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MODELLING OF A PROBLEM OF PHASE TRANSITIONS AT NOT ISOTHERMAL FILTRATION AND QUALITATIVE PROPERTIES OF THE DECISION

1. STATEMENT OF A PROBLEM

Organic laws of preservation of mechanics of the continuous environment can be written down as system of the divergent equations:

$$\frac{\partial F}{\partial t} + div\left(F \cdot v - G\right) = X \qquad (1.1)$$

In particular, in the law of conservation of energy

$$F = \rho U,$$

where:

- ρ – density of environment;
- U – specific internal energy;
- G = $-q$ where $q = -\chi \nabla \theta$ – a vector of a stream;
- G = $P : D$ (double convolution tensor's P with tensor D), where P – tensor of pressure, and D – tensor of speeds of deformation.

It appears, limiting values of the specified functions on a surface Γ_γ (on a surface of strong explosion) not any, and satisfies to system of the equations on "strong" break

$$\left[F \cdot \left(v \cdot \upsilon - V_\upsilon\right) - G \cdot \upsilon\right] = 0 \qquad (1.2)$$

where:

- V_υ – speed of moving of section,
- $\Gamma(t)$ – a hypersurface Γ_γ a plane (t = const) in a direction of a normal υ to this section.

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The equations (1.2) follow from the equations (1) and assumptions of structure of movement. Really, each equation (1) is equivalent to integrated identity:

$$\iint\left(\left(G-F\cdot v\right)\cdot\nabla\varphi-F\cdot\frac{\partial\varphi}{\partial t}-X\cdot\varphi\right)dxdt=0 \tag{1.3}$$

With any smooth finite in Ω_γ function φ. Let Q – any area containing, Γ_γ and:

$$Q=Q^{+}\cup Q^{-}\cup\sigma \tag{1.4}$$

where: σ – a part of surface Γ, and in Q^+ and Q^- functions F, G, v are continuously differentiated and satisfy to the equation (1.1) in usual sense. The overall objective to find out, whether the generalized movements with a transitive phase, for problem of Stephen with convective heat transfer, a nonzero measure borrowing set are possible.

It appears, that such variants are possible. Only the conclusion of corresponding mathematical model from (1.1)–(1.3) below is given, and its correctness in work completely is not resulted. Then qualitative properties of decisions are considered.

Let's take advantage of the above-stated designations and from (1.1) we shall receive the following equation:

$$\frac{\partial U}{\partial t}+\vec{u}\cdot\nabla U=div(\chi\cdot\nabla\theta)+f(x,t) \tag{1.5}$$

For simplicity we shall be limited to a case of one spatial variable.

So, let the area $\Omega_\gamma^-=\left(\left(x,t\right)\big|_+ -1<x<R^-\left(t\right),0<t<T\right)$ is borrowed with a firm phase, and area $\Omega_\gamma^+=\left(\left(x,t\right)\big|_+ R^+\left(t\right)<x<1,0<t<T\right)$ – a liquid phase, and area $\Omega_T^+=\left(\left(x,t\right)\big|_+ R^-\left(t\right)<x<R^+,0<t<T\right)$ – a transitive phase. Functions $R^-\left(t\right)$, $R^+\left(t\right)$ are assumed continuously differentiated. By definition of movement with strong break everywhere outside of lines $x=R^-(t)$ and $x=R^+(t)$ movement continuous, i.e. in areas Ω_T^- and Ω_T^+ the equation (1.5) is equivalent to the equation

$$\frac{\partial\theta}{\partial t}+\vec{u}\cdot\nabla\theta=div\left(\chi\cdot\nabla\right)+f\left(x,t\right) \tag{1.6}$$

And in area Ω_T^* – to the equation generally (i.e. at $n>1$)

$$\frac{\partial U}{\partial t}+\vec{u}\cdot\nabla U=f\left(x,t\right) \tag{1.7}$$

And functions θ and U in the corresponding closed areas possess the continuous derivatives which are included in the equation (1.6) and (1.7). We shall notice, that the aprioristic assumption of smoothness of functions θ and U in the specified areas none principal. This smoothness follows from the assumption of structure of the decision of the equation (1.5) and differential properties of decisions of the parabolic equations of the certain smoothness of functions χ, f. If to take into account, that speed of moving of a surface $x = R(t)$ is a derivative on time of functions $R(t)$, and that the limit of specific internal energy on the part of a liquid phase on a line $x = R^+(t)$ is equal to zero, and on the part of a firm phase on a line $x = R^-(t)$ – to size L we shall receive from (1.2) following parities{ratio}:

$$\left(U\left(R^-(t)+0,t\right)+L\right)\left(\frac{dR^-}{dt}-b^*\varphi\left(R^-(t),t\right)\right)=\chi_S\cdot\frac{\partial\theta}{\partial x}\left(R^-(t)-0,t\right) \tag{1.8}$$

$$\left(U\left(R^+(t)-0,t\right)+L\right)\left(\frac{dR^+}{dt}-b^*\varphi\left(R^+(t),t\right)\right)=\chi_L\cdot\frac{\partial\theta}{\partial x}\left(R^+(t)+0,t\right) \tag{1.9}$$

Where: $F=U, G=\chi\frac{\partial\theta}{\partial x}, \vec{u}\cdot\vec{v}=b^*\cdot\varphi(x,t)$ – the charge of a mix:

$$b^*={(a_ik_i)}\big/{(k_1+k_2)}, i=1,2.$$

Together with entry conditions on borders $X = \pm 1$ it is not enough these conditions for short circuit of mathematical model. Additional assumptions are necessary.

We research occurrence of a transitive phase, when

$$R^-(0)==R^+(0)^{x_0}\quad \text{And}\quad R^-(0)<R^+(0)\; at\; t>0 \tag{1.10}$$

It appears, in this case under certain conditions on the given problems the boundary condition (1.8) or (1.9) breaks up on two independent conditions, and a initial problem – to the consecutive decision of three independent problems.

Under the assumption the temperature θ nonpositive in area Ω_T also is equal to zero on its right border $x = R^+(t)$. Hence,

$$\frac{\partial\theta}{\partial x}\left(R^-(t)-0,t\right)\geq 0 \tag{1.11}$$

Similarly

$$\frac{\partial\theta}{\partial x}\left(R^+(t)+0,t\right)\geq 0 \tag{1.12}$$

In a transitive phase specific internal energy accepts values from an interval $(-L; 0)$, i.e.

$$\left(U\left(R^-(t)+0,t\right)+L\right)\geq 0,\quad \left(U\left(R^+(t)-0,t\right)+L\right)\leq 0 \tag{1.13}$$

Comparing (1.9) with (1.12) and (1.13), we see, that

$$\frac{dR^+}{dt}(t)\leq b^*\cdot\varphi\left(R^+(t),t\right) \tag{1.14}$$

Further, comparing (1.8) with (1.11) and (1.13), we see, that

$$\frac{dR^-}{dt}(t) \geq b^* \cdot \varphi\left(R^-(t),t\right) \tag{1.15}$$

Then from (1.10) follows, that at least on a small interval of time

$$b^* \cdot \varphi\left(R^-(t),t\right) \leq \frac{dR^-}{dt} < \frac{dR^+}{dt} \leq b^* \cdot \varphi\left(R^+(t),t\right) \tag{1.16}$$

Let's assume the opposite, let (coincides with physical statement)

$$\frac{dR^-}{dt} < b^* \cdot \varphi\left(R^-(t),t\right) \leq b^* \cdot \varphi\left(R^+(t),t\right) < \frac{dR^+}{dt} \tag{1.17}$$

Valid (1.8), (1.11) and (1.13) last inequality is possible, if only

$$\left(U\left(R^-(t)+0,t\right)+L\right)=0, \quad \frac{\partial\theta}{\partial x}\left(R^-(t)-0,t\right)=0 \tag{1.18}$$

Besides from (1.9), (1.12) and (1.13) inequality (1.17) it is fair only at

$$U\left(R^+(t)-0,t\right)=0, \quad \frac{\partial\theta}{\partial x}\left(R^+(t)+0,t\right)=0 \tag{1.19}$$

To find out, when what variant is realized, differentiated equality $\theta\left(R^-(t)-0,t\right)=0$ on time:

$$\frac{\partial\theta}{\partial x}\left(R^-(t)-0,t\right) \cdot \frac{dR^-}{dt}(t) + \frac{\partial\theta}{\partial t}\left(R^-(t)-0,t\right)=0$$

If parities{ratio} (1.18) from the second equation (1.18), last equality and the equation (1.6) follows, that are carried out

$$\frac{\partial^2\theta}{\partial x^2}\left(R^-(t)-0,t\right) = -\frac{1}{\chi} f\left(R^-(t),t\right)$$

For function θ(x, t) at everyone fixed t at $x < R^-(t)$ fair an inequality

$$\theta(x,t) = \frac{\partial^2\theta}{\partial x^2}\left(R^-(t)-0,t\right) \cdot \frac{\left|x-R(t)\right|^2}{2} + O\left(\left|x-R(t)\right|^2\right)$$

Taking into account last two ratio and strict negativity of temperature at $x < R^-(t)$, we see, that first variant (eq. (1.18)) it is realized at $f(x_0, 0) > 0$.

In the above-stated statement theorems of existence and uniqueness are received, and also qualitative properties of decisions are investigated. With the help of automodelling decisions positions of free border and its behaviors are investigated at unlimited increase of time.

2. ASYMPTOTIC BEHAVIOR ON TIME OF THE DECISION OF A HOMOGENEOUS PROBLEM SUCH AS STEPHEN

Let during the initial moment of time a liquid phase (i.e. water in an oil collector) occupies area $\Omega^{+}(0)=\{x|0<x<x_0\}$, and a firm phase (i.e. it is conditional – *парафинированная oil*) – area $\Omega^{-}(0)=\{x|x_0<x<\infty\}$. Thus it is supposed, that the firm phase is at the temperature equal to temperature of fusion, i.e. everywhere in $\Omega^{-}(0)$ the temperature θ_0 is equal to zero, and specific internal energy U_0 is equal a minus to unit.

Then in area $\Omega_T^{+}=\left((x,t)\ \middle|\ 0<x<R(t)\ ,0<t<T\right)$ the temperature $\theta(x,t)$ satisfies to the equation:

$$\frac{\partial}{\partial t}\Phi(\theta)+v\frac{\partial}{\partial x}\Phi(\theta)=\frac{\partial^2\theta}{\partial x^2},\ (x,t)\in\Omega_T^{+} \tag{1.20}$$

And to two conditions on free border:

$$\theta\big|_{x=R(t)}=0,\ \frac{dR}{dt}=-\frac{\partial\theta}{\partial x}\bigg|_{x=R(t)},\quad t\in(0,T) \tag{1.21}$$

$$\theta\big|_{x=0}=\theta^{0}(t),\ \text{либо}\ \left(\frac{\partial\theta}{\partial x}+в(t)\cdot\theta\right)\Big|_{x=0}=\theta'(t) \tag{1.22}$$

At the initial moment of time it is conditional

$$R(0)=x_0,\ \theta(x.0)=\theta_0(x),\quad x\in\Omega^{+}(0) \tag{1.23}$$

Theorem 1. Let $\Phi\in C^2[0,\beta],\ \Phi'(s)>0$ at $s\in[0,\beta]$. Then the problem (1.20)–(1.23) with the data $\theta_0(t)=\text{const}>0, x_0=0, U_0(x)=-1$ at $x\in(0,\infty)$ has the unique decision of a kind $R_0(t)=D_*(\beta)(t+1)^{1/2}\,\theta_*(x,t)=u\left(x(t+1)^{-1/2},\beta\right)$, with function $D_*(\beta)$.

Continuously dependent on parameter β and such, that $\lim D_*(\beta)=0$ at $\beta\to 0$.

Proof. Function $U(\xi,\beta),\ \xi=\dfrac{x}{\sqrt{t+1}}$ and parameter $D_*=D_*(\beta)$ are subject to definition from conditions

$$u''+\left(\frac{\xi}{2}-V\cdot\sqrt{t+1}\right)\cdot a(u)\cdot u'=0,\ a(u)=\Phi'(\theta),\ \xi\in(0,D_*) \tag{1.24}$$

$$u(0,\beta)=\beta,\ \ u(D_*,\beta)=0 \tag{1.25}$$

$$\frac{du}{d\xi}(D_*,\beta)=-\frac{1}{2}\cdot D_* \tag{1.26}$$

Let's show, that for everyone $D > 0$ there will be even one function $V(\xi)$ satisfying the equation (1.24) and regional conditions (1.25). Further, calculating a derivative $dV/d\xi$ in a point $\xi = D$ and substituting it in the left part (1.26) we shall receive the equation which D_* decision determines the decision of a problem{task} (1.24)–(1.26).

For definition of function $V(\xi)$ we shall consider a linear boundary problem

$$\frac{d^2\tilde{V}}{d\xi^2}+\left[\frac{\xi}{2}-v\sqrt{t+1}\right]a\left(f\left(\varsigma\right)\right)\frac{d\tilde{V}}{d\xi}=0,\ \ \tilde{V}\left(0\right)=\beta,\ \ \tilde{V}\left(D\right)=0$$

Where argument in factor *and* is any non-negative function $f(\xi)$, continuous on an interval $(0, D)$ and limited there a constant β. The decision of last is given by the formula

$$\tilde{V}\left(\xi\right)=\beta\cdot\frac{\int\limits_{\xi}^{D}\exp\left\{-\int\limits_{0}^{\tau}\left[\frac{s}{2}-v\sqrt{t+1}\right]\cdot a\left(f\left(s\right)\right)ds\right\}d\tau}{\int\limits_{0}^{D}\exp\left\{-\int\limits_{0}^{\tau}\left[\frac{s}{2}-v\sqrt{t+1}\right]\cdot a\left(f\left(s\right)\right)ds\right\}d\tau} \tag{1.27}$$

The right part of last expression is the continuous operator $\Phi(f)$ determined on convex closed set

$$\mathrm{M}=\left\{\xi\in\left(0,D\right)\middle|\left\|f\left(\xi\right)\right\|_{C[0,D]}\le\beta\right\}\ \text{и}\ \ \Phi:\mathrm{M}\to\mathrm{M}$$

From that derivatives $\tilde{V}\left(\xi\right)$ are in regular intervals limited:

$$\left|\frac{d\tilde{V}}{d\xi}\right|\le 2\beta e^{kD_* v\sqrt{T+1}}\left(\int\limits_{0}^{D}\exp\left[\int\limits_{0}^{\tau}(v\sqrt{t+1}-\frac{s}{2})a\left(f\left(s\right)\right)ds\right]d\tau\right)^{-1}$$

Where $k=\max\limits_{[0,\beta]}a\left(f\left(s\right)\right)$, the operator $\Phi(f)$ quite continuous on set M. Under theorem of Shauder there will be even one motionless point V of the operator $\Phi:\ V=\Phi\left(V\right)$. Function $V(\xi)$ satisfies to the equation (1.24) and to conditions (1.25). The equation $\frac{dV\left(D\right)}{d\xi}=-\frac{D_*}{2}$ has even one decision $D_*>0$ as for $V(\xi)$ fair the representation similar (1.27), from which inequalities are easily deduced

$$-2\beta \cdot e^{a_0 D_* v\sqrt{T+1}}\left(\int_0^D \exp\left\{\int_0^\tau \left[v\sqrt{t+1}-\frac{s}{2}\right]\cdot a\left(f\left(s\right)\right)ds\right\}d\tau\right)^{-1} \le \frac{dV}{d\xi}(D) \le$$

$$\le 2\beta \cdot e^{kD_* v\sqrt{T+1}}\left(\int_0^D \exp\left\{\int_0^\tau \left[v\sqrt{t+1}-\frac{s}{2}\right]\cdot a\left(f\left(s\right)\right)ds\right\}d\tau\right)^{-1}.$$

Uniqueness of the found automodelling decision follows that function $U_*\left(x,t\right)$ equal $\Phi\left(\theta_*\left(x,t\right)\right)$ at $0<x<R\left(t\right)$ and the minus to unit at $x > R(t)$, is the unique limited generalized decision of Stephen problem with convective heat transfer.

With multiplication the equation (1.24) on ξ integration in parts in limits from 0 up to D^* with use of conditions (1.25) and (1.26) we shall receive

$$\frac{1}{2}D_*^2\left(\beta\right)+\int_0^{D_*(\beta)}\left(\xi-v\sqrt{t+1}\right)\Phi\left[u\left(\xi,\beta\right)\right]d\xi-\beta \quad (1.28)$$

Which equality confirms continuous dependence $D_*(\beta)$ on parameter β, i.e. consequence from the theorem of continuous dependence of decisions of the ordinary differential equations from parameter.

At research asymptotic behavior of the decision of an initial problem we shall be limited to a case, when

$$\lim_{t\to\infty}\theta^0\left(t\right)=\beta_1 \lim_{t\to\infty}v\left(t\right)=\beta_2, \quad 0<\beta_i<\infty, \quad i=1,2 \quad (1.29)$$

Theorem 2. Conditions of the Theorem 1 and conditions (1.29). Let are executed with $\beta > 0$. Then for the decision $R(t)$ of a problem{task} (1.20)–(1.23) equality is fair

$$\lim_{t\to\infty}\left(t+1\right)^{-1/2}R\left(t\right)=D_*\left(\beta\right).$$

where $D_*(\beta)$ – the decision of a problem{task} (1.24)–(1.26).

Proof. For an estimation of the decision $R(t)$ from below it is fixed any way positively small number ε and let t_0 so big, that

$$0<\beta-\varepsilon\le\theta^0\left(t\right)\le\beta+\varepsilon \quad \text{At} \quad t\in\left(t_0-1,\infty\right) \quad (1.30)$$

Function $\theta_*\left(x,t\right)=u\left(x\left(t+1-t_0\right),\beta-\varepsilon\right)$ defines the generalized decision $U_*\left(x,t\right)$ which does not surpass function $U^0\left(t\right)=\Phi\left[\theta^0\left(t\right)\right]$ on border $x = 0$ and the function $U_0\left(x\right)$, equal $\Phi\left[\theta\left(x,t_0\right)\right]$ at $0<x<R\left(t\right)$ and a minus to unit at $x>R\left(t\right), t>t_0$, is own decision of a problem{task} (1.20)–(1.23) and coincides with functions $U^0\left(t\right)$ and $U_0\left(x\right)$ accordingly at $x = 0$ and at $t = t_0$ it is possible to take advantage of the theorem of comparison of the generalized decisions of a problem (1.20)–(1.23), received by the author, and to approve, that

$$R(t)\geq(t+1-t_0)^{1/2}\cdot D_*(\beta-\varepsilon),\quad U(x,t)\geq U_*(x,t),\quad t>t_0-1 \tag{1.31}$$

For an estimation of function $R(t)$ from above we shall take advantage of identity

$$B(t)=B(t_0-1)+\int\limits_{t_0-1}^{t}\left[\theta^0(\tau)+\int\limits_0^{R(\tau)}vU(x,\tau)dx\right]d\tau \tag{1.32}$$

Where $B(t)=\frac{1}{2}R^2(t)+\int\limits_0^{R(t)}vU(x,\tau)dx.$

Equality (1.32) turns out after multiplication of the equation (1.20) on x and integration in parts.

Let's estimate $B(t)$ from below with the help of inequalities (1.31):

$$\int\limits_0^{R(t)}xU(x,t)dx\geq\int\limits_0^{R_*(t)}xU_*(x,t)dx=(t+1-t_0)\int\limits_0^{D_*(\beta-\varepsilon)}\xi\Phi\left[u(\xi,\beta-\varepsilon)\right]d\xi.$$

The estimation from above for the second composed in (1.32) is carried out{spent} as follows:

$$\int\limits_{t_0-1}^{t}\left[\theta^0(\tau)+\int\limits_0^{R(\tau)}v(\tau)U(x,\tau)dx\right]d\tau\leq\int\limits_{t_0-1}^{t}\left[\theta^0(\tau)+v(\tau)|U|_{2,\Omega_T}R(\tau)\right]d\tau\leq$$

$$\leq(\beta+\varepsilon)(t+1-t_0)+M\int\limits_{t_0-1}^{t}v(\tau)R(\tau)d\tau$$

where $M=|U|_{2,\Omega_T}$.

On the other hand under the theorem of average

$$M\int\limits_{t_0-1}^{t}v(\tau)R(\tau)d\tau=M\cdot(t+1-t_0)\cdot v(t_0)R(t_0),$$

where $t_0\in(t_0,t)$.

Hence

$$\int_{t_0-1}^{t}\left[\theta^0(\tau)+\int_0^{R(\tau)} v(\tau)U(x,\tau)dx\right]d\tau \le (\beta+\varepsilon)(t+1-t_0)+ \tag{1.33}$$

$$+M\cdot(t+1-t_0)\cdot v(t_*)R(t_*)=\left(\beta+\varepsilon+M\cdot v(t_*)R(t_*)\right)(t+1-t_*)$$

Further we shall take advantage of that at big enough t_* and $\forall\varepsilon_1>0:\ |v(t)|<\varepsilon_1$. Then (1.33) – we shall limit from above:

$$\left(\beta+\varepsilon+M\cdot v(t_*)R(t_*)\right)(t+1-t_*)\le(\beta+\varepsilon_2)(t+1-t_*) \tag{1.34}$$

$$\varepsilon_2=\max(\varepsilon,\varepsilon_1)$$

Recollecting identity (1.28) and an inequality (1.34), we have

$$\frac{1}{2}R^2(t)(t+1-t_0)\left(\beta-\varepsilon-\frac{1}{2}D_*^2\cdot(\beta-\varepsilon)\right)\le$$

$$\le B(t_0-1)+\int_{t_0-1}^{t}\left[\theta^0(\tau)+\int_0^{R(\tau)} v(\tau)U(x,\tau)dx\right]d\tau\le(\beta+\varepsilon_2)(t+1-t_0)$$

Passing in last ratio to a limit at $t\to\infty$, we shall receive

$$\overline{\lim_{t\to\infty}}\left(t^{-1}R^2(t)\right)\le D_*^2(\beta-\varepsilon)+2(\varepsilon+\varepsilon_2).$$

On the other hand, from (1.31) follows, that

$$D_*^2(\beta-\varepsilon)\le\underline{\lim_{t\to\infty}}\left(t^{-1}R^2(t)\right)$$

Last two inequalities, a continuity of function $D_*(\beta)$ and an any choice of size ε_2 prove the statement of the theorem.