, $S_{\gamma^+}$ is unitarily equivalent to the symmetric operator $S'_+ = i \frac{d}{dx}$, $\mathcal{D}(S'_+) = \{u \in W_2^1(\mathbb{R}) : u(0) = 0\}$ in $L_2(\mathbb{R})$ [16, Subsection 3.4].

Recalling the decomposition (2.13) of $S$, where the simple symmetric operator $S_{\gamma^-} = S \mid_{\mathcal{D}_\gamma}$ also has deficiency indices $(1, 1)$ and zero characteristic function, we conclude that $S$ is unitarily equivalent to the symmetric operator $S'$ defined by (3.7). This establishes the equivalence of (ii) and (iii).

Assume again that the Weyl function of $S$ is a constant. Then the Straus characteristic function of $S$ is zero. In that case, Corollary 3.2 in [15] yields that $A \in \Xi_{\vec{\alpha}}$ if and only if the corresponding subspace $M$ in (3.3) coincides with linear span

\[
M = \text{span}\{d_1, d_2\}, \quad d_1 = e_{++} + e^{i(\phi + \gamma)}e_{+-}, \quad d_2 = e_{--} + e^{i(\phi - \gamma)}e_{-+}, \tag{3.8}
\]

where $\phi, \gamma \in [0, 2\pi)$ are two arbitrary parameters. Thus the set $\Xi_{\vec{\alpha}}$ is described by two independent parameters $\phi$ and $\gamma$.

Due to the proof of Theorem 3.3, the operator $A \in \Xi_{\vec{\alpha}}$ belongs to the subset $\Xi_{\vec{\alpha}, \vec{\beta}}$ if and only if the subspace $M$ in (3.8) is neutral with respect to $[\cdot, \cdot]_{J_{\vec{\beta}}\mathbb{Z}}$. Repeating the argumentation above, we conclude that the neutrality of $M$ is equivalent to the existence of nontrivial solution $\beta_2, \beta_3$ of the system (cf. (3.6))

\[
\begin{align*}
\cos(\phi + \gamma)\beta_2 - \sin(\phi + \gamma)\beta_3 &= 0, \\
\cos(\phi - \gamma)\beta_2 + \sin(\phi - \gamma)\beta_3 &= 0.
\end{align*} \tag{3.9}
\]
The determinant of (3.9) is \( \sin 2\phi \). Therefore, there are no nontrivial solutions for \( \phi \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\} \). This means the existence of operators \( A \in \Xi_\alpha \) which, simultaneously, do not belong to \( \bigcup \Xi_{\alpha,\beta} \). Thus, we establish the equivalence of (ii) and (i). Corollary 3.5 is proved.

4. Consider the one-dimensional Schrödinger differential expression

\[
l(\phi)(x) = -\phi''(x) + q(x)\phi(x), \quad x \in \mathbb{R},
\]

where \( q \) is an even real-valued measurable function that has a non-integrable singularity at zero and is integrable on every finite subinterval of \( \mathbb{R} \setminus \{0\} \).

Assume in what follows that the potential \( q(x) \) is in the limit point case at \( x \to \pm\infty \) and is in the limit-circle case at \( x = 0 \). Denote by \( \mathcal{D} \) the set of all functions \( \phi(x) \in L_2(\mathbb{R}) \) such that \( \phi \) and \( \phi' \) are absolutely continuous on every finite subinterval of \( \mathbb{R} \setminus \{0\} \) and \( l(\phi) \in L_2(\mathbb{R}) \). On \( \mathcal{D} \) we define the operator \( L \) as follows:

\[
L \phi = l(\phi), \quad \forall \phi \in \mathcal{D}.
\]

The operator \( L \) commutes with the space parity operator \( \mathcal{P} \phi(x) = \phi(-x) \) and with the operator of multiplication by \( (\text{sgn } x)I \). These operators are anti-commuting fundamental symmetries in \( L_2(\mathbb{R}) \). Therefore, \( L \) commutes with elements of the Clifford algebra \( Cl_2(\mathcal{P}, (\text{sgn } x)I) \). However, \( L \) is not a symmetric operator.

Denote for brevity \( J_{\gamma} = (\text{sgn } x)I \). Then, the decomposition (2.12) takes the form

\[
L_2(\mathbb{R}) = L_2(\mathbb{R}^+) \oplus L_2(\mathbb{R}^-)
\]

and with respect to it

\[
L = \begin{pmatrix} L_+ & 0 \\ 0 & \mathcal{P}L_+\mathcal{P} \end{pmatrix}, \quad L_+ = L \mid_{L_2(\mathbb{R}^+)}.
\]

The operator \( L_+ \) is the maximal operator for differential expression \( l(\phi) \) considered on semi-axes \( \mathbb{R}^+ = (0, \infty) \). Denote by \( S_+ \) the minimal operator generated \( l(\phi) \) in \( L_2(\mathbb{R}^+) \). The symmetric operator \( S_+ \) has deficiency indices \( (1,1) \).

Let \( (\mathbb{C}, \Gamma_0^+, \Gamma_1^+) \) be an arbitrary boundary triplet of \( L_+ = S_+^* \) in \( L_2(\mathbb{R}^+) \). Then, the boundary triplet \( (\mathbb{C}, \Gamma_0, \Gamma_1) \) determined by (2.27) with \( J_{\gamma} = \mathcal{P} \) and \( N = \mathbb{C} \) is a boundary triplet of \( L = S^* \) in the space \( L_2(\mathbb{R}) \). Here \( S = \begin{pmatrix} S_+ & 0 \\ 0 & \mathcal{P}S_+\mathcal{P} \end{pmatrix} \) is the symmetric operator in \( L_2(\mathbb{R}) \) (cf. (3.11)) with deficiency indices \( (2,2) \).

Let \( J_{\bar{\alpha}} \) be an arbitrary fundamental symmetry from \( Cl_2(\mathcal{P}, (\text{sgn } x)I) \). By Theorem 2.15, \( J_{\bar{\alpha}} \)-self-adjoint extensions \( A \in \Sigma_{J_{\bar{\alpha}}} \) of \( S \) are defined as the restrictions of \( L \):

\[
A = L \mid \{ f \in \mathcal{D} : U(J_{\bar{\alpha}}\Gamma_1 + i\Gamma_0)f = (J_{\bar{\alpha}}\Gamma_1 - i\Gamma_0)f \},
\]

where \( U \) runs the set of \( 2 \times 2 \)-unitary matrices. The operators \( A \) can be interpreted as \( J_{\bar{\alpha}} \)-self-adjoint operator realizations of differential expression (3.10) in \( L_2(\mathbb{R}) \).

Since the sets \( \Sigma_{J_{\bar{\alpha}}} \) are unitarily equivalent for different \( \bar{\alpha} \in \mathbb{S}^2 \) (Theorem 2.9) one can set \( J_{\bar{\alpha}} = \mathcal{P} \) for definiteness.
Proposition 3.6. The collection of all $\cal P$-self-adjoint extensions $A \in \Sigma_\cal P$ with empty resolvent set coincides with the restrictions of $L$ onto the sets of functions $f \in \cal D$ satisfying the condition
\[
\begin{pmatrix}
  i \sin \theta & 1 - \cos \theta \\
  1 + \cos \theta & -i \sin \theta
\end{pmatrix}
\Gamma_1 f = i
\begin{pmatrix}
  1 + \cos \theta & -i \sin \theta \\
  i \sin \theta & 1 - \cos \theta
\end{pmatrix}
\Gamma_0 f,
\forall \theta \in [0, 2\pi).
\]

Proof. Since $J_\vec{\tau} = \cal P$ and $J_\vec{\gamma} = (\text{sgn } x)I$, relations (2.26), (2.28) mean that $\sigma_1 \Gamma_j = \Gamma_j \cal P$ and $\sigma_3 \Gamma_j = \Gamma_j (\text{sgn } x)I$ ($j = 0, 1$), where $\sigma_1$ and $\sigma_3$ are Pauli matrices. Therefore, the ‘image’ of the Clifford algebra $\text{Cl}_2(\cal P, (\text{sgn } x)I)$ coincides with $\text{Cl}_2(\sigma_1, \sigma_3)$ in $\mathbb{C}^2$ (see Remark 2.13).

The fundamental symmetries $J_\vec{\beta} \in \text{Cl}_2(\sigma_1, \sigma_3)$ anti-commuting with $\sigma_1$ have the form $J_\vec{\beta} = \beta_2 \sigma_2 + \beta_3 \sigma_3$, $\beta_2^2 + \beta_3^2 = 1$, where $\sigma_2 = i \sigma_1 \sigma_3$. Hence,
\[
J_\vec{\beta} = \begin{pmatrix}
  \beta_3 & -i \beta_2 \\
  i \beta_2 & -\beta_3
\end{pmatrix} = \begin{pmatrix}
  \cos \theta & -i \sin \theta \\
  i \sin \theta & -\cos \theta
\end{pmatrix}, \quad \theta \in [0, 2\pi).
\]

Here, we set $\beta_3 = \cos \theta$ and $\beta_2 = \sin \theta$ (since $\beta_2^2 + \beta_3^2 = 1$). Applying Corollary 3.4 and rewriting (2.31) in the form (2.37) with $J_\vec{\alpha} = \sigma_1$, $J_\vec{\gamma} = J_\vec{\beta}$ (here $J_\vec{\beta}$ is determined by (3.13)), we complete the proof of Proposition 3.6.

Remark 3.7. To apply Proposition 3.6 for concrete potentials $q(x)$ in (3.13) one needs only to construct a boundary triplet $(\mathbb{C}^2, \Gamma_0, \Gamma_1)$ of $L$ with the help of a boundary triplet $(\mathbb{C}, \Gamma_0^+, \Gamma_1^+)$ of the differential expression (3.10) on semi-axis $\mathbb{R}_+$ (see (2.27)). To do that one can use [14], where simple explicit formulas for operators $\Gamma_j^+$ constructed in terms of asymptotic behavior of $q(x)$ as $x \to 0$ were obtained for great number of singular potentials.

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